CHAPTER I

PRELIMINARIES

In this chapter, we will recall, without proof, some notions and facts from topology and real analysis.

The materials of this chapter we drawn from references [2],[3],[5],[6],[8] and [11].

1.1 Topological Spaces

1.1.1 Definition. A topology on a set E is a set $\mathcal{T}$ of subsets of E satisfying the following two conditions :

(i) The union of any family $(F_\lambda)_{\lambda \in \Lambda}$ of sets belonging to $\mathcal{T}$ belongs to $\mathcal{T}$;

(ii) The set E belongs to $\mathcal{T}$, and the intersection of any two sets belonging to $\mathcal{T}$ belongs to $\mathcal{T}$.

1.1.2 Definition. A set E with a topology $\mathcal{T}$ defined on it is called a topological space , denoted by $(E, \mathcal{T})$; we shall often omit $\mathcal{T}$ and refer to E as a topological space.

1.1.3 Definition. The subsets of E which belong to $\mathcal{T}$ are called the open sets of the topological space E.

1.1.4 Example. The set $\mathcal{B}$ of open sets in a metric space E satisfies the two conditions of (1.1.1). This topology $\mathcal{B}$ is called the topology of the metric space E (or is said to be defined by the distance given on E).

1.1.5 <u>Definition</u>. A topological space E is said to be <u>metrizable</u> if its topology can be defined by a distance on E. (and then this topology is also said to be metrizable).

1.1.6 <u>Definition</u>. A topological space E is said to be <u>Hausdorff</u> (or <u>separated</u>), if it satisfies the following " Hausdorff axiom " :

Given any two distinct points x, y in E, there exists a neighborhood U of x and a neighborhood V of y which do not intersect.

1.1.7 <u>Example</u>. Every metrizable space is Hausdorff.

1.1.8 <u>Definition</u>. A topological space E is called <u>connected</u> when it is not the union of two non-empty disjoint open sets. This is equivalent to saying that E contains no proper non-empty sets which are both open and closed.

1.1.9 <u>Example</u>. The n-dimensional real Euclidean space $\mathbb{R}^n$ is connected.

1.1.10 <u>Definition</u>. A subset F of a topological space E is called <u>compact</u> if every cover of F by open subsets of E contains a finite sub-cover.

1.1.11 <u>Definition</u>. A subset F of a topological space E is called <u>relatively compact</u> if its closure $\bar{F}$ is compact.

1.1.12 <u>Examples</u>. A closed bounded subset of $\mathbb{R}^n$ is compact. A bounded set of $\mathbb{R}^n$ is relatively compact.

1.1.13 <u>Definition</u>. A topological space E is said to be <u>locally compact</u> if each point of the space has a compact neighborhood.

1.1.14 <u>Example</u>. $\mathbb{R}^n$ is locally compact but not compact.

1.2 Vector Spaces

1.2.1 <u>Definitions</u>. Let K be an arbitrary (commutative) field with elements (called scalars) α, β, ... , with zero element 0 and identity element 1. A <u>vector space</u> over K (or <u>linear space</u> over K) is a set E with elements (called points or vectors) x, y, z,..., which has the following properties :

(i) For every two elements $x, y \in E$ a sum $x + y$ is defined in E ; under this addition, E is an abelian group, i.e. for all $x, y, z \in E$ we have

(a) $x + y = y + x$,

(b) $x + (y + z) = (x + y) + z$,

(c) There exists $0 \in E$ with $x + 0 = x$ for all $x \in E$,

(d) There exists for each x E an $x' \in E$ with $x + x' = 0$.

(ii) For every $\alpha \in K$ and every $x \in E$ the product $\alpha x = x \alpha$ of α with x is defined as an element of E, and for all $x, y \in E$, $\alpha, \beta \in K$ we have

(e) $x(\alpha + \beta) = x\alpha + x\beta$,

(f) $(x+y)\alpha = x\alpha + y\alpha$,

(g) $x(\alpha\beta) = (x\alpha)\beta$,

(h) $x \cdot 1 = x$.

If K is the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, then E is called a <u>real</u> or <u>complex vector space</u> respectively.

A subset F of elements of a vector space E is a vector space provided that whenever it contains x and y it also contains $x\alpha + y\beta$, for arbitrary α, β in K. F is then called a <u>linear subspace</u> (or simply <u>subspace</u>) of E.

1.2.2 <u>Examples</u>. (i) The n-dimensional real Euclidean space $\mathbb{R}^n$ is the set of all n-tuples $x = (x_1, \ldots, x_n)$ of real numbers, where addition and multiplication by a scalar $\alpha \in \mathbb{R}$ are defined by

$$(x_1,\ldots,x_n)+(y_1,\ldots,y_n) = (x_1+ y_1,\ldots, x_n+ y_n) ,$$

$$\alpha(x_1,\ldots,x_n) = (\alpha x_1,\ldots, \alpha x_n).$$

The zero vector is $(0,\ldots, 0)$. The properties of a real vector space can be easily verified.

(ii) Let Ω be an open subset of $\mathbb{R}^n$. The set $C(\Omega)$ of all continuous real-value functions on Ω is a vector space.

(iii) The set $\overset{\infty}{C}(\Omega)$ of all infinitely differentiable real-value functions on Ω is a subspace of $C(\Omega)$.

1.3 Linear Maps

1.3.1 Definition. Let E and F be two vector spaces over the same field of scalars K. We recall that a map f from E into F is said to be linear if it satisfies the identity

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all $x, y \in E$ and $\alpha, \beta \in K$.

1.3.2 Definition. Let E be a vector space over the field K. A linear form (or linear functional) on E is a map f from E into K which satisfies the identity

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all $x, y \in E$ and $\alpha, \beta \in K$.

1.4 Topological Vector Spaces

1.4.1 Definitions. Let E be a vector space over K. A topology on E is said to be compatable with the vector space structure if the mapping $(x,y) \longmapsto x + y$ of $E \times E$ into E, and $(\alpha, x) \longmapsto \alpha x$ of $K \times E$ into E are continuous. A vector space endowed with a topology compatible with its vector space structure is called a topological vector space.

1.4.2 <u>Example</u>. A normed vector space, equipped with the topology defined by its norm, is a topological vector space.

1.4.3 <u>Definition</u>. A real-valued function q defined on a vector space E is called a <u>semi-norm</u> (psurdo-norm, pre-norm) on E, of the following conditions are satisfied :

(i) $q(x) \geq 0$ for all $x \in E$,

(ii) $q(\alpha x) = |\alpha| q(x)$ for any $x \in E$ and any scalar $\alpha \in K$,

(iii) $q(x+y) \leq q(x)+q(y)$ (subadditivity) for any pair $x,y \in E$.

1.4.4 <u>Example</u>. In the n-dimensional Euclidean space, define

$$q(x) = \max_{1 \leq j \leq n} |x_j| .$$ Then $q(x)$ is a semi-norm.

1.4.5 <u>Remark</u>. A semi-norm q on a vector space E is a norm if for all $x \in E$, $q(x) = 0$ implies $x = 0$.