CHAPTER III

HYPERGRAPHS WITH PRESCRIBED NEIGHBOURHOOD STRUCTURES

In this chapter we define neighbourhood hypergraphs and discuss the problem on realizability of a given family of hypergraphs as neighbourhood hypergraphs.

3.1 Neighbourhood Hypergraphs.

Let $H = (V, \mathcal{E})$ be a hypergraph. For each vertex v in H, we associate a hypergraph $vH = (vV, v\mathcal{E})$, where

$$v\mathcal{E} = \{E-\{v\} / E \in \mathcal{E},\ v \in E \text{ and } E-\{v\} \neq \emptyset\}$$

and

$$vV = \cup v\mathcal{E} .$$

The hypergraph vH will be called the *neighbourhood hypergraph of H at v*.

Let u, v be vertices in H. If u is a vertex in vH, then u(vH), the neighbourhood hypergraph of vH at u, will be denoted by uvH.

3.1.1 <u>Proposition</u> Let $H = (V, \mathcal{E})$ be a hypergraph. Let u, v be distinct vertices in H. Then u is a vertex in vH, if and only if, v is a vertex in uH. Furthermore, when this is the case we have uvH = vuH and $d_{uH}(v) = d_{vH}(u)$.

<u>Proof.</u> Let $H = (V, \mathcal{E})$ be a hypergraph. Let u, v be distinct vertices in H. Assume that u is a vertex in vH. Hence u belongs to vV. Therefore u belongs to vE for some edge vE in $v\mathcal{E}$. Hence $vE \cup \{v\}$ is in $\mathcal{E}$ and u belongs to $vE \cup \{v\}$. Therefore u, v belongs to $vE \cup \{v\}$. Hence $(vE \cup \{v\}) - \{u\}$ is in $u\mathcal{E}$ and v belongs to $(vE \cup \{v\}) - \{u\}$. Therefore v belongs to $\cup u\mathcal{E}$, i.e. v belongs to uV. Hence v is a vertex in uH.

Similarly we can show that if v is a vertex in uH then u is a vertex in vH.

Suppose that u is a vertex in vH. Observe that

$$\begin{aligned} uvE \in u(v\mathcal{E}) &\Leftrightarrow uvE \cup \{u\} \in v\mathcal{E}, \\ &\Leftrightarrow (uvE \cup \{u\}) \cup \{v\} \in \mathcal{E}, \\ &\Leftrightarrow (uvE \cup \{v\}) \cup \{u\} \in \mathcal{E}, \\ &\Leftrightarrow uvE \cup \{v\} \in u\mathcal{E}, \\ &\Leftrightarrow uvE \in v(u\mathcal{E}). \end{aligned}$$

Hence $u(v\mathcal{E}) = v(u\mathcal{E})$, it follows that $u(vV) = v(uV)$. Therefore uvH = vuH. Next, we observe that

$$
\begin{aligned}
d_{uH}(v) &= |\{uE \in u\mathcal{E} / v \in uE\}| ,\\
&= |\{E \in \mathcal{E} / u \in E,\ E-\{u\} \neq \emptyset \text{ and } v \in E-\{u\}\}| ,\\
&= |\{E \in \mathcal{E} / u \in E \text{ and } v \in E\}| ,\\
&= |\{E \in \mathcal{E} / v \in E,\ E-\{v\} \neq \emptyset \text{ and } u \in E-\{v\}\}| ,\\
&= |\{vE \in v\mathcal{E} / u \in vE\}| ,\\
&= d_{vH}(u).
\end{aligned}
$$

Therefore $d_{uH}(v) = d_{vH}(u)$. #

3.1.2 <u>Proposition</u> Let $H = (V,\mathcal{E})$ and $H' = (V',\mathcal{E}')$ be hypergraphs. Let α be an isomorphism from H to H'. Let v be any vertex in H. Then $vH \cong \alpha(v)H'$.

<u>Proof.</u> Let α be an isomorphism from $H = (V,\mathcal{E})$ to $H' = (V',\mathcal{E}')$. Let v be any vertex in H. Let ρ be the restriction of α to vV. A straightforward verification shows that ρ is an isomorphism from vH to $\alpha(v)H'$. Hence $vH \cong \alpha(v)H'$. #

3.2 <u>Realizations.</u>

Let $\Gamma = (K_v)_{v \in I}$ be a finite family of hypergraphs. A hypergraph $H = (V,\mathcal{E})$ will be said to be a <u>realization of Γ</u> if there exists a bijection σ from I to V such that $K_v \cong \sigma(v)H$ for all v in I. When a realization of the family Γ exists, we say that Γ is <u>realizable</u>.

In the case Γ is the empty family, i.e. when $I = \emptyset$, it can be verified that the empty hypergraph $(\emptyset,\emptyset)$ is a realization of Γ. Hence in the sequal we shall be interested in non-empty families of hypergraphs only.

In general, a family of hypergraphs $\Gamma = (K_v)_{v \varepsilon I}$ may or may not have a realization. If Γ is realizable, its realization may or may not be unique. These will be illustrated by examples.

3.2.1 Proposition. Let $\Gamma = (K_v)_{v \varepsilon I}$ be a family of hypergraphs. Let

$$J = \{v \varepsilon I / K_v \neq (\emptyset,\emptyset)\} .$$

If the family $\Gamma' = (K_v)_{v \varepsilon J}$ is realizable, then Γ is also realizable.

Proof. Let $\Gamma = (K_v)_{v \varepsilon I}$ be a family of hypergraphs. Let

$$J = \{v \varepsilon I / K_v \neq (\emptyset,\emptyset)\} .$$

Suppose that the family $\Gamma' = (K_v)_{v \varepsilon J}$ is realizable. Let $H = (V,\mathcal{E})$ be a realization of Γ'. Hence there exists a bijection σ from J to V such that $K_v \cong \sigma(v)H$ for all v in J. Let

$$V^* = V \cup (I-J)$$

and

$$\mathcal{E}^* = \mathcal{E} \cup \{\{v\} / v \varepsilon I-J\}.$$

It can be seen that $\cup \mathcal{E}^* = V^*$. Hence $H^* = (V^*, \mathcal{E}^*)$ is a hypergraph .

Define $\sigma^* : I \to V^*$ by

$$\sigma^*(v) = \begin{cases} v & \text{if } v \notin J, \\ \sigma(v) & \text{if } v \in J. \end{cases}$$

A straightforward verification shows that σ^* is a bijection from I to V^* such that $K_v \cong \sigma^*(v)H^*$ for all v in I. Hence H^* is a realization of Γ , i.e. Γ is realizable. #

From Proposition 3.2.1 , we see that it suffices to consider only families of non-empty hypergraphs. Hence in the sequal we shall assume that all hypergraphs of any family are non-empty hypergraphs.

Let $\Gamma = (K_v)_{v \in I}$ be a family of hypergraphs. A hypergraph $H = (V, \mathcal{E})$ will be said to be a <u>proper realization of Γ</u> if H is a realization of Γ and $|E| \geq 2$ for all edges E in $\mathcal{E}$.

3.2.2 <u>Proposition</u>. Let $\Gamma = (K_v)_{v \in I}$ be a family of hypergraphs. Then Γ is realizable if and only if it has a proper realization.

<u>Proof.</u> Let $\Gamma = (K_v)_{v \in I}$ be a family of hypergraphs. If Γ has a proper realization, it is clear that Γ is realizable.

Suppose that Γ is realizable. Let $H = (V, \mathcal{E})$ be a realization of Γ. Let

$$\mathcal{E}^* = \{E / E \in \mathcal{E} \text{ and } |E| \geq 2\}.$$

Let u be any element in V. Since $uH = (uV, u\mathcal{E})$ is isomorphic to some K_v, which is non-empty, hence $u\mathcal{E} \neq \emptyset$. Choose an element uE in $u\mathcal{E}$. Hence u is not in uE and $uE \cup \{u\}$ is in $\mathcal{E}$. Therefore $|uE \cup \{u\}| \geqslant 2$. Hence $uE \cup \{u\}$ is in $\mathcal{E}^*$. Therefore u is in $\cup\mathcal{E}^*$. Hence $V \subseteq \cup\mathcal{E}^*$. Clearly, from definition of $\mathcal{E}^*$, we have $\cup\mathcal{E}^* \subseteq \cup\mathcal{E} = V$. Therefore $V = \cup\mathcal{E}^*$. Hence $H^* = (V, \mathcal{E}^*)$ is a hypergraph. Observe that

$$
\begin{aligned}
u\mathcal{E} &= \{E-\{u\}/E \in \mathcal{E},\ u \in E \text{ and } E-\{u\} \neq \emptyset\}, \\
&= \{E-\{u\}/E \in \mathcal{E},\ u \in E \text{ and } |E| \geqslant 2\}, \\
&= \{E-\{u\}/E \in \mathcal{E}^* \text{ and } u \in E\}, \\
&= \{E-\{u\}/E \in \mathcal{E}^*,\ u \in E \text{ and } E-\{u\} \neq \emptyset\}, \\
&= u\mathcal{E}^*.
\end{aligned}
$$

Hence $\cup u\mathcal{E} = \cup u\mathcal{E}^*$. Therefore $uH = uH^*$. Hence $uH = uH^*$ for all u in V. Therefore $H^* = (V, \mathcal{E}^*)$ is a realization of Γ. Since $|E| \geqslant 2$ for all edges E in $\mathcal{E}^*$, hence H^* is a proper realization of Γ. #

Let $\Gamma = (K_v)_{v \in I}$ be a family of hypergraphs. To determine that Γ is realizable, by Proposition 3.2.2 , it suffices to find only a proper realization of Γ. Hence in the sequal any realization of a given family of hypergraphs Γ we mean a proper realization.

3.2.3 <u>Proposition.</u> Let $\Gamma = (K_v)_{v \in I}$ be a family of hypergraphs. If Γ is realizable, then there exists a realization $H = (I, \mathcal{E})$ such that $K_v \cong vH$ for all v in I.

<u>Proof.</u> Assume that $\Gamma = (K_v)_{v \in I}$ be a realizable family of hypergraphs. Let $H^* = (V^*, \mathcal{E}^*)$ be a realization of Γ. Hence there exists a bijection σ from I to V^* such that $K_v \cong \sigma(v)H^*$ for all v in I. Clearly, σ^{-1} is a bijection from V^* to I. Let

$$\mathcal{E} = \{\sigma^{-1}[E] \,/\, E \in \mathcal{E}^*\} .$$

Hence, by Proposition 2.2.1 , σ^{-1} is an isomorphism from H^* to $H = (I, \mathcal{E})$, i.e. $H^* \cong H$. By Proposition 3.1.2 , we see that for all v in I,

$$\begin{aligned} K_v &\cong \sigma(v)H^* , \\ &\cong \sigma^{-1}(\sigma(v))H , \\ &= vH . \end{aligned}$$

Hence $H = (I, \mathcal{E})$ is a realization of Γ such that $K_v \dot{\cong} vH$ for all v in I. #

From Proposition 3.2.3 , we have

3.2.4. <u>Proposition.</u> Let $\Gamma = (K_v)_{v \in I}$ be a family of hypergraphs. If $H^* = (V^*, \mathcal{E}^*)$ is a realization of Γ , then there exists a realization $H = (I, \mathcal{E})$ isomorphic to H^* such that $K_v \dot{\cong} vH$ for all v in I.

3.3 Γ-injections.

In this section we introduce the concept of Γ-injection and relate realizability of a given family of hypergraphs with the existence of what is called "compatible full family of Γ-injections".

Let $\Gamma = (K_v)_{v \varepsilon I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a family of hypergraphs. For each v in I, a one-to-one function α_v from W_v into $I-\{v\}$ will be called an <u>(Γ, v)-injection</u> or simply <u>Γ-injection.</u> Two (Γ, v)-injections α_v and α_v' will be said to be <u>equivalent</u> if there exists an automorphism Θ_v in K_v such that $\alpha_v' = \alpha_v \circ \Theta_v$. It can be verified that being equivalent is an equivalence relation. For each (Γ, v)-injection α_v and each u in I, we let

$$\mathcal{S}(\alpha_v, u) = \{\{v\} \cup \alpha_v[F_v] / F_v \varepsilon \mathcal{F}_v \text{ and } u \varepsilon \alpha_v[F_v]\}.$$

Two Γ-injections α_v and α_u are said to be <u>compatible</u> if

$$\mathcal{S}(\alpha_v, u) = \mathcal{S}(\alpha_u, v) .$$

By a <u>family of Γ-injections</u> we mean any family $A_J = (\alpha_v)_{v \varepsilon J}$, where $J \subseteq I$ and α_v is an (Γ, v)-injection for all v in J. In the case $J = I$, the family A_J is said to be a <u>full family of Γ-injections.</u> A family of Γ-injections $A_J = (\alpha_v)_{v \varepsilon J}$ is said to be <u>compatible</u> if α_v and α_u are compatible for all u, v in J. Two families of Γ-injections $A_J = (\alpha_v)_{v \varepsilon J}$ and $A_J' = (\alpha_v')_{v \varepsilon J}$ are said to be <u>equivalent</u> if α_v and α_v' are equivalent for all v in J. For each family of Γ-injections $A_J = (\alpha_v)_{v \varepsilon J}$, we let

$$\mathcal{E}^{A_J} = \{\{v\} \cup \alpha_v[F_v] / v \varepsilon J \text{ and } F_v \varepsilon \mathcal{F}_v\} .$$

Clearly, $J = \cup \mathcal{E}^{A_J}$, hence $H^{A_J} = (J, \mathcal{E}^{A_J})$ is a hypergraph. The hypergraph H^{A_J} will be called the <u>hypergraph induced by the family of Γ-injections</u> A_J.

3.3.1 Theorem. Let $\Gamma = (K_v)_{v \in I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a realizable family of hypergraphs. Let $H = (I, \mathcal{E})$ be a realization of Γ. For each v in I, let α_v be any isomorphism from K_v to vH. Then $A = (\alpha_v)_{v \in I}$ is a compatible full family of Γ-injections and H is the hypergraph induced by the family A.

Proof. Let $\Gamma = (K_v)_{v \in I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a realizable family of hypergraphs. Let $H = (I, \mathcal{E})$ be a realization of Γ. For each v in I, let α_v be an isomorphism from K_v to vH. For any v in I, it can be seen that $vI \subseteq I-\{v\}$, then α_v is a one-to-one function from W_v into $I-\{v\}$, i.e. α_v is an (Γ, v)-injection. Then $A = (\alpha_v)_{v \in I}$ is a full family of Γ-injections.

Let u, v be distinct elements in I. Let E be any set in $\mathcal{S}(\alpha_u, v)$. Hence

$$E = \{u\} \cup \alpha_u[F_u]$$

for some F_u in $\mathcal{F}_u$ such that v belongs to $\alpha_u[F_u]$. Since α_u is an isomorphism from K_u to uH, hence $\alpha_u[F_u]$ belongs to $u\mathcal{E}$. Hence, it can be seen that $\{u\} \cup \alpha_u[F_u]$ belongs to $\mathcal{E}$, i.e. E belongs to $\mathcal{E}$. Since u, v belong to E and u, v are distinct, we obtain $E-\{v\} \neq \emptyset$. Hence $E-\{v\}$ belongs to $v\mathcal{E}$ and u belongs to $E-\{v\}$. Since α_v is an isomorphism from K_v to vH,

$$E-\{v\} = \alpha_v[F_v]$$

for some F_v in $\mathcal{F}_v$. Hence

$$\{v\} \cup \alpha_v[F_v] = \{v\} \cup (E-\{v\}),$$

$$= E .$$

Since u is in E-{v}, hence u belongs to $\alpha_v[F_v]$. Hence $\{v\} \cup \alpha_v[F_v]$ belongs to $\mathcal{S}(\alpha_v,u)$, i.e. E belongs to $\mathcal{S}(\alpha_v,u)$. Hence

$$\mathcal{S}(\alpha_u,v) \subseteq \mathcal{S}(\alpha_v,u) .$$

Similarly we can show that

$$\mathcal{S}(\alpha_v,u) \subseteq \mathcal{S}(\alpha_u,v) .$$

Hence $\mathcal{S}(\alpha_v,u) = \mathcal{S}(\alpha_u,v)$, i.e. α_v and α_u are compatible. Hence $A = (\alpha_v)_{v \in I}$ is a compatible full family of Γ-injections.

Next, we show that H is the hypergraph induced by the family A. Observe that

$$E \in \mathcal{E}^A \Leftrightarrow E = \{v\} \cup \alpha_v[F_v] \text{ for some } v \text{ in } I \text{ and } F_v \text{ in } \mathcal{F}_v,$$

$$\Leftrightarrow E-\{v\} = \alpha_v[F_v] \text{ for some } v \text{ in } I \text{ and } F_v \text{ in } \mathcal{F}_v,$$

$$\Leftrightarrow E-\{v\} \in v\mathcal{E} \text{ for some } v \text{ in } I,$$

$$\Leftrightarrow E \in \mathcal{E}.$$

Hence $\mathcal{E}^A = \mathcal{E}$. Therefore

$$H = (I,\mathcal{E}),$$

$$= (I,\mathcal{E}^A),$$

$$= H^A.$$

Hence H is the hypergraph induced by the family A. #

3.3.2 <u>Corollary</u> Let $\Gamma = (K_v)_{v \varepsilon I}$ be a family of hypergraphs. If H^* is a realization of Γ, then H^* is isomorphic to some hypergraph induced by a compatible full family of Γ-injections.

<u>Proof.</u> Let $\Gamma = (K_v)_{v \varepsilon I}$ be a family of hypergraphs. Suppose that H^* is a realization of Γ. By Proposition 3.2.4., there exists a realization $H = (I, \mathcal{E})$ isomorphic to H^* such that

$$K_v \cong vH$$

for all v in I. For each v in I, let α_v be an isomorphism from K_v to vH. By Theorem 3.3.1., $A = (\alpha_v)_{v \varepsilon I}$ is a compatible full family of Γ-injections and H is the hypergraph induced by the family A. Hence H^* is isomorphic to the hypergraph induced by the compatible full family of Γ-injections A. #

3.3.3. <u>Theorem</u> Let $\Gamma = (K_v)_{v \varepsilon I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a family of hypergraphs. If there exists a compatible full family $A = (\alpha_v)_{v \varepsilon I}$ of Γ-injections, then Γ is realizable and H^A, the hypergraph induced by the family A, is a realization of Γ. Furthermore each α_v is an isomorphism from K_v to vH^A.

<u>Proof.</u> Let $\Gamma = (K_v)_{v \varepsilon I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a family of hypergraphs. Suppose that there exists a compatible full family $A = (\alpha_v)_{v \varepsilon I}$ of Γ-injections. Then the hypergraph induced by the family A is $H^A = (I, \mathcal{E}^A)$, where

$$\mathcal{E}^A = \{\{u\} \cup \alpha_u[F_u] \,/\, u \in I \text{ and } F_u \in \mathcal{F}_u\}.$$

Let v be any element in I. We shall show that

$$\alpha_v[\mathcal{F}_v] = v\mathcal{E}^A. \qquad \text{.........................(1)}$$

Let F_v be an edge in $\mathcal{F}_v$. Then $\{v\} \cup \alpha_v[F_v]$ belongs to $\mathcal{E}^A$. It follows that $\alpha_v[F_v]$ belongs to $v\mathcal{E}^A$. Hence, we have

$$\alpha_v[\mathcal{F}_v] \subseteq v\mathcal{E}^A. \qquad \text{.........................(2)}$$

Let vE be an edge in $v\mathcal{E}^A$. Then $\{v\} \cup vE$ belongs to $\mathcal{E}^A$. Hence

$$\{v\} \cup vE = \{u\} \cup \alpha_u[F_u]. \qquad \text{.........................(3)}$$

for some u in I and some F_u in $\mathcal{F}_u$.

<u>Case 1.</u> If v = u, then $vE = \alpha_v[F_v]$. Hence vE belongs to $\alpha_v[\mathcal{F}_v]$.

<u>Case 2.</u> If $v \neq u$, then v belongs to $\alpha_u[F_u]$. From the definition of $\mathcal{S}(\alpha_u, v)$, we see that $\{u\} \cup \alpha_u[F_u]$ belongs to $\mathcal{S}(\alpha_u, v)$. Since the family A is compatible, hence α_v and α_u are compatible. Hence

$$\mathcal{S}(\alpha_u, v) = \mathcal{S}(\alpha_v, u).$$

Hence $\{u\} \cup \alpha_u[F_u]$ belongs to $\mathcal{S}(\alpha_v, u)$. Hence

$$\{u\} \cup \alpha_u[F_u] = \{v\} \cup \alpha_v[F_v]. \qquad \text{.................(4)}$$

for some F_v in $\mathcal{F}_v$ such that u belongs to $\alpha_v[F_v]$. From (3) and (4), we have

$$\{v\} \cup vE = \{v\} \cup \alpha_v[F_v].$$

Therefore $vE = \alpha_v[F_v]$. Hence vE belongs to $\alpha_v[\mathcal{F}_v]$.
Hence, in any case, we have

$$v\mathcal{E}^A \subseteq \alpha_v[\mathcal{F}_v]. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

Hence, from (2) and (5), we have (1). Observe that

$$\begin{aligned} \alpha_v[W_v] &= \alpha_v[\cup\mathcal{F}_v], \\ &= \cup\alpha_v[\mathcal{F}_v], \\ &= \cup v\mathcal{E}^A, \\ &= vI. \end{aligned}$$

Hence α_v is a function from W_v onto vI. Since α_v is an (Γ, v)-injection, α_v is a one-to-one function from W_v into $I-\{v\}$. Since $vI \subsetneq I-\{v\}$, hence α_v is a one-to-one function from W_v onto vI. From $\alpha_v[\mathcal{F}_v] = v\mathcal{E}^A$, we have

F_v belongs to $\mathcal{F}_v$ if and only if $\alpha_v[F_v]$ belongs to $v\mathcal{E}^A$.

Hence α_v is an isomorphism from K_v to vH^A, i.e. $K_v \cong vH^A$.
Hence

$$K_v \cong vH^A$$

for all v in I. Therefore H^A is a realization of Γ and each α_v is an isomorphism from K_v to vH^A. #

From Theorem 3.3.1 and Theorem 3.3.3 we have

3.3.4 Theorem. Let $\Gamma = (K_v)_{v \varepsilon I}$ be a family of hypergraphs. Then Γ is realizable if and only if there exists a compatible full family $A = (\alpha_v)_{v \varepsilon I}$ of Γ-injections.

3.3.5 Theorem. Let $\Gamma = (K_v)_{v \varepsilon I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a family of hypergraphs. Let $A = (\alpha_v)_{v \varepsilon I}$ and $A' = (\alpha'_v)_{v \varepsilon I}$ be equivalent families of Γ-injections. If A is compatible, then A' is also compatible.

Proof. Let $\Gamma = (K_v)_{v \varepsilon I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a family of hypergraphs. Let $A = (\alpha_v)_{v \varepsilon I}$ and $A' = (\alpha'_v)_{v \varepsilon I}$ be equivalent families of Γ-injections. Suppose that A is compatible.

Let u, v be any elements in I. We shall show that

$$\mathcal{S}(\alpha_u, v) = \mathcal{S}(\alpha'_u, v). \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Let E be any set in $\mathcal{S}(\alpha_u, v)$. Hence

$$E = \{u\} \cup \alpha_u[F_u]$$

for some F_u in $\mathcal{F}_u$ such that v belongs to $\alpha_u[F_u]$. Since A and A' are equivalent, α_u and α'_u are equivalent. Hence there exists Θ_u an automorphism in K_u such that $\alpha'_u = \alpha_u \circ \Theta_u$. Since Θ_u is an automorphism in K_u, $F_u = \Theta_u[F'_u]$ for some F'_u in $\mathcal{F}_u$. Therefore $\Theta_u[F'_u]$ is in $\mathcal{F}_u$. Observe that

$$\alpha_u[F_u] = \alpha_u[\Theta_u[F'_u]],$$

$$= (\alpha_u \circ \Theta_u)[F'_u],$$

$$= \alpha'_u [F'_u].$$

Therefore $E = \{u\} \cup \alpha'_u[F'_u]$. Since v is in $\alpha_u[F_u]$, hence v belongs to $\alpha'_u[F'_u]$. Hence $\{u\} \cup \alpha'_u[F'_u]$ belongs to $\mathcal{S}(\alpha'_u, v)$. Therefore E belongs to $\mathcal{S}(\alpha'_u, v)$. Hence

$$\mathcal{S}(\alpha_u, v) \subseteq \mathcal{S}(\alpha'_u, v) .$$

Similarly we can show that

$$\mathcal{S}(\alpha'_u, v) \subseteq \mathcal{S}(\alpha_u, v) .$$

Therefore we have (1). Similarly we can show that

$$\mathcal{S}(\alpha_v, u) = \mathcal{S}(\alpha'_v, u) . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Since A is compatible, hence α_v and α_u are compatible. Therefore

$$\mathcal{S}(\alpha_u, v) = \mathcal{S}(\alpha_v, u) .$$

Hence, by (1) and (2), we have

$$\mathcal{S}(\alpha'_u, v) = \mathcal{S}(\alpha'_v, u) .$$

Therefore α'_u and α'_v are compatible. Hence A' is compatible. #

3.3.6 <u>Theorem</u>. Let $\Gamma = (K_v)_{v \in I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a family of hypergraphs. Let $A = (\alpha_v)_{v \in I}$ and $A' = (\alpha'_v)_{v \in I}$ be compatible full families of Γ-injections. Then A and A' are equivalent if and only if $H^A = H^{A'}$.

<u>Proof.</u> Let $\Gamma = (K_v)_{v \in I}$, where $K_v = (W_v, \mathcal{F}_v)$ for all v in I, be a family of hypergraphs. Let $A = (\alpha_v)_{v \in I}$ and $A' = (\alpha'_v)_{v \in I}$ be compatible full families of Γ-injections.

Assume that A and A' are equivalent. Hence for each v in I, there exists an automorphism θ_v in K_v such that $\alpha'_v = \alpha_v \circ \theta_v$. Observe that

$$\begin{aligned} \mathcal{E}^A &= \{\{v\} \cup \alpha_v[F_v] / v \in I \text{ and } F_v \in \mathcal{F}_v\}, \\ &= \{\{v\} \cup \alpha_v[\theta_v[F_v]] / v \in I \text{ and } F_v \in \mathcal{F}_v\}, \\ &= \{\{v\} \cup (\alpha_v \circ \theta_v)[F_v] / v \in I \text{ and } F_v \in \mathcal{F}_v\}, \\ &= \{\{v\} \cup \alpha'_v[F_v] / v \in I \text{ and } F_v \in \mathcal{F}_v\}, \\ &= \mathcal{E}^{A'}. \end{aligned}$$

Hence, we have $H^A = H^{A'}$.

Assume that $H^A = H^{A'}$. Let v be any element in I. By Theorem 3.3.3 , α_v and α'_v are isomorphisms from K_v to vH^A and $vH^{A'}$ respectively. Since $H^A = H^{A'}$, hence $vH^A = vH^{A'}$. A straightforward verification shows that $\alpha_v^{-1} \circ \alpha'_v$ is an automorphism in K_v. Since $\alpha'_v = \alpha_v \circ (\alpha_v^{-1} \circ \alpha'_v)$ and $\alpha_v^{-1} \circ \alpha'_v$ is an automorphism in K_v, hence α_v and α'_v are equivalent. Hence A and A' are equivalent. #

3.4 Examples.

In this section we illustrate how we can apply our results to obtain all non-isomorphic realizations of a given family of hypergraphs. For a given family $\Gamma = (K_v)_{v \in I}$, where $K_v = (W_v, \mathcal{F}_v)$ and $|W_v| < |I|$ for all v in I we do the followings:

(1) For each v in I, we determine all the (Γ, v)-injections, then pick out exactly one representative from each equivalence class of these (Γ, v)-injentions. These representatives form a set of (Γ, v)-injections with the property that any (Γ, v)-injection is equivalent to exactly one in the set. Such a set will be refered to as a complete set of inequivalent (Γ, v)-injections.

(2) From the complete set of inequivalent (Γ, v)-injections obtains in (1), we form inequivalent full families of Γ-injections, in all possible ways. Then determine whether each is compatible. In doing so, we obtain all inequivalent compatible full families of Γ-injections.

(3) For each inequivalent compatible full family of Γ-injections A obtained in (2), we obtain, by Theorem 3.3.3, the hypergraph H^A which is a realization of Γ. These H^A's include, up to isomorphism, all realization of Γ. It may happen that some or all of the H^A obtains are isomorphic.

3.4.1 Example. Let $I = \{1,2,3,4,5\}$. For each v in I, let $K_v = (W_v, \mathcal{F}_v)$, where

$$W_1 = W_2 = W_3 = \{1,2\},$$
$$\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \{\{1\},\{2\}\};$$
$$W_4 = W_5 = \{1\},$$
$$\mathcal{F}_4 = \mathcal{F}_5 = \{\{1\}\}.$$

Let $\Gamma = (K_v)_{v \in I}$. We shall determine all the realization of Γ, if any exists.

First, we determine all $(\Gamma,1)$-injections. Since $W_1 = \{1,2\}$ and $I-\{1\} = \{2,3,4,5\}$, hence there are exactly 12 such $(\Gamma,1)$-injections. We shall denote these 12 $(\Gamma,1)$-injections by α_1^i, $i = 1,2,3,\ldots,12$. Their values are given in the following table:

Table 1.

x	$\alpha_1^1(x)$	$\alpha_1^2(x)$	$\alpha_1^3(x)$	$\alpha_1^4(x)$	$\alpha_1^5(x)$	$\alpha_1^6(x)$	$\alpha_1^7(x)$	$\alpha_1^8(x)$	$\alpha_1^9(x)$	$\alpha_1^{10}(x)$	$\alpha_1^{11}(x)$	$\alpha_1^{12}(x)$
1	2	2	2	3	3	4	3	4	5	4	5	5
2	3	4	5	4	5	5	2	2	2	3	3	4

It can be verified that these 12 $(\Gamma,1)$-injections are partitioned into 6 equivalent classes:

$$\{\alpha_1^i, \alpha_1^{i+6}\}, \quad i = 1,2,3,4,5,6.$$

The following tablesgive a complete set of inequivalent $(\Gamma,1)$-injections;

Table 2.

x	$\alpha_1^1(x)$	$\alpha_1^2(x)$	$\alpha_1^3(x)$	$\alpha_1^4(x)$	$\alpha_1^5(x)$	$\alpha_1^6(x)$
1	2	2	2	3	3	4
2	3	4	5	4	5	5

We do in the same manner as above obtains Table 3,4,5 and 6 give complete set of inequivalent $(\Gamma,2)$-injections, $(\Gamma,3)$-injections, $(\Gamma,4)$-injections and $(\Gamma,5)$-injections respectively.

Table 3.

x	$\alpha_2^1(x)$	$\alpha_2^2(x)$	$\alpha_2^3(x)$	$\alpha_2^4(x)$	$\alpha_2^5(x)$	$\alpha_2^6(x)$
1	1	1	1	3	3	4
2	3	4	5	4	5	5

Table 4.

x	$\alpha_3^1(x)$	$\alpha_3^2(x)$	$\alpha_3^3(x)$	$\alpha_3^4(x)$	$\alpha_3^5(x)$	$\alpha_3^6(x)$
1	1	1	1	2	2	4
2	2	4	5	4	5	5

Table 5.

x	$\alpha_4^1(x)$	$\alpha_4^2(x)$	$\alpha_4^3(x)$	$\alpha_4^4(x)$
1	1	2	3	5

Table 6.

x	$\alpha_5^1(x)$	$\alpha_5^2(x)$	$\alpha_5^3(x)$	$\alpha_5^4(x)$
1	1	2	3	4

It can be verified that there does not exist any compatible full family of Γ-injections of the followings types:

Type 1 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^1, \alpha_5^4)$,

Type 2 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^2, \alpha_5^4)$,

Type 3 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^3, \alpha_5^4)$,

Type 4 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^4, \alpha_5^1)$,

Type 5 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^4, \alpha_5^2)$,

Type 6 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^4, \alpha_5^3)$,

Type 7 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^1, \alpha_5^1)$,

Type 8 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^2, \alpha_5^2)$,

Type 9 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^3, \alpha_5^3)$.

Verifications that there does not exist any compatible full family of Γ-injections of type 1 and type 7 are given in Appendic 2. Verifications that there does not exist any compatible full family of Γ-injections of types 2-6 and 8-9 are similar to those of type 1 and 7 respectively.

There remains the following types of full family of Γ-injections to be considered:

Type 10 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^1, \alpha_5^2)$,

Type 11 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^1, \alpha_5^3)$,

Type 12 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^2, \alpha_5^1)$,

Type 13 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^2, \alpha_5^3)$,

Type 14 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^3, \alpha_5^1)$,

Type 15 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^3, \alpha_5^2)$,

Type 16 $(\alpha_1^i, \alpha_2^j, \alpha_3^k, \alpha_4^4, \alpha_5^4)$,

where $i,j,k = 1,2,3,4,5,6$.

By inspection (see Appendix 2) it turns out that for each of these types we can find a unique compatible full family of Γ-injections. They are the followings:

$$A_{10} = (\alpha_1^4, \alpha_2^5, \alpha_3^1, \alpha_4^1, \alpha_5^2) ,$$

$$A_{11} = (\alpha_1^2, \alpha_2^1, \alpha_3^5, \alpha_4^1, \alpha_5^3) ,$$

$$A_{12} = (\alpha_1^5, \alpha_2^4, \alpha_3^1, \alpha_4^2, \alpha_5^1) ,$$

$$A_{13} = (\alpha_1^1, \alpha_2^2, \alpha_3^3, \alpha_4^2, \alpha_5^3) ,$$

$$A_{14} = (\alpha_1^3, \alpha_2^1, \alpha_3^4, \alpha_4^3, \alpha_5^1) ,$$

$$A_{15} = (\alpha_1^1, \alpha_2^3, \alpha_3^2, \alpha_4^3, \alpha_5^2)$$

and

$$A_{16} = (\alpha_1^1, \alpha_2^1, \alpha_3^1, \alpha_4^4, \alpha_5^4) .$$

Hence A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15} and A_{16} are the only inequivalent compatible full families of Γ-injections. Hence, by Theorem 3.3.3 , the family Γ is realizable.

Next, we shall determine all distinct, up to isomorphism, realizations of Γ. Observe that

$$\mathcal{E}^{A_{10}} = \{\{1,3\},\{1,4\},\{2,3\},\{2,5\}\} ,$$

$$\mathcal{E}^{A_{11}} = \{\{1,2\},\{1,4\},\{2,3\},\{3,5\}\} ,$$

$$\mathcal{E}^{A_{12}} = \{\{1,3\},\{1,5\},\{2,3\},\{2,4\}\} ,$$

$$\mathcal{E}^{A_{13}} = \{\{1,2\},\{1,3\},\{2,4\},\{3,5\}\} ,$$

$$\mathcal{E}^{A_{14}} = \{\{1,2\},\{1,5\},\{2,3\},\{3,4\}\} ,$$

$$\mathcal{E}^{A_{15}} = \{\{1,2\},\{1,3\},\{2,5\},\{3,4\}\}$$

and

$$\mathcal{E}^{A_{16}} = \{\{1,2\},\{1,3\},\{2,3\},\{4,5\}\} .$$

Hence, by Theorem 3.3.3.,

$$H^{A_i} = (I, \mathcal{E}^{A_i}),$$

$i = 10,11,\ldots,16$, are realizations of Γ. Observe that

$$H^{A_i} \cong H^{A_j},$$

for all i, $j = 10,11,\ldots,15$, and $H^{A_{10}} \not\cong H^{A_{16}}$.

Let H be any realization of Γ. By Corollary 3.3.2., H must isomorphic to H^A, a hypergraph induced by a compatible full family of Γ-injections, for some compatible full family of Γ-injections A. Hence, A must be equivalent to exactly one of A_i, $i = 10,11,\ldots,16$. Hence, by Theorem 3.3.6.,

$$H^A = H^{A_i},$$

for some $i = 10,11,\ldots,16$. Hence, H must be isomorphic to exactly one of $H^{A_{10}}$ and $H^{A_{16}}$. Therefore, up to isomorphism, $H^{A_{10}}$ and $H^{A_{16}}$ are the only two distinct realizations of Γ.

3.4.2 <u>Example.</u> Let $I = \{1,2,3,4,5,6\}$ and

$$K = (\{1,2,3,4,5\},\{\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{1,5\}\}).$$

Let $\Gamma = (K_v)_{v \in I}$, where $K_v = (W_v, \mathcal{E}_v)$ are equal to K for all v in I.

We shall determine all the realizations of Γ, if any exists. We do this in the same manner as in Example 3.4.1. We have the following results:

(1) For i = 1,2,3,4,5,6, Table i+6 gives a complete set of inequivalent (Γ,i)-injections.

Table 7.

x	$\alpha_1^1(x)$	$\alpha_1^2(x)$	$\alpha_1^3(x)$	$\alpha_1^4(x)$	$\alpha_1^5(x)$	$\alpha_1^6(x)$	$\alpha_1^7(x)$	$\alpha_1^8(x)$	$\alpha_1^9(x)$	$\alpha_1^{10}(x)$	$\alpha_1^{11}(x)$	$\alpha_1^{12}(x)$
1	2	2	2	2	2	2	2	2	2	2	2	2
2	3	3	3	3	3	3	4	4	4	4	5	5
3	4	4	5	5	6	6	3	3	5	6	3	4
4	5	6	4	6	4	5	5	6	3	3	4	3
5	6	5	6	4	5	4	6	5	6	5	6	6

Table 8.

x	$\alpha_2^1(x)$	$\alpha_2^2(x)$	$\alpha_2^3(x)$	$\alpha_2^4(x)$	$\alpha_2^5(x)$	$\alpha_2^6(x)$	$\alpha_2^7(x)$	$\alpha_2^8(x)$	$\alpha_2^9(x)$	$\alpha_2^{10}(x)$	$\alpha_2^{11}(x)$	$\alpha_2^{12}(x)$
1	1	1	1	1	1	1	1	1	1	1	1	1
2	3	3	3	3	3	3	4	4	4	4	5	5
3	4	4	5	5	6	6	3	3	5	6	3	4
4	5	6	4	6	4	5	5	6	3	3	4	3
5	6	5	6	4	5	4	6	5	6	5	6	6

Table 9.

x	$\alpha_3^1(x)$	$\alpha_3^2(x)$	$\alpha_3^3(x)$	$\alpha_3^4(x)$	$\alpha_3^5(x)$	$\alpha_3^6(x)$	$\alpha_3^7(x)$	$\alpha_3^8(x)$	$\alpha_3^9(x)$	$\alpha_3^{10}(x)$	$\alpha_3^{11}(x)$	$\alpha_3^{12}(x)$
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	4	4	4	4	5	5
3	4	4	5	5	6	6	2	2	5	6	2	4
4	5	6	4	6	4	5	5	6	2	2	4	2
5	6	5	6	4	5	4	6	5	6	5	6	6

Table 10.

x	$\alpha_4^1(x)$	$\alpha_4^2(x)$	$\alpha_4^3(x)$	$\alpha_4^4(x)$	$\alpha_4^5(x)$	$\alpha_4^6(x)$	$\alpha_4^7(x)$	$\alpha_4^8(x)$	$\alpha_4^9(x)$	$\alpha_4^{10}(x)$	$\alpha_4^{11}(x)$	$\alpha_4^{12}(x)$
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	3	3	3	3	5	5
3	3	3	5	5	6	6	2	2	5	6	2	3
4	5	6	3	6	3	5	5	6	2	2	3	2
5	6	5	6	3	5	3	6	5	6	5	6	6

Table 11.

x	$\alpha_5^1(x)$	$\alpha_5^2(x)$	$\alpha_5^3(x)$	$\alpha_5^4(x)$	$\alpha_5^5(x)$	$\alpha_5^6(x)$	$\alpha_5^7(x)$	$\alpha_5^8(x)$	$\alpha_5^9(x)$	$\alpha_5^{10}(x)$	$\alpha_5^{11}(x)$	$\alpha_5^{12}(x)$
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	3	3	3	3	4	4
3	3	3	4	4	6	6	2	2	4	6	2	3
4	4	6	3	6	3	4	4	6	2	2	3	2
5	6	4	6	3	4	3	6	4	6	4	6	6

Table 12.

x	$\alpha_6^1(x)$	$\alpha_6^2(x)$	$\alpha_6^3(x)$	$\alpha_6^4(x)$	$\alpha_6^5(x)$	$\alpha_6^6(x)$	$\alpha_6^7(x)$	$\alpha_6^8(x)$	$\alpha_6^9(x)$	$\alpha_6^{10}(x)$	$\alpha_6^{11}(x)$	$\alpha_6^{12}(x)$
1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	3	3	3	3	4	4
3	3	3	4	4	5	5	2	2	4	5	2	3
4	4	5	3	5	3	4	4	5	2	2	3	2
5	5	4	5	3	4	3	5	4	5	4	5	5

(2) From the six complete sets of inequivalent Γ-injections we found that the following families A_i, $i = 1,2,\ldots,12$, comprise inequivalent compatible full families of Γ-injections such that any compatible full family of Γ-injections must be equivalent to one of these. They are given below:

$$A_1 = (\alpha_1^1, \alpha_2^3, \alpha_3^4, \alpha_4^{10}, \alpha_5^{11}, \alpha_6^3),$$

$$A_2 = (\alpha_1^2, \alpha_2^5, \alpha_3^6, \alpha_4^9, \alpha_5^3, \alpha_6^{11}),$$

$$A_3 = (\alpha_1^3, \alpha_2^1, \alpha_3^2, \alpha_4^{11}, \alpha_5^{10}, \alpha_6^5),,$$

$$A_4 = (\alpha_1^4, \alpha_2^6, \alpha_3^5, \alpha_4^3, \alpha_5^9, \alpha_6^{12}),$$

$$A_5 = (\alpha_1^5, \alpha_2^2, \alpha_3^1, \alpha_4^{12}, \alpha_5^5, \alpha_6^{10}),$$

$$A_6 = (\alpha_1^6, \alpha_2^4, \alpha_3^3, \alpha_4^5, \alpha_5^{12}, \alpha_6^9),$$

$$A_7 = (\alpha_1^7, \alpha_2^9, \alpha_3^{10}, \alpha_4^4, \alpha_5^7, \alpha_6^1),$$

$$A_8 = (\alpha_1^8, \alpha_2^{10}, \alpha_3^9, \alpha_4^6, \alpha_5^1, \alpha_6^7),$$

$$A_9 = (\alpha_1^9, \alpha_2^7, \alpha_3^{11}, \alpha_4^2, \alpha_5^8, \alpha_6^6),$$

$$A_{10} = (\alpha_1^{10}, \alpha_2^8, \alpha_3^{12}, \alpha_4^1, \alpha_5^6, \alpha_6^8),$$

$$A_{11} = (\alpha_1^{11}, \alpha_2^{12}, \alpha_3^8, \alpha_4^7, \alpha_5^4, \alpha_6^2),$$

$$A_{12} = (\alpha_1^{12}, \alpha_2^{11}, \alpha_3^7, \alpha_4^8, \alpha_5^2, \alpha_6^4),$$

(3) From the above 12 compatible full families of Γ-injections A_i, $i = 1,2,\ldots,12$. We obtain, by theorem 3.3.3 , the 12 realizations $H^{A_i} = (I, \mathcal{E}^{A_i})$, $i = 1,2,\ldots,12$, where

$\mathcal{E}^{A_1} = \{\{1,2,3\},\{1,3,4\},\{1,4,5\},\{1,5,6\},\{1,2,6\},\{2,3,5\},$
$\{2,4,5\},\{2,4,6\},\{3,4,6\},\{3,5,6\}\}$,

$\mathcal{E}^{A_2} = \{\{1,2,3\},\{1,3,4\},\{1,4,6\},\{1,5,6\},\{1,2,5\},\{2,3,6\},$
$\{2,4,5\},\{2,4,6\},\{3,4,5\},\{3,5,6\}\}$,

$\mathcal{E}^{A_3} = \{\{1,2,3\},\{1,3,5\},\{1,4,5\},\{1,4,6\},\{1,2,6\},\{2,3,4\},$
$\{2,4,5\},\{2,5,6\},\{3,4,6\},\{3,5,6\}\}$,

$\mathcal{E}^{A_4} = \{\{1,2,3\},\{1,3,5\},\{1,4,6\},\{1,5,6\},\{1,2,4\},\{2,3,6\},$
$\{2,4,5\},\{2,5,6\},\{3,4,5\},\{3,4,6\}\}$,

$\mathcal{E}^{A_5} = \{\{1,2,3\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,2,5\},\{2,3,4\},$
$\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,5,6\}\}$,

$\mathcal{E}^{A_6} = \{\{1,2,3\},\{1,3,6\},\{1,4,5\},\{1,5,6\},\{1,2,4\},\{2,3,5\},$
$\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\}\}$,

$\mathcal{E}^{A_7} = \{\{1,2,4\},\{1,3,4\},\{1,3,5\},\{1,5,6\},\{1,2,6\},\{2,3,5\},$
$\{2,3,6\},\{2,4,5\},\{3,4,6\},\{4,5,6\}\}$,

$\mathcal{E}^{A_8}$ = {{1,2,4},{1,3,4},{1,3,6},{1,5,6},{1,2,5},{2,3,5},
{2,3,6},{2,4,6},{3,4,5},{4,5,6}},

$\mathcal{E}^{A_9}$ = {{1,2,4},{1,4,5},{1,3,5},{1,3,6},{1,2,6},{2,3,4},
{2,3,5},{2,5,6},{3,4,6},{4,5,6}},

$\mathcal{E}^{A_{10}}$ = {{1,2,4},{1,4,6},{1,3,6},{1,3,5},{1,2,5},{2,3,4},
{2,3,6},{2,5,6},{3,4,5},{4,5,6}},

$\mathcal{E}^{A_{11}}$ = {{1,2,5},{1,3,5},{1,3,4},{1,4,6},{1,2,6},{2,3,4},
{2,3,6},{2,4,5},{3,5,6},{4,5,6}},

$\mathcal{E}^{A_{12}}$ = {{1,2,5},{1,4,5},{1,3,4},{1,3,6},{1,2,6},{2,3,4},
{2,3,5},{2,4,6},{3,5,6},{4,5,6}}.

(4) All the realizations H^{A_i}, $i = 1,2,\ldots,12$, obtained above, are isomorphic. This can be seen by establishing an isomorphism ψ_i from H^{A_1} to H^{A_i}, $i = 2,3,\ldots,12$. Table 13 gives such isomorphisms.

Table 13.

x	$\psi_2(x)$	$\psi_3(x)$	$\psi_4(x)$	$\psi_5(x)$	$\psi_6(x)$	$\psi_7(x)$	$\psi_8(x)$	$\psi_9(x)$	$\psi_{10}(x)$	$\psi_{11}(x)$	$\psi_{12}(x)$
1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2
3	3	3	3	3	3	4	4	4	4	5	5
4	4	5	5	6	6	3	3	5	6	3	4
5	6	4	6	4	5	5	6	3	3	4	3
6	5	6	4	5	4	6	5	6	5	6	6

Hence, up to isomorphism, H^{A_1} is the only realization of Γ.