CHAPTER II

THE SPACE $\mathcal{D}(\Omega)$ AND SCHWARTZ FUNCTIONS

In this chapter we first recall some notations in the real n-dimensional Euclidean space $\mathbb{R}^n$. Later on we are going to study the space $\mathcal{D}(\Omega)$ and Schwartz functions which will be used constantly in the succeeding chapters.

The materials of this chapter are drawn from references [4], [5], [7], [9], [10] and [11].

For function defined on $\mathbb{R}^n$ we need a concise notation for partial derivatives. First we denote $\partial/\partial x_j$ by ∂_j and then we write, for each family $r = (r_1, \ldots, r_n)$ of non-negative integer, $\partial^r = \partial_1^{r_1} \partial_2^{r_2} \ldots \partial_n^{r_n}$. The symbol r is called a multi-index, and its order is $|r| = r_1 + \ldots + r_n$.

2.1 The Space $\mathcal{D}(\Omega)$

2.1.1 <u>Definition</u>. Let φ be a real-valued function defined on an open subset Ω of $\mathbb{R}^n$. The <u>support</u> (or carrier) of φ, denoted by Supp (φ), is the closure of the set on which its valued are different from zero.

2.1.2 <u>Remark</u>. For any two real-valued functions φ and ψ, defined on an open subset Ω of $\mathbb{R}^n$, and any real number $\alpha \neq 0$,

$$\mathrm{Supp}(\varphi + \psi) \subsetneq \mathrm{Supp}(\varphi) \cup \mathrm{Supp}(\psi),$$
$$\mathrm{Supp}(\alpha\varphi) = \mathrm{Supp}(\varphi).$$

Proof : For each x belongs to Ω such that $(\varphi+\psi)(x) = \varphi(x)+\psi(x) \neq 0$, we have that either $\varphi(x) \neq 0$ or $\psi(x) \neq 0$. Then by using the facts, $A \subsetneq B$ implies $\bar{A} \subseteq \bar{B}$ and $\overline{A \cup B} = \bar{A} \cup \bar{B}$, we get the first result.

For any $\alpha \neq 0$ and for each x belongs to Ω such that $\alpha\varphi(x) \neq 0$, we have that $\varphi(x) \neq 0$. And for each x such that $\varphi(x) \neq 0$, we have that $\alpha\varphi(x) \neq 0$, $\alpha \neq 0$. Hence the second result.

2.1.3 Notations. Let Ω be an open subset of $\mathbb{R}^n$. The infinitely differentiable functions on Ω with compact support form a vector space, denoted by $\mathcal{D}(\Omega)$.

For each compact subset K of Ω, these functions φ of $\mathcal{D}(\Omega)$ for which $\mathrm{Supp}(\varphi) \subsetneq K$ form a vector subspace of $\mathcal{D}(\Omega)$, which we shall denote by $\mathcal{D}(\Omega; K)$.

2.1.4 Remark. $\mathcal{D}(\Omega)$ is the union of its subspace $\mathcal{D}(\Omega; K)$ as K varies over all the compact subsets of Ω.

Proof : It suffices to prove that $\mathcal{D}(\Omega) \subseteq \bigcup_{K \subset \Omega} \mathcal{D}(\Omega; K)$.

For every $\varphi \in \mathcal{D}(\Omega)$, φ has the compact support K, say. Then $\varphi \in \mathcal{D}(\Omega; K)$.

2.1.5 Theorem. For any compact subset K of Ω and any non-negative integer m, let

$$p_{m,K}(\varphi) = \max_{|r| \leq m} \sup_{x \in K} \left|\partial^r \varphi(x)\right| \qquad (\varphi \in \mathcal{D}(\Omega; K)),$$

(note that in the bracket means " for all $\varphi \in \mathcal{D}(\Omega ; K)$ ").

Then $p_{m,K}$ is a norm on $\mathcal{D}(\Omega ; K)$.

Proof : It follows immediately from the properties of absolute values and the definition.

2.1.6 Theorem. For all $\varphi_1, \varphi_2 \in \mathcal{D}(\Omega ; K)$, let

$$d(\varphi_1, \varphi_2) = \sum_{m=0}^{+\infty} \frac{p_{m,K}(\varphi_1 - \varphi_2)}{2^m \left[1+p_{m,K}(\varphi_1 - \varphi_2)\right]},$$

where $p_{m,K}$ is defined as in (2.1.5). Then d is a metric on $\mathcal{D}(\Omega;K)$. From now on we shall use the metric d to define a topology for $\mathcal{D}(\Omega;K)$.

Proof : Since $\frac{t}{1+t} \leq 1$ for all $t \geq 0$, we see that for every non-negative integer m,

$$\frac{p_{m,K}(\varphi_1 - \varphi_2)}{2^m \left[1+p_{m,K}(\varphi_1 - \varphi_2)\right]} \leq \frac{1}{2^m} .$$

And since the series $\sum_{m=0}^{+\infty} 1/2^m$ converges, we set that

$$d(\varphi_1,\varphi_2) = \sum_{m=0}^{+\infty} \frac{p_{m,K}(\varphi_1-\varphi_2)}{2^m\left[1+p_{m,K}(\varphi_1-\varphi_2)\right]} \leq \sum_{m=0}^{+\infty} \frac{1}{2^m} < +\infty \quad (\varphi_1,\varphi_2 \in \mathcal{D}(\Omega;K)).$$

The conditions $d(\varphi_1,\varphi_2) \geq 0$, $d(\varphi_1,\varphi_2) = 0$ iff $\varphi_1 = \varphi_2$, and $d(\varphi_1, \varphi_2) = d(\varphi_2, \varphi_1)$ are obvious. We must therefore ckeck the triangular inequality. The result will follow if we prove that if a, b, c are three non-negative numbers and if

$$(*) \qquad c \leq a + b,$$

then

$$(**) \qquad c/(1+c) \leq a/(1+a) + b/(1+b).$$

If c or a+b are equal to zero, there is nothing to prove so that we may assume that none of these two numbers is equal to zero. Then (*) is equivalent to

$$(a+b)^{-1} \leq 1/c ,$$

which implies

$$(1+1/c)^{-1} \leq (1+1/(a+b))^{-1} = a/(1+a+b) + b/(1+a+b).$$

The left-hand side is $c/(1+c)$; the right-hand side is obviously at most equal to

$$a/(1+a) + b/(1+b) ,$$

whence (**). This proves that d is a metric.

2.1.7 Remark. The metric d is translation invariant, i.e.,

$$d(\varphi_1,\varphi_2) = d(\varphi_1-\varphi_2, 0) \qquad (\varphi_1,\varphi_2 \in \mathcal{D}(\Omega;K)).$$

Proof. For all $\varphi_1, \varphi_2 \in \mathcal{D}(\Omega;K)$,

$$d(\varphi_1,\varphi_2) = \sum_{m=o}^{+\infty} \frac{p_{m,K}(\varphi_1-\varphi_2)}{2^m\left[1+p_{m,K}(\varphi_1-\varphi_2)\right]}$$

$$= \sum_{m=0}^{+\infty} \frac{p_{m,K}((\varphi_1-\varphi_2)-0)}{2^m\left[1+p_{m,K}((\varphi_1-\varphi_2)-0)\right]}$$

$$= d(\varphi_1-\varphi_2,\ 0).$$

2.1.8 Lemma. Let $B(\varphi_o, \varepsilon)$ be the open ball in $\mathcal{D}(\Omega;K)$ with centre φ_o and radius ε.

(i) Then there exists an interger $m_o \geq 0$ and $\delta_o(\varepsilon) > 0$ such that $\left\{\varphi \in \mathcal{D}(\Omega;K) : p_{m_o,K}(\varphi-\varphi_o) < \delta_o\right\} \subset B(\varphi_o, \varepsilon)$.

(ii) If (φ_j) is a sequence in $\mathcal{D}(\Omega;K)$ such that for every multi-index r, $(\partial^r\varphi_j)$ converges to $\partial^r\varphi_o$ as $j \to +\infty$, uniformly on K, then there exists j_o such that for all $j > j_o$, $\varphi_j \in B(\varphi_o, \varepsilon)$.

Proof : (i) Since the series $\sum_{m=0}^{+\infty} 1/2^m$ converges, for any given $\varepsilon > 0$, we may find an integer $m_o \geq 0$ such that

$$\sum_{m=m_o+1}^{+\infty} 1/2^m \quad < \quad \varepsilon/2 .$$

And since $(\frac{t}{1+t}) \to 0$ as $t \to 0$, there exists a $\delta_o(\varepsilon) > 0$ such that

$$\frac{p_{m_o,K}(\varphi-\varphi_o)}{1+p_{m_o,K}(\varphi-\varphi_o)} \quad < \quad \varepsilon/4$$

whenever $\qquad p_{m_o,K}(\varphi-\varphi_o) < \delta_o(\varepsilon)$.

As $p_{m,K}$ is nondecreasing with m and $\frac{t}{1+t}$ is increasing with $t(t \geq 0)$, we have

$$\frac{p_{m,K}(\varphi-\varphi_o)}{1+p_{m,K}(\varphi-\varphi_o)} \leq \frac{p_{m_o,K}(\varphi-\varphi_o)}{1+p_{m_o,K}(\varphi-\varphi_o)} < \varepsilon/4 \qquad (m \leq m_o).$$

Thus $\displaystyle\sum_{m=o}^{m_o} \frac{p_{m,K}(\varphi-\varphi_o)}{2^m\left[1+p_{m,K}(\varphi-\varphi_o)\right]} \leq \sum_{m=o}^{m_o} \frac{\varepsilon}{4\cdot 2^m} < \frac{\varepsilon}{4}\cdot 2 = \frac{\varepsilon}{2}$.

Therefore

$$\sum_{m=o}^{+\infty} \frac{p_{m,K}(\varphi-\varphi_o)}{2^m\left[1+p_{m,K}(\varphi-\varphi_o)\right]} = \sum_{m=o}^{m_o} \frac{p_{m,K}(\varphi-\varphi_o)}{2^m\left[1+p_{m,K}(\varphi-\varphi_o)\right]} + \sum_{m=m_o+1}^{+\infty} \frac{p_{m,K}(\varphi-\varphi_o)}{2^m\left[1+p_{m,K}(\varphi-\varphi_o)\right]}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

whenever $\; p_{m_o,K}(\varphi-\varphi_o) < \delta_o$.

(ii) Let $B(\varphi_o,\varepsilon)$ be the open bell in $\mathcal{D}(\Omega;K)$ with centre φ_o and radius ε. Let m_o and $\delta_o(\varepsilon)$ be as in (i) so that $d(\varphi,\varphi_o) < \varepsilon$ whenever $p_{m_o,K}(\varphi-\varphi_o) < \delta_o$. Since for every multi-index r, $(\partial^r\varphi_j) \longrightarrow \partial^r\varphi_o$ as $j \to +\infty$, uniformly on K, for $\delta_o > 0$, there exists j_o such that for all $j > j_o$

$$p_{m_o,K}(\varphi_j-\varphi_o) < \delta_o$$

and thus for all $j > j_o$,

$$\varphi_j \in B(\varphi_o,\varepsilon).$$

2.1.9 Lemma. Under this topology, a sequence (φ_j) in $\mathcal{D}(\Omega;K)$ converges to φ_o in $\mathcal{D}(\Omega;K)$ iff for every multi-index r, $(\partial^r\varphi_j)$ converges to $\partial^r\varphi_o$ as $j \to +\infty$ uniformly on K.

Proof : Necessity. Let r be an arbitrary fixed multi-index, let $|r| = m_o$, and let $\varepsilon > 0$. By the hypothesis on (φ_j), there exists j_o such that for all $j > j_o$,

$$d(\varphi_j,\varphi_o) = \sum_{m=o}^{+\infty} \frac{p_{m,K}(\varphi_j-\varphi_o)}{2^m\left[1+p_{m,K}(\varphi_j-\varphi_o)\right]} < \frac{\varepsilon}{2^{m_o}(1+\varepsilon)},$$

which implies that

$$\frac{p_{m_o,K}(\varphi_j-\varphi_o)}{1+p_{m_o,K}(\varphi_j-\varphi_o)} < \frac{\varepsilon}{1+\varepsilon}.$$

Therefore $p_{m_o,K}(\varphi_j-\varphi_o) < \varepsilon$, which implies that $|\partial^r\varphi_j(x)-\partial^r\varphi_o(x)|<\varepsilon$ for all $x \in K$ and all $j > j_o$. We conclude that $(\partial^r\varphi_j) \to \partial^r\varphi_o$ as $j \to +\infty$, uniformly on K.

Sufficiency. By (2.1.8 (ii)), there exists a j_o such that that for all $j > j_o$, $\varphi_j \in B(\varphi_o, \varepsilon)$. That is, $(\varphi_j) \to \varphi_o$ in $\mathcal{D}(\Omega;K)$.

2.1.10 Theorem. Let $\mathcal{D}(\Omega;K)$ be a space as defined in (2.1.6) and let T be a linear form on $\mathcal{D}(\Omega;K)$. Then the following conditions are equivalent :

(i) T is continuous on $\mathcal{D}(\Omega;K)$.

(ii) There exists a non-negative integer m_o and a positive constant β such that

$$|T(\varphi)| \leq \beta\, p_{m_o,K}(\varphi) \qquad (\varphi \in \mathcal{D}(\Omega;K)).$$

(iii) If (φ_j) is a sequence in $\mathcal{D}(\Omega;K)$ and tends to zero in $\mathcal{D}(\Omega;K)$, then $(T(\varphi_j))$ tends to zero as $j \to +\infty$.

<u>Proof</u> : <u>(i) implies (ii)</u>. Let φ_o be any element of $\mathcal{D}(\Omega;K)$. Because of the translation invariant character of d, we may assume $\varphi_o = 0$. Then, by the continuity of T, for any given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for all $\varphi \in \mathcal{D}(\Omega;K)$, $|T(\varphi)| < \varepsilon$ (*) whenever $d(\varphi, 0) < \delta$. Consider $B(0,\delta)$; by (2.1.8 (i)), there exists an integer $m_o \geq 0$ and a $\delta_o(\delta) > 0$ such that

$$\left\{\varphi \in \mathcal{D}(\Omega;K) \;:\; p_{m_o,K}(\varphi) < \delta_o\right\} \subset B(0,\delta).$$

Thus for every $\varphi \in \mathcal{D}(\Omega;K)$ such that $p_{m_o,K}\left(\dfrac{\delta_o\varphi}{2\,p_{m_o,K}(\varphi)}\right) < \delta_o$,

we have (*) and by the linearity of T that $\dfrac{\delta_o}{2\,p_{m_o,K}(\varphi)}|T(\varphi)| < \varepsilon$.

Choosing $\beta = 2\varepsilon/\delta_o$, we have

$$|T(\varphi)| < \beta\, p_{m_o,K}(\varphi) \qquad (\varphi \in \mathcal{D}(\Omega;K)).$$

Since $0 \in \mathcal{D}(\Omega;K)$ and $p_{m_o,K}(0) = 0$, we can conclude that there exists an integer $m_o \geq 0$ and a constant $\beta > 0$ such that

$$|T(\varphi)| \leq \beta\, p_{m_o,K}(\varphi) \qquad (\varphi \in \mathcal{D}(\Omega;K)).$$

<u>(ii) implies (iii)</u>. Assume (ii); i.e., there exists an integer $m_o \geq 0$, and a constant $\beta > 0$ such that $|T(\varphi)| \leq \beta\, p_{m_o,K}(\varphi)$ $(\varphi \in \mathcal{D}(\Omega;K))$. Let (φ_j) be any sequence in $\mathcal{D}(\Omega;K)$ which tends to zero. Then for any given $\varepsilon > 0$, there exists a j_o such that for all $j > j_o$,

$$d(\varphi_j, 0) = \sum_{m=o}^{+\infty} \frac{p_{m,K}(\varphi_j)}{2^m\left[1+p_{m,K}(\varphi_j)\right]} < \frac{\varepsilon}{2^{m_o}(\beta+\varepsilon)}$$

which implies that

$$\frac{p_{m_o,K}(\varphi_j)}{2^{m_o}\left[1+p_{m_o,K}(\varphi_j)\right]} < \frac{\varepsilon}{2^{m_o}(\beta+\varepsilon)}$$

or $$p_{m_o,K}(\varphi_j) < \varepsilon/\beta.$$

Thus $$|T(\varphi_j)| < \varepsilon \qquad \text{for all } j > j_o.$$

This means that $(T(\varphi_j)) \to 0$ as $j \to +\infty$.

(iii) implies (i). Suppose T is not continuous at φ_0. Because of the translation invariant character of d, we may assume $\varphi_0 = 0$. Then there exists $\varepsilon > 0$ such that for any $\delta > 0$, there is $\varphi \in \mathcal{D}(\Omega;K)$ such that $d(\varphi,0) \leq \delta$ and $|T(\varphi)| > \varepsilon$. Choose $\delta_j = \frac{1}{j}$, $j = 1, 2, 3, \ldots$. Then for each δ_j, there exists $\varphi_j \in \mathcal{D}(\Omega;K)$ such that $d(\varphi_j,0) \leq \delta_j$ and $|T(\varphi_j)| > \varepsilon$. For any given $\delta_0 > 0$, there exists δ_{j_0} such that $\delta_0 \geq \delta_{j_0}$ and therefore for every $j > j_0$, we have that $d(\varphi_j,0) \leq \delta_0$. This means that there exists $(\varphi_j) \to 0$, but $(T(\varphi_j)) \not\to 0$, which contradicts (iii).

2.2 Schwartz Functions

2.2.1 Definition. On $\mathbb{R}$, we define the function

$$\sigma(t) = \begin{cases} \exp(-1/t) & \text{for } t > 0 \\ 0 & \text{for } t \leq 0. \end{cases}$$

We can proved, by induction on the order, that derivatives of σ of all order exist, and are zero, at $t = 0$. Hence σ is infinitely differentiable.

Next we define on $\mathbb{R}^n$ the function

$$\sigma_1(x) = \alpha\, \sigma(1 - \|x\|^2) = \begin{cases} \alpha \exp\left(-\dfrac{1}{1-\|x\|^2}\right) & (\|x\| < 1) \\ 0 & (\|x\| \geq 1). \end{cases}$$

The constant α is defined by

$$\alpha = \left(\int_{\|x\| < 1} \exp\left(-\frac{1}{1-\|x\|^2} \right) dx \right)^{-1},$$

so that we have

$$\int_{\mathbb{R}^n} \sigma_1(x)dx = 1.$$

and hence σ_1 is infinitely differentiable.

For any $\varepsilon > 0$, we put

$$\sigma_\varepsilon(x) = \varepsilon^{-n} \sigma_1(x/\varepsilon) \qquad (x \in \mathbb{R}^n).$$

Throughout this thesis the function σ_ε will be called the Schwartz function.

2.2.2 Remarks. (i) $\displaystyle\int_{\mathbb{R}^n} \sigma_1(x)dx = \alpha S_n \int_0^1 r^{n-1} \exp\left(-\frac{1}{1-r^2}\right) dr,$

where S_n denotes the surface area of a unit sphere.

(ii) For any $\varepsilon > 0$, $\sigma_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} \sigma_\varepsilon(x)dx = 1.$$

(iii) If $B = B(x_0,\varepsilon)$ is a ball such that $\bar{B} \subset \Omega$, then for all $x \in \Omega$, $x \longmapsto \sigma_\varepsilon(x-x_0)$ belongs to $\mathcal{D}(\Omega;\bar{B})$.

<u>Proof</u> : (i) $$\int_{\mathbb{R}^n} \sigma_1(x)dx = \int_{\bar{B}(0,1)} \sigma_1(x)dx$$

$$= \alpha \int_{\bar{B}(0,1)} \exp\left(-\frac{1}{1-\|x\|^2}\right) dx$$

$$= \alpha \int_0^1 r^{n-1}\left(\int_{\|\theta\|=1} \exp\left(-\frac{1}{1-r^2}\right)ds(\theta)\right) dr,$$

where $ds(\theta)$ is the surface area element on the sphere $S(0,r)$.

$$= \alpha \int_0^1 r^{n-1} \exp\left(-\frac{1}{1-r^2}\right)\left(\int_{\|\theta\|=1} ds(\theta)\right) dr$$

$$= \alpha\, S_n \int_0^1 r^{n-1} \exp\left(-\frac{1}{1-r^2}\right) dr.$$

(ii) $\sigma_\varepsilon(x) \in \mathcal{D}(\mathbb{R}^n)$, since $\sigma_1(x)$ is infinitely differentiable, and by changing the variable, we have

$$\int_{\mathbb{R}^n} \sigma_\varepsilon(x)dx = \int_{\mathbb{R}^n} \varepsilon^{-n}\sigma_1(x/\varepsilon)dx = \int_{\mathbb{R}^n} \sigma_1(x/\varepsilon)d(x/\varepsilon) = 1.$$

(iii) Since $\sigma_\varepsilon(x) > 0$ $(\|x\| < \varepsilon)$, and $\sigma_\varepsilon(x) = 0$ $(\|x\| \geq \varepsilon)$, we have that

$$\operatorname{Supp}(\sigma_\varepsilon(x)) = \left\{ x \in \mathbb{R}^n : \|x\| \leq \varepsilon \right\}.$$

Thus

$$\operatorname{Supp}(\sigma_\varepsilon(x-x_0)) = \left\{ x \in \mathbb{R}^n : \|x-x_0\| \leq \varepsilon \right\} = \bar{B}(x_0,\varepsilon).$$

That is, if $\bar{B} \subset \Omega$, then for all $x \in \Omega$, $x \mapsto \sigma_\varepsilon(x-x_o)$ belongs to $\mathcal{D}(\Omega;\bar{B})$.

2.2.3 <u>Theorem.</u> Let f be any continuous function with compact support K contained in Ω. Let δ be the distance from K to the complement of Ω, and for any ε, $0 < \varepsilon < \delta$, let

$$K_\varepsilon = \left\{ x \in \Omega : \text{dist } (x,K) \leq \varepsilon \right\}.$$

Then $f_\varepsilon(x) = \int_{\|y-x\| \leq \varepsilon} f(y)\sigma_\varepsilon(x-y)dy = \int_{\|y\| \leq \varepsilon} f(x-y)\sigma_\varepsilon(y)dy \quad (x \in \Omega)$

belongs to $\mathcal{D}(\Omega;K_\varepsilon)$. Further $f_\varepsilon \to f$ as $\varepsilon \to 0^+$, uniformly on Ω.

<u>Proof</u> : For any ε, $0 < \varepsilon < \delta$, we define the regularization f_ε of f by

$$f_\varepsilon(x) = \int_\Omega f(y)\sigma_\varepsilon(x-y)dy = \int_\Omega f(x-y)\sigma_\varepsilon(y)dy,$$

or

$$f_\varepsilon(x) = \int_{\|y-x\| \leq \varepsilon} f(y)\sigma_\varepsilon(x-y)dy = \int_{\|y\| \leq \varepsilon} f(x-y)\sigma_\varepsilon(y)dy \quad (x \in \Omega).$$

The integral is convergent since f and σ_ε have compact support, we can differentiate f_ε. Then for any multi-index r, we have

$$\partial^r f_\varepsilon(x) = \partial_x^r f_\varepsilon(x) = \int_\Omega f(y)\, \partial_x^r \sigma_\varepsilon(x-y)dy .$$

That is, $f_\varepsilon \in \mathcal{D}(\Omega)$, hence $f_\varepsilon \in \mathcal{D}(\Omega;K\ )$, since $\mathrm{Supp}(f_\varepsilon) \subseteq K\ $.

By (2.2.2(ii)), we have

$$
\begin{aligned}
|f_\varepsilon(x)-f(x)| &= \left| \int_{\|y\|<\varepsilon} f(x-y)\sigma_\varepsilon(y)dy - f(x) \int_{\|y\|<\varepsilon} \sigma_\varepsilon(y)dy \right| \\
&= \left| \int_{\|y\|<\varepsilon} \left[f(x-y)-f(x) \right] \sigma_\varepsilon(y)dy \right| \\
&\leq \int_{\|y\|<\varepsilon} | f(x-y)-f(x) | \sigma_\varepsilon(y)dy \\
&\leq \sup_{\|y\|<\varepsilon} | f(x-y)-f(x) | \int_{\|y\|<\varepsilon} \sigma_\varepsilon(y)dy \\
&\leq \sup_{\|y\|<\varepsilon} | f(x-y)-f(x) | .
\end{aligned}
$$

But now, by the (uniform) continuity of f, the right side tends to zero, uniformly in x, as $\varepsilon \to 0^+$. The proof is complete.

2.2.4 Remark. The function f in the theorem can be uniformly approximated by functions of $\mathcal{D}(\Omega)$ with supports contained in a given compact neighbourhood of K.

2.2.5 Theorem. If K is compact and contained in the open set Ω, there is a function $\varphi \in \mathcal{D}(\Omega)$ taking the value 1 in a neigh neighbourhood of K and lying between 0 and 1 on Ω.

Proof : Let δ be the distance from K to the complement of Ω, and for any ε, $0 < \varepsilon < \delta$, let

$$K_\varepsilon = \left\{ x \in \Omega \ : \ \text{dist}(x,K) \leq \varepsilon \right\}.$$

Then by the Urysohn's lemma ([9]), there is a continuous function, say f ; taking the value 1 on $K_{\varepsilon/2}$, the value 0 outside $K_{3\varepsilon/4}$, and values between 0 and 1 in the annular region between.

If $0 < \varepsilon_1 < \varepsilon/4$, by (2.2.3), we can find the function

$$f_{\varepsilon_1}(x) = \int_{\|y\| \leq \varepsilon_1} f(x-y)\,\sigma_{\varepsilon_1}(y)dy \qquad (x \in \Omega)$$

which belongs to $\mathcal{D}(\Omega;K_{\varepsilon_1})$. Set $\varphi(x) = f_{\varepsilon_1}(x)$. Then φ has all the required properties, i.e.,

(i) $\varphi \in \mathcal{D}(\Omega;K_{\varepsilon_1}) \subset \mathcal{D}(\Omega)$,

(ii) $\varphi(x) \equiv 1$ on K_{ε_1}, since $f(x) \equiv 1$ on K_{ε_1} and (2.2.2 (ii)),

(iii) $0 \leq \varphi(x) \leq 1$ $(x \in \Omega)$, since $0 \leq f(x) \leq 1$ $(x \in \Omega)$.

2.2.6 <u>Theorem</u>. Suppose that the compact set K is contained in the union of the open sets $\Omega_1, \Omega_2, \ldots, \Omega_m$. Then there are non-negative functions $\varphi_j \in \mathcal{D}(\Omega_j)$ such that

$$\varphi(x) = \sum_{j=1}^{m} \varphi_j(x) \begin{cases} \leq 1 & (x \in \Omega = \bigcup_{j=1}^{m} \Omega_j). \\ = 1 & (x \in K). \end{cases}$$

<u>Proof</u> : Let K_1 be any compact neighbourhood of $K - \bigcup_{j=2}^{m} \Omega_j$, contained in Ω_1, let K_2 be a compact neighbourhood of $K - (K_1 \cup \bigcup_{j=3}^{m} \Omega_j)$, contained in Ω_2 , and so on. Then $K \subsetneq \bigcup_{j=1}^{m} K_j$

and $K_i \cap K_j = \emptyset$ for $i \neq j$. By (2.2.5), there are functions $\psi_j \in \mathcal{D}(\Omega_j)$ lying between 0 and 1 on Ω_j and taking the value 1 on K_j. Put

$$\varphi_1 = \psi_1,\ \varphi_2 = \psi_2(1-\psi_1), \dots, \varphi_m = \psi_m(1-\psi_1)(1-\psi_2)\dots(1-\psi_{m-1}).$$

Then all the conditions are satisfied, because

(i) φ_j are non-negative, since $0 \leq \psi_j \leq 1$,

(ii) $\varphi_j \in \mathcal{D}(\Omega_j)$, since $\mathrm{Supp}\,(\varphi_j) \subseteq \mathrm{Supp}\,(\psi_j)$,

(iii) $$\varphi = \sum_{j=1}^{m} \varphi_j = \psi_1 + \psi_2(1-\psi_1) + \dots + \psi_m(1-\psi_1)(1-\psi_1)\dots(1-\psi_{m-1})$$

$$= 1-1+\psi_1+\psi_2(1-\psi_1)+\dots+\psi_m(1-\psi_1)(1-\psi_2)\dots(1-\psi_{m-1})$$

$$= 1-(1-\psi_1)+\psi_2(1-\psi_1)+\dots+\psi_m(1-\psi_1)(1-\psi_2)\dots(1-\psi_{m-1})$$

$$= 1-(1-\psi_1)(1-\psi_2)\dots(1-\psi_m).$$