CHAPTER III

A MORE REALISTIC MATHEMATICAL MODEL FOR THE TURBULENT DIFFUSION OF SMOKE FROM A CONTINUOUS POINT SOURCE

Let us consider a new model. We shall replace the squares in the XZ-plane by hexagons using the coordinates ξ, η with oblique axes as shown in figure 3.1(a). For each step in time, assume that a particle is emitted from a hexagon, called the source, at the origin.

At each step assume that the combined effect of the wind and turbulence causes the particle to move from its original hexagon to one of the three hexagons adjacent to it on the right hand side as shown in figure 3.1(b). It is assumed that the turbulent motion of the particle is random with fixed equal transition probabilities for each of the three possible directions.

If for each step, we draw a separate random number to indicate the turbulent motion of each particle according to the rule in table 3.1, then each probability of moving up, sideways or down is equal to $\frac{1}{3}$.

We can find the probability distribution for a single particle in each step as shown in fugures 3.2 (a),(b),(c),(d),(e), and (f).

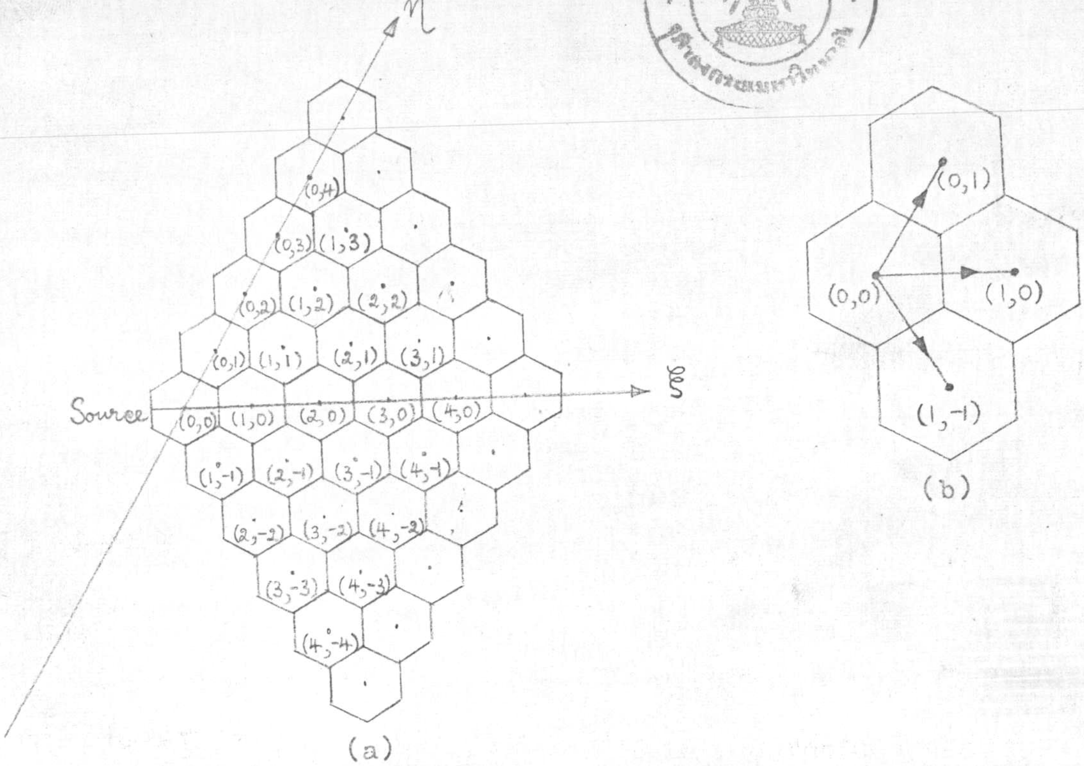


Figure 3.1

Random Numbers. Rules.

Random Numbers	Rules
1, 2, 3	up
4, 5, 6	sideways
7, 8, 9	down

(Number 0 is ignored)

Table 3.1

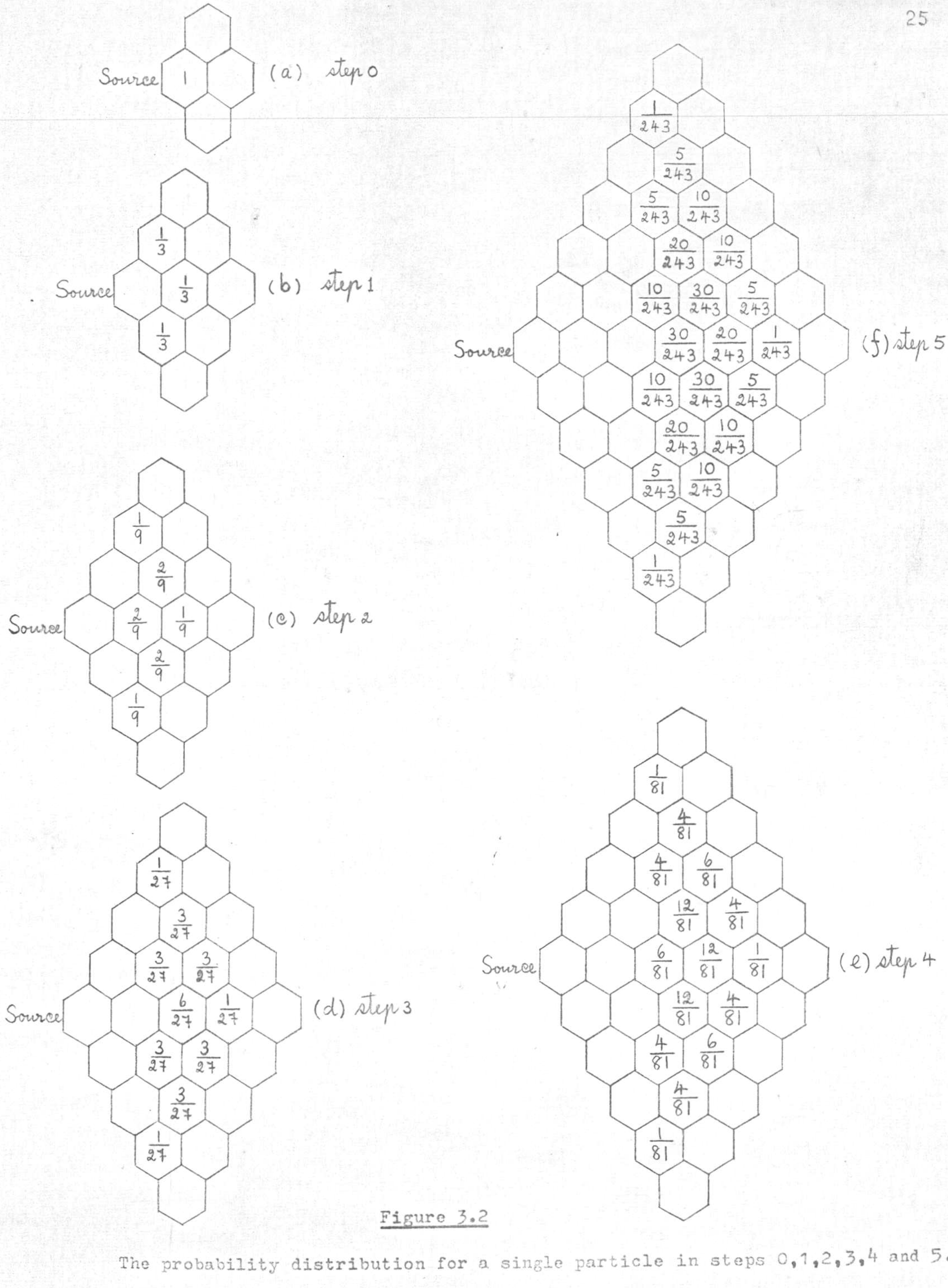
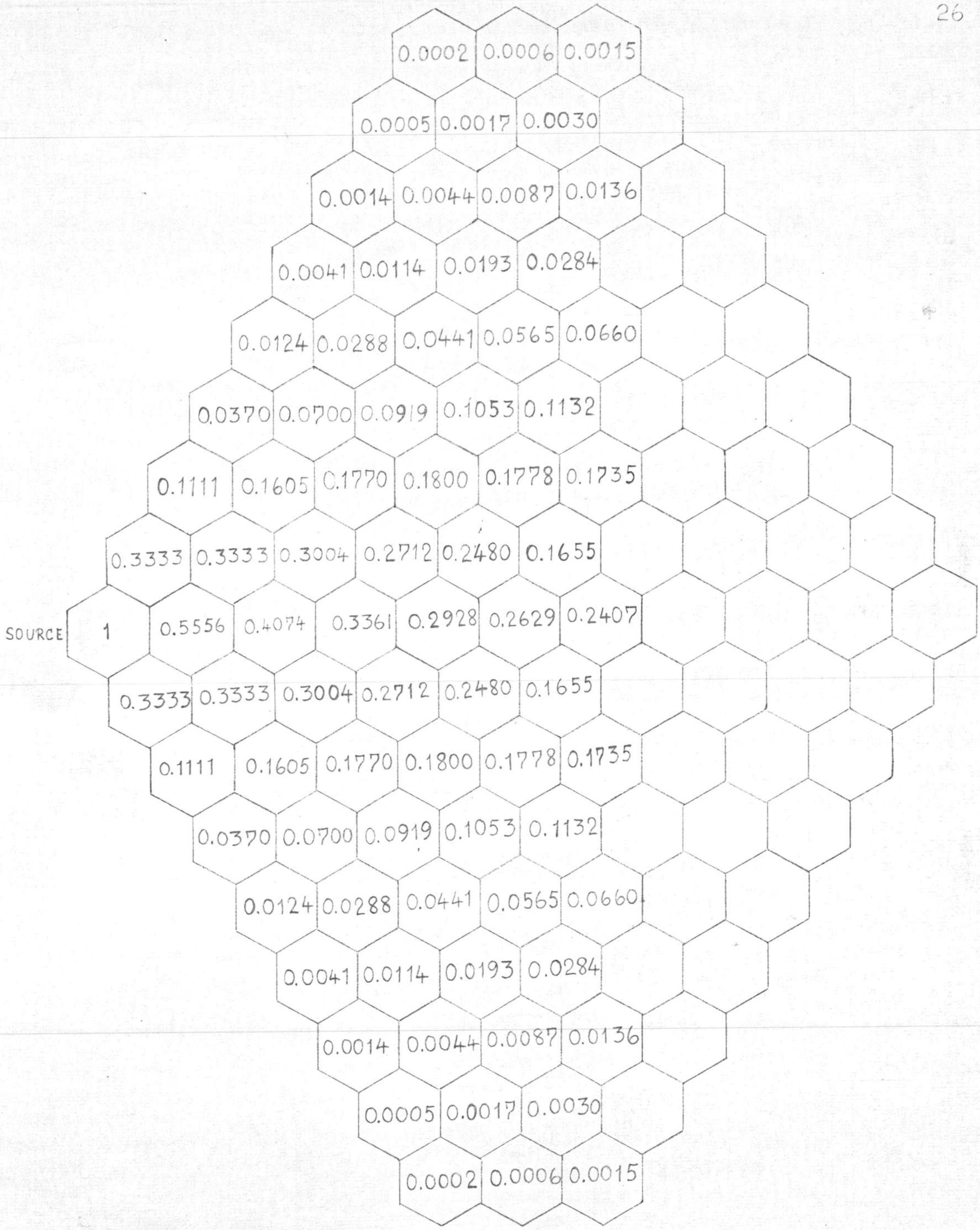


Figure 3.2

The probability distribution for a single particle in steps 0,1,2,3,4 and 5.



<u>Figure 3.3</u>

The probability distribution for the whole system of particles.

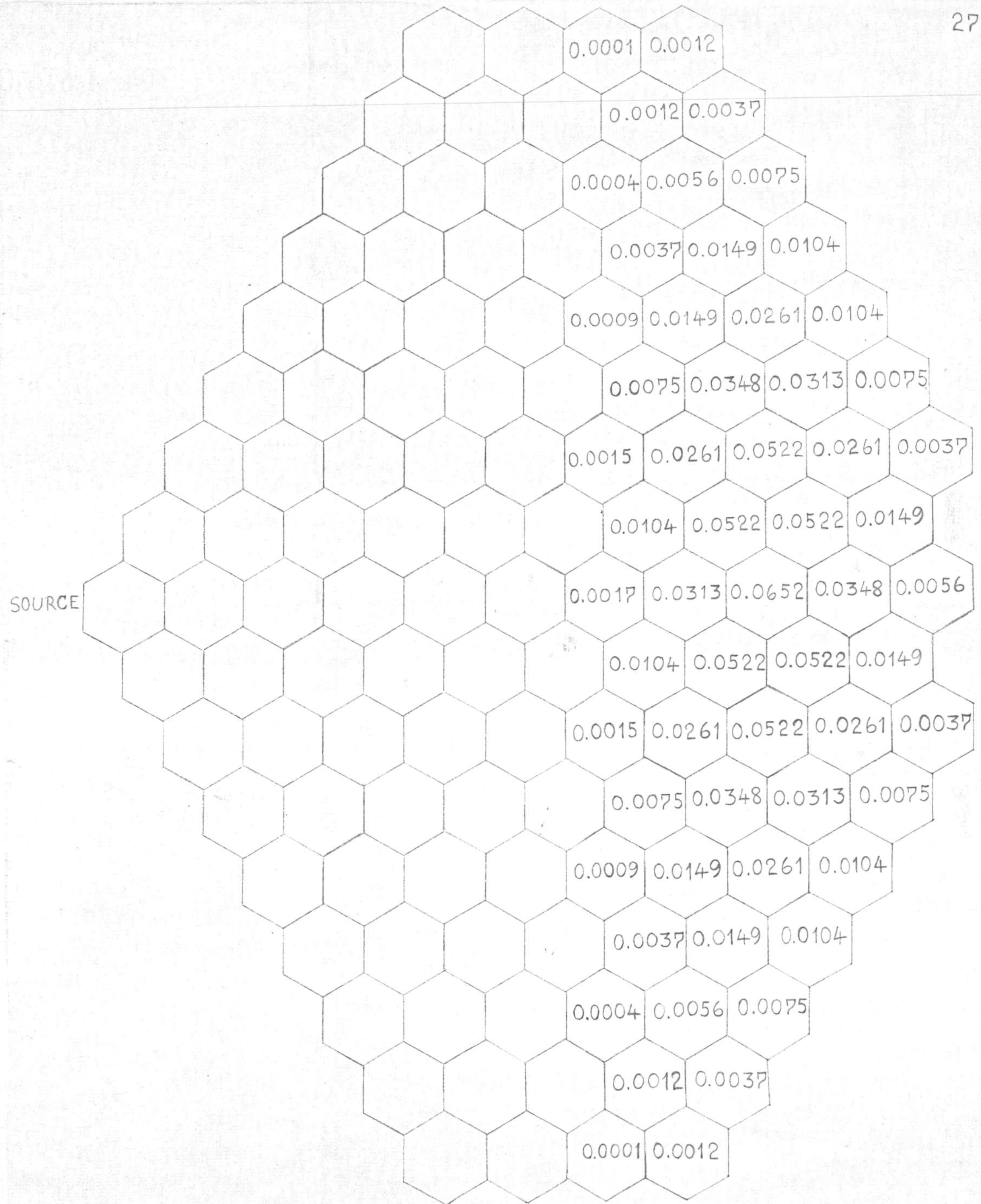


Figure 3.4

The probability distribution of a single particle in the twelveth step. (Non-zero values also occur beyond the boundary of the diagram.)

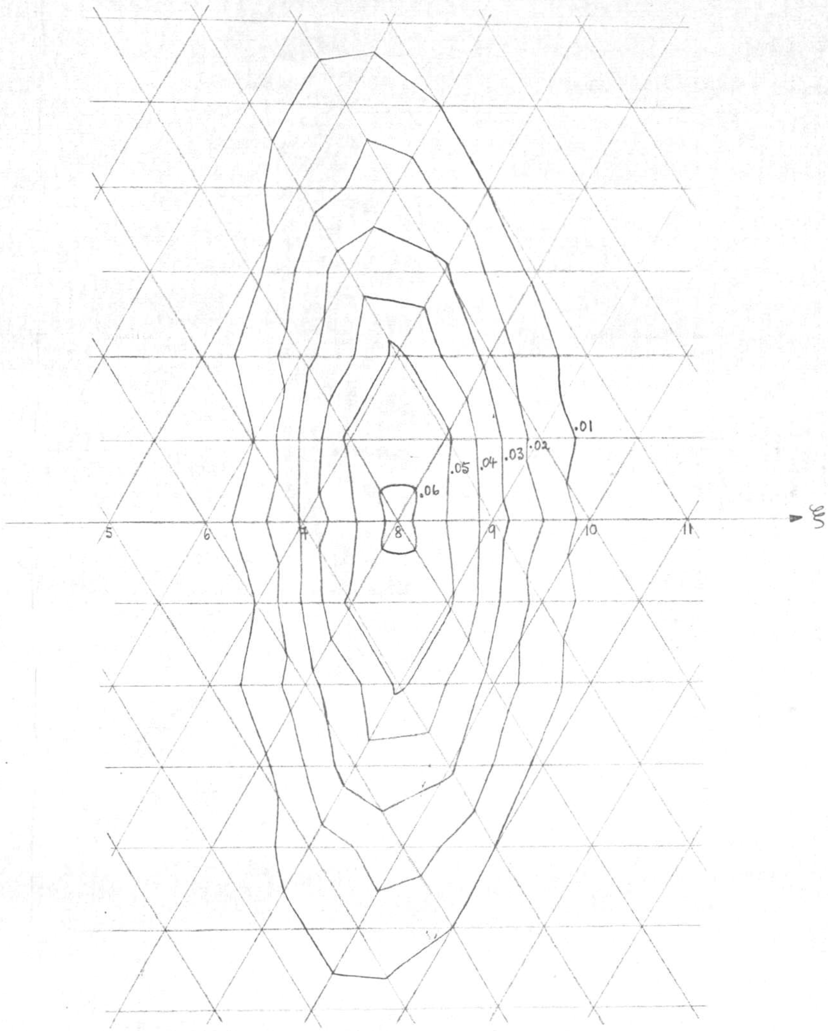


Figure 3.5

The contours of the distribution of a single particle in the twelveth step.

In figures 3.2 we notice that, the denominator in each step is equal to 3^n, where n is the number of steps, i.e. n = 0,1,2,...

After the twelveth step the probability distribution for the whole system of particles is shown in fugure 3.3, and the contours of the distribution of a single particle in the twelveth step (shown in figure 3.4) are shown in figure 3.5, which are obtained by linear interpolation between the data points. Finally we can find the distribution function as follows :

We calculate the probability distribution along the ξ-axis by assuming the following model :

A particle is emitted at time zero at the origin. At each step in time the particle remains stationary with the probability $\frac{1}{3}$ and moves one unit in the ξ-direction with the probability $\frac{2}{3}$. The probability distribution at successive steps are as shown in table 3.2.

From table 3.2 we can see that :

(i) the sum of the probability values in each row is equal to unity.

(ii) the average distance $\bar{\xi}$ in each step is equal to $\frac{2}{3}$ n, where n is the number of steps. For example, in the second step we have the average distance

$$\bar{\xi} = 0 \times \frac{1}{9} + 1 \times \frac{4}{9} + 2 \times \frac{4}{9} = \frac{12}{9} = \frac{4}{3} = \frac{2}{3}n.$$

$n \backslash \xi$	0	1	2	3	4	5	---
0	1	0	0	0	0	0	---
1	$\frac{1}{3}$	$\frac{2}{3}$	0	0	0	0	---
2	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{4}{9}$	0	0	0	---
3	$\frac{1}{27}$	$\frac{6}{27}$	$\frac{12}{27}$	$\frac{8}{27}$	0	0	---
4	$\frac{1}{81}$	$\frac{8}{81}$	$\frac{24}{81}$	$\frac{32}{81}$	$\frac{16}{81}$	0	---
5	$\frac{1}{243}$	$\frac{10}{243}$	$\frac{40}{243}$	$\frac{80}{243}$	$\frac{80}{243}$	$\frac{32}{243}$	---
---	---	---	---	---	---	---	---

<u>Table 3.2</u>

Let $P_n(\xi)$ be the probability that after n steps the random particle is ξ steps from the origin ; that is

$$P_n(\xi) \quad = \quad {}^nC_{n-\xi}\left(\frac{2}{3}\right)^{\xi}\left(\frac{1}{3}\right)^{n-\xi} \qquad .... (3.1)$$

Let $p = \frac{2}{3}$ and $q = \frac{1}{3}$. From (3.1) we obtain

$$P_n(\xi) \quad = \quad \frac{n!}{(n-\xi)!\,\xi!}\, p^{\xi}\, q^{n-\xi} \quad , ... (3.2)$$

which is the binomial distribution.

Let n tend to infinity. As the number of steps n increases, we expect that also the numbers ξ and $(n-\xi)$ will increase, so that

$$n \longrightarrow \infty\, , \quad \xi \longrightarrow \infty \, , \quad n-\xi \longrightarrow \infty \quad ....(3.3)$$

Then we express the factorials in (3.2) by means of Stirling's formula

$$r! \simeq (2\pi r)^{1/2} r^r e^{-r} , \quad \text{as } r \longrightarrow \infty .$$

From (3.2), we get

$$P_n(\xi) \simeq \left\{ \frac{n}{2\pi \xi (n-\xi)} \right\}^{1/2} \left(\frac{np}{\xi} \right)^{\xi} \left(\frac{nq}{n-\xi} \right)^{n-\xi} \quad (3.4)$$

The last two factors on the right are equal to unity for $\xi = np$, and their product decreases as $|\xi - np|$ increases. Therefore, it is natural to replace ξ by the new variable

$$\delta = \xi - np \quad (3.5)$$

From (3.5), $\xi = np + \delta$, $n - \xi = nq - \delta$,

so that (3.4) becomes

$$P_n(\xi) \simeq \left\{ \frac{n}{2\pi (np+\delta)(nq-\delta)} \right\}^{1/2} \frac{1}{\left(1+\frac{\delta}{np}\right)^{np+\delta} \left(1-\frac{\delta}{nq}\right)^{nq-\delta}} \quad (3.6)$$

To evaluate the last fraction we use logarithms. In the interval $|\delta| \leqslant npq$ we may use Taylor's expansion and find for the logarithm of the denominator

$$(np+\delta)\ln\left(1+\frac{\delta}{np}\right)+(nq-\delta)\ln\left(1-\frac{\delta}{nq}\right)$$

$$= (np+\delta)\left(\frac{\delta}{np} - \frac{\delta^2}{2n^2p^2} + \frac{\delta^3}{3n^3p^3} - \ldots\right) + (nq-\delta)\left(-\frac{\delta}{nq} - \frac{\delta^2}{2n^2q^2} - \frac{\delta^3}{3n^3q^3} - \ldots\right) \quad (3.7)$$

Reordering the terms according to powers of δ , we get

$$\frac{\delta^2}{2n}\left(\frac{1}{p} + \frac{1}{q}\right) - \frac{\delta^3}{6n^2}\left(\frac{1}{p^2} - \frac{1}{q^2}\right) + \ldots \quad (3.8)$$

Suppose that ξ increases with n in such a manner that

$$\frac{\delta^3}{n^2} \longrightarrow 0 \qquad (3.9)$$

(In this case also $\frac{\delta}{n} \longrightarrow 0$ so that (3.3) holds and the expansion (3.7) is justified.)

From (3.9), the term within braces in (3.6) is equivalent to $(2\pi n p q)^{-1/2}$. The logarithm of the denominator in (3.6) is given by (3.8), but in view of (3.9) all terms except the first one may be neglected; the first term equals $\frac{\delta^2}{2npq}$.
Combining these results, we have

$$P_n(\xi) \simeq \frac{1}{\sqrt{2\pi npq}} e^{-(\xi-np)^2/2npq} \quad {}^* , \qquad (3.10)$$

where $\delta = \xi - np$,

which is known as a Gaussian distribution.
Substituting $p = \frac{2}{3}$ and $q = \frac{1}{3}$ into (3.10), we get

$$P_n(\xi) \simeq \frac{3}{2(n\pi)^{1/2}} e^{-9(\xi-np)^2/4n} .$$

The average distance $\bar{\xi}$ is

$$\bar{\xi} = \int_{-\infty}^{\infty} \xi P(\xi) d\xi$$

$$= \frac{3}{2(n\pi)^{1/2}} \int_{-\infty}^{\infty} \xi e^{-9(\xi-np)^2/4n} d\xi$$

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* See from William Feller, An Introduction to Probability Theory and its Applications, Volume 1, Chapter 7.

$$= \frac{3}{2(n\pi)^{1/2}} \int_{-\infty}^{\infty} (np+y)e^{-9y^2/4n} \, dy$$

$$= \frac{3}{2(n\pi)^{1/2}} \left[np \int_{-\infty}^{\infty} e^{-9y^2/4n} \, dy + \int_{-\infty}^{\infty} ye^{-9y^2/4n} \, dy \right].$$

Since $\int_{-\infty}^{\infty} ye^{-9y^2/4n} \, dy = 0.$

Therefore $$\bar{\xi} = \frac{3np}{2(n\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-9y^2/4n} \, dy$$

$$= \left(\frac{n}{\pi}\right)^{1/2} \times 2 \int_{0}^{\infty} e^{-9y^2/4n} \, dy \text{ , where } p = \frac{2}{3} \text{ ,}$$

$$= \left(\frac{n}{\pi}\right)^{1/2} \times 2 \times \frac{1}{2} \sqrt{\frac{\pi}{\frac{9}{4n}}} \; ^{*}$$

$$= \frac{2}{3} n \, .$$

The same result as (ii).

If n is small then we can prove that $\bar{\xi} = \frac{2}{3} n$ by using $P_n(\xi)$ from equation (3.2) using mathematical induction, i.e. we have to prove that

$$\bar{\xi} = \sum_{\xi=0}^{\xi=n} \xi P_n(\xi) = \sum_{\xi=0}^{\xi=n} \frac{\xi \, n!}{(n-\xi)!\,\xi!} \left(\frac{2}{3}\right)^{\xi} \left(\frac{1}{3}\right)^{n-\xi} = \frac{2}{3} n \, .$$

If $n = 0$, then $\bar{\xi} = 0 = \frac{2}{3} n$.

Assume that it is true for $n = k$,

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* See appendix A.

i.e. $$\bar{\xi} = \sum_{\xi=0}^{\xi=k} \frac{\xi\, k!}{(k-\xi)!\, \xi!}\left(\frac{2}{3}\right)^{\xi}\left(\frac{1}{3}\right)^{k-\xi} = \frac{2}{3}k$$

$$\sum_{\xi=0}^{\xi=k-1} \frac{\xi\, k!}{(k-\xi)!\, \xi!}\left(\frac{2}{3}\right)^{\xi}\left(\frac{1}{3}\right)^{k-\xi} + k\left(\frac{2}{3}\right)^{k} = \frac{2}{3}k$$

$$\sum_{\xi=0}^{\xi=k-1} \frac{\xi\, k!}{(k-\xi)!\, \xi!}\left(\frac{2}{3}\right)^{\xi}\left(\frac{1}{3}\right)^{k-\xi} = \frac{2}{3}k - k\left(\frac{2}{3}\right)^{k} \qquad (3.11)$$

To prove that it is also true for $n = k+1$ we have to prove that

$$\bar{\xi} = \sum_{\xi=0}^{\xi=k+1} \frac{\xi\,(k+1)!}{(k-\xi+1)!\, \xi!}\left(\frac{2}{3}\right)^{\xi}\left(\frac{1}{3}\right)^{k-\xi+1} = \frac{2}{3}(k+1).$$

Proof :

$$\bar{\xi} = \sum_{\xi=0}^{\xi=k} \frac{\xi\,(k+1)!}{(k-\xi+1)!\, \xi!}\left(\frac{2}{3}\right)^{\xi}\left(\frac{1}{3}\right)^{k-\xi+1} + (k+1)\left(\frac{2}{3}\right)^{k+1}.$$

Let $k + 1 = h$, i.e. $k = h - 1$, so that we get

$$\bar{\xi} = \sum_{\xi=0}^{\xi=h-1} \frac{\xi\, h!}{(h-\xi)!\, \xi!}\left(\frac{2}{3}\right)^{\xi}\left(\frac{1}{3}\right)^{h-\xi} + h\left(\frac{2}{3}\right)^{h}$$

$$= \frac{2}{3}h - h\left(\frac{2}{3}\right)^{h} + h\left(\frac{2}{3}\right)^{h}, \qquad \text{from } (3.11)$$

$$= \frac{2}{3}h \qquad = \frac{2}{3}(k+1).$$

Therefore $\bar{\xi} = \frac{2}{3}n$ for all $n \geqslant 0$.

The probability values for single particles referred to the η -axis and the number of steps n are as follow :

η \ n	0	1	2	3	4	5	...
5						$\frac{1}{243}$	...
4					$\frac{1}{81}$	$\frac{5}{243}$	...
3				$\frac{1}{27}$	$\frac{4}{81}$	$\frac{15}{243}$	...
2			$\frac{1}{9}$	$\frac{3}{27}$	$\frac{10}{81}$	$\frac{30}{243}$	...
1		$\frac{1}{3}$	$\frac{2}{9}$	$\frac{6}{27}$	$\frac{16}{81}$	$\frac{45}{243}$	...
0	1	$\frac{1}{3}$	$\frac{3}{9}$	$\frac{7}{27}$	$\frac{19}{81}$	$\frac{51}{243}$	...
-1		$\frac{1}{3}$	$\frac{2}{9}$	$\frac{6}{27}$	$\frac{16}{81}$	$\frac{45}{243}$	...
-2			$\frac{1}{9}$	$\frac{3}{27}$	$\frac{10}{81}$	$\frac{30}{243}$	...
-3				$\frac{1}{27}$	$\frac{4}{81}$	$\frac{15}{243}$	...
-4					$\frac{1}{81}$	$\frac{5}{243}$	...
-5						$\frac{1}{243}$	...

Table 3.3

From table 3.3, we can see that :

(i) each probability value of moving up, sideways or down from left to right is equal to $\frac{1}{3}$.

(ii) the sum of the probability values in each column is equal to unity because it is the value of $\left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right)^n$, where n = 0, 1, 2, ...

(iii) the maximum values of the probability lie on the ξ -axis and the probability values are also symmetrical about this line.

(iv) the average distance $\bar{\eta}$ in each step is equal to zero. For example, in the third step we have the average distance

$$\bar{\eta} = 3\left(\frac{1}{27}\right) + 2\left(\frac{3}{27}\right) + 1\left(\frac{6}{27}\right) + 0\left(\frac{7}{27}\right) + (-1)\left(\frac{6}{27}\right) + (-2)\left(\frac{3}{24}\right) + (-3)\left(\frac{1}{27}\right)$$

$$= 0.$$

$\eta \backslash n$	0	1	2	3	4	5
	$p(0,\eta)$	$p(1,\eta)$	$p(2,\eta)$	$p(3,\eta)$	$p(4,\eta)$	$p(5,\eta)$
5						$\frac{1}{243}$
4					$\frac{1}{81}$	$\frac{5}{243}$
3				$\frac{1}{27}$	$\frac{4}{81}$	$\frac{15}{243}$
2			$\frac{1}{9}$	$\frac{3}{27}$	$\frac{10}{81}$	$\frac{30}{243}$
1		$\frac{1}{3}$	$\frac{2}{9}$	$\frac{6}{27}$	$\frac{16}{81}$	$\frac{45}{243}$
0	1	$\frac{1}{3}$	$\frac{3}{9}$	$\frac{7}{27}$	$\frac{19}{81}$	$\frac{51}{243}$
-1		$\frac{1}{3}$	$\frac{2}{9}$	$\frac{6}{27}$	$\frac{16}{81}$	$\frac{45}{243}$
-2			$\frac{1}{9}$	$\frac{3}{27}$	$\frac{10}{81}$	$\frac{30}{243}$
-3				$\frac{1}{27}$	$\frac{4}{81}$	$\frac{15}{243}$
-4					$\frac{1}{81}$	$\frac{5}{243}$
-5						$\frac{1}{243}$

(a)

$\eta \backslash n$	0	1	2	3	4	5
	$p(0,\eta)$	$\frac{1}{3}p(0,\eta)+p(1,\eta) = \frac{1}{3}\,{}^{2}C_{1-\eta}$	$\frac{1}{3}p(1,\eta)+p(2,\eta)$	$\frac{1}{3}p(2,\eta)+p(3,\eta)$	$\frac{1}{3}p(3,\eta)+p(4,\eta)$	$\frac{1}{3}p(4,\eta)+p(5,\eta)$
5						$\frac{1}{243}$
4					$\frac{1}{81}$	$\frac{6}{243}$
3				$\frac{1}{27}$	$\frac{5}{81}$	$\frac{19}{243}$
2			$\frac{1}{9}$	$\frac{4}{27}$	$\frac{13}{81}$	$\frac{40}{243}$
1		$\frac{1}{3}$	$\frac{3}{9}$	$\frac{8}{27}$	$\frac{22}{81}$	$\frac{61}{243}$
0	1	$\frac{2}{3}$	$\frac{4}{9}$	$\frac{10}{27}$	$\frac{26}{81}$	$\frac{70}{243}$
-1		$\frac{1}{3}$	$\frac{3}{9}$	$\frac{8}{27}$	$\frac{22}{81}$	$\frac{61}{243}$
-2			$\frac{1}{9}$	$\frac{4}{27}$	$\frac{13}{81}$	$\frac{40}{243}$
-3				$\frac{1}{27}$	$\frac{5}{81}$	$\frac{19}{243}$
-4					$\frac{1}{81}$	$\frac{6}{243}$
-5						$\frac{1}{243}$

(b)

Table 3.4

n \ η	-5	-4	-3	-2	-1	0	1	2	3	4	5	
0						1						$p(0,\eta)$
1					$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$					$\frac{1}{3}\,{}^{2}C_{1-\eta}$
2				$\frac{1}{9}$	$\frac{4}{9}$	$\frac{6}{9}$	$\frac{4}{9}$	$\frac{1}{9}$				$\frac{1}{3^2}\,{}^{2}C_{1-\eta}+\frac{1}{3}p(1,\eta)+p(2,\eta) = \frac{1}{3^2}\,{}^{4}C_{2-\eta}$
3			$\frac{1}{27}$	$\frac{5}{27}$	$\frac{11}{27}$	$\frac{14}{27}$	$\frac{11}{27}$	$\frac{5}{27}$	$\frac{1}{27}$			$\frac{1}{3^2}p(1,\eta)+\frac{2}{3}p(2,\eta)+p(3,\eta)$
4		$\frac{1}{81}$	$\frac{6}{81}$	$\frac{17}{81}$	$\frac{30}{81}$	$\frac{36}{81}$	$\frac{30}{81}$	$\frac{17}{81}$	$\frac{6}{81}$	$\frac{1}{81}$		$\frac{1}{3^2}p(2,\eta)+\frac{2}{3}p(3,\eta)+p(4,\eta)$
5	$\frac{1}{243}$	$\frac{7}{243}$	$\frac{24}{243}$	$\frac{53}{243}$	$\frac{83}{243}$	$\frac{96}{243}$	$\frac{83}{243}$	$\frac{53}{243}$	$\frac{24}{243}$	$\frac{7}{243}$	$\frac{1}{243}$	$\frac{1}{3^2}p(3,\eta)+\frac{2}{3}p(4,\eta)+p(5,\eta)$

(c)

n \ η	-5	-4	-3	-2	-1	0	1	2	3	4	5	
0						1						$p(0,\eta)$
1					$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$					$\frac{1}{3}\,{}^{2}C_{1-\eta}$
2				$\frac{1}{9}$	$\frac{4}{9}$	$\frac{6}{9}$	$\frac{4}{9}$	$\frac{1}{9}$				$\frac{1}{3^2}\,{}^{4}C_{2-\eta}$
3			$\frac{1}{27}$	$\frac{6}{27}$	$\frac{15}{27}$	$\frac{20}{27}$	$\frac{15}{27}$	$\frac{6}{27}$	$\frac{1}{27}$			$\frac{1}{3^3}\,{}^{4}C_{2-\eta}+\frac{1}{3^2}p(1,\eta)+\frac{2}{3}p(2,\eta)+p(3,\eta) = \frac{1}{3^3}\,{}^{6}C_{3-\eta}$
4		$\frac{1}{81}$	$\frac{7}{81}$	$\frac{22}{81}$	$\frac{41}{81}$	$\frac{50}{81}$	$\frac{41}{81}$	$\frac{22}{81}$	$\frac{7}{81}$	$\frac{1}{81}$		$\frac{1}{3^3}p(1,\eta)+\frac{3}{3^2}p(2,\eta)+\frac{3}{3}p(3,\eta)+p(4,\eta)$
5	$\frac{1}{243}$	$\frac{8}{243}$	$\frac{30}{243}$	$\frac{70}{243}$	$\frac{113}{243}$	$\frac{132}{243}$	$\frac{113}{243}$	$\frac{70}{243}$	$\frac{30}{243}$	$\frac{8}{243}$	$\frac{1}{243}$	$\frac{1}{3^3}p(2,\eta)+\frac{3}{3^2}p(3,\eta)+\frac{3}{3}p(4,\eta)+p(5,\eta).$

(d)

Table 3.4

n \ η	-5	-4	-3	-2	-1	0	1	2	3	4	5	
0						1						$p(0,\eta)$
1					$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$					$\frac{1}{3}\,{}^2C_{1-\eta}$
2				$\frac{1}{9}$	$\frac{4}{9}$	$\frac{6}{9}$	$\frac{4}{9}$	$\frac{1}{9}$				$\frac{1}{3^2}\,{}^4C_{2-\eta}$
3			$\frac{1}{27}$	$\frac{6}{27}$	$\frac{15}{27}$	$\frac{20}{27}$	$\frac{15}{27}$	$\frac{6}{27}$	$\frac{1}{27}$			$\frac{1}{3^3}\,{}^6C_{3-\eta}$
4		$\frac{1}{81}$	$\frac{8}{81}$	$\frac{28}{81}$	$\frac{56}{81}$	$\frac{70}{81}$	$\frac{56}{81}$	$\frac{28}{81}$	$\frac{8}{81}$	$\frac{1}{81}$		$\frac{1}{3^4}\,{}^6C_{3-\eta}+\frac{1}{3^3}p(1,\eta)+\frac{3}{3^2}p(2,\eta)+\frac{3}{3}p(3,\eta)+p(4,\eta)=\frac{1}{3^4}\,{}^8C_{4-\eta}$
5	$\frac{1}{243}$	$\frac{9}{243}$	$\frac{37}{243}$	$\frac{92}{243}$	$\frac{154}{243}$	$\frac{182}{243}$	$\frac{154}{243}$	$\frac{92}{243}$	$\frac{37}{243}$	$\frac{9}{243}$	$\frac{1}{243}$	$\frac{1}{3^4}p(1,\eta)+\frac{4}{3^3}p(2,\eta)+\frac{6}{3^2}p(3,\eta)+\frac{4}{3}p(4,\eta)+p(5,\eta)$

(e)

n \ η	-5	-4	-3	-2	-1	0	1	2	3	4	5	
0						1						$p(0,\eta)$
1					$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{3}$					$\frac{1}{3}\,{}^2C_{1-\eta}$
2				$\frac{1}{9}$	$\frac{4}{9}$	$\frac{6}{9}$	$\frac{4}{9}$	$\frac{1}{9}$				$\frac{1}{3^2}\,{}^4C_{2-\eta}$
3			$\frac{1}{27}$	$\frac{6}{27}$	$\frac{15}{27}$	$\frac{20}{27}$	$\frac{15}{27}$	$\frac{6}{27}$	$\frac{1}{27}$			$\frac{1}{3^3}\,{}^6C_{3-\eta}$
4		$\frac{1}{81}$	$\frac{8}{81}$	$\frac{28}{81}$	$\frac{56}{81}$	$\frac{70}{81}$	$\frac{56}{81}$	$\frac{28}{81}$	$\frac{8}{81}$	$\frac{1}{81}$		$\frac{1}{3^4}\,{}^8C_{4-\eta}$
5	$\frac{1}{243}$	$\frac{10}{243}$	$\frac{45}{243}$	$\frac{120}{243}$	$\frac{210}{243}$	$\frac{252}{243}$	$\frac{210}{243}$	$\frac{120}{243}$	$\frac{45}{243}$	$\frac{10}{243}$	$\frac{1}{243}$	$\frac{1}{3^5}\,{}^8C_{4-\eta}+\frac{1}{3^4}p(1,\eta)+\frac{4}{3^3}p(2,\eta)+\frac{6}{3^2}p(3,\eta)+\frac{4}{3}p(4,\eta)+p(5,\eta)=\frac{1}{3^5}\,{}^{10}($

(f)

Table 3.4

Let $p(n,\eta)$ be the probability that after n steps the particle is η steps from the origin.

Table 3.4 (a), (b), (c), (d), (e) and (f) show how can we find the value of $p(n,\eta)$. We can see that from table 3.4 (a), we get

$$p(0,\eta) = \begin{cases} 1, & \text{when } \eta = 0 \\ 0, & \text{when } \eta \neq 0, \end{cases}$$

from table 3.4 (b), we get

$$\begin{aligned} p(1,\eta) &= \frac{1}{3}\,{}^2C_{1-\eta} - \frac{1}{3}\,p(0,\eta) \\ &= \frac{1}{3}\left[{}^2C_{1-\eta} - p(0,\eta)\right] \\ &= \frac{1}{3}\left[{}^1C_0\,{}^2C_{1-\eta} - {}^1C_1 p(0,\eta)\right], \end{aligned}$$

from table 3.4 (c), we get

$$\begin{aligned} p(2,\eta) &= \frac{1}{3^2}\,{}^4C_{2-\eta} - \frac{1}{3^2}\,{}^2C_{1-\eta} - \frac{1}{3}\,p(1,\eta) \\ &= \frac{1}{3^2}\,{}^4C_{2-\eta} - \frac{2}{3^2}\,{}^2C_{1-\eta} + \frac{1}{3^2}p(0,\eta) \\ &= \frac{1}{3^2}\left[{}^2C_0\,{}^4C_{2-\eta} - {}^2C_1\,{}^2C_{1-\eta} + {}^2C_2 p(0,\eta)\right], \end{aligned}$$

from table 3.4 (d), we get

$$\begin{aligned} p(3,\eta) &= \frac{1}{3^3}\,{}^6C_{3-\eta} - \frac{1}{3^3}\,{}^4C_{2-\eta} - \frac{1}{3^2}\,p(1,\eta) - \frac{2}{3}\,p(2,\eta) \\ &= \frac{1}{3^3}\,{}^6C_{3-\eta} - \frac{3}{3^3}\,{}^4C_{2-\eta} + \frac{3}{3^3}\,{}^2C_{1-\eta} - \frac{1}{3^3}\,p(0,\eta) \\ &= \frac{1}{3^3}\left[{}^3C_0\,{}^6C_{3-\eta} - {}^3C_1\,{}^4C_{2-\eta} + {}^3C_2\,{}^2C_{1-\eta} - {}^3C_3\,p(0,\eta)\right], \end{aligned}$$

from table 3.4 (e), we get

$$p(4,\eta) = \frac{1}{3^4}\,{}^8C_{4-\eta} - \frac{1}{3^4}\,{}^6C_{3-\eta} - \frac{1}{3^3}p(1,\eta) - \frac{3}{3^2}p(2,\eta) - \frac{3}{3}\,p(1,\eta)$$

$$= \frac{1}{3^4}\,{}^8C_{4-\eta} - \frac{4}{3^4}\,{}^6C_{3-\eta} + \frac{6}{3^4}\,{}^4C_{2-\eta} - \frac{4}{3^4}\,{}^2C_{1-\eta} + \frac{1}{3^4}p(0,\eta)$$

$$= \frac{1}{3^4}\left[{}^4C_0\,{}^8C_{4-\eta} - {}^4C_1\,{}^6C_{3-\eta} + {}^4C_2\,{}^4C_{2-\eta} - {}^4C_3\,{}^2C_{1-\eta} + {}^4C_4 p(0,\eta)\right],$$

from table 3.4 (f), we get

$$p(5,\eta) = \frac{1}{3^5}\,{}^{10}C_{5-\eta} - \frac{1}{3^5}\,{}^8C_{4-\eta} - \frac{1}{3^4}p(1,\eta) - \frac{4}{3^3}p(2,\eta) - \frac{6}{3^2}p(3,\eta) - \frac{4}{3}p(4,\eta)$$

$$= \frac{1}{3^5}\,{}^{10}C_{5-\eta} - \frac{5}{3^5}\,{}^8C_{4-\eta} + \frac{10}{3^5}\,{}^6C_{3-\eta} - \frac{10}{3^5}\,{}^4C_{2-\eta} + \frac{5}{3^5}\,{}^2C_{1-\eta} - \frac{1}{3^5}p(0,\eta)$$

$$= \frac{1}{3^5}\left[{}^5C_0\,{}^{10}C_{5-\eta} - {}^5C_1\,{}^8C_{4-\eta} + {}^5C_2\,{}^6C_{3-\eta} - {}^5C_3\,{}^4C_{2-\eta} + {}^5C_4\,{}^2C_{1-\eta} - {}^5C_5 p(0,\eta)\right],$$

and so on.

Therefore, we obtain

$$p(n,\eta) = \frac{1}{3^n}\left[\sum_{i=0}^{i=n-1} (-1)^i\ {}^nC_i\ {}^{2(n-i)}C_{(n-i)-\eta} + (-1)^n\ p(0,\eta)\right],$$

$$\text{or } p(n,\eta) = \frac{1}{3^n}\left[\sum_{i=0}^{i=n-1} \frac{(-i)^i n!(2\overline{n-i})!}{i!(n-i)!(n-i-\eta)!(n-i+\eta)!} + (-1)^n p(0,\eta)\right] \quad \ldots(3.12)$$

$$\text{where}\quad p(0,\eta) = \begin{cases} 1, & \text{when } \eta = 0 \\ 0, & \text{when } \eta \neq 0. \end{cases}$$

Hence, the average distance $\bar{\eta}$ is

$$\bar{\eta} = \sum_{\eta=-n}^{\eta=+n} \eta\ p(n,\eta).$$

Substituting $p(n,\eta)$ from (3.12), we obtain

$$\bar{\eta} = \frac{1}{3^n} \sum_{\eta=-n}^{\eta=+n} \eta \left[\sum_{i=0}^{i=n-1} \frac{(-1)^i \, n!(2n-i)!}{i!(n-i)!(n-i-\eta)!(n-i+\eta)!} + (-1)^n p(0,\eta) \right]$$

$$= 0.$$

The same result as (iv) before.

In this case, the formula for $p(n,\eta)$ in equation (3.12) is so complicated that we shall not find the formula for $p(n,\eta)$ when n is very large. Now we know the probabilities $P_n(\xi)$ and $p(n,\eta)$ that after n steps in the time a particle is ξ steps and η steps from the origin respectively. In this method we cannot find the probability values of a particle in each hexagon as in figure 3.3. But we can find these values by the following procedure.

Let $P_n(\xi,\eta)$ be the probability that after n steps a particle is at the point (ξ,η) in the $\xi\eta$-plane. Then, according to figure 3.2, we have

$$P_n(\xi,\eta) = \frac{1}{3^n} \, {}^nC_\xi \, {}^\xi C_{n-\xi-\eta} \quad , \quad \text{where } \xi = 0,1,2,\ldots,n, \text{ and } \eta = 0,\pm1,\pm2,\ldots,\pm n.$$

Therefore, we can find the probability values as in figure 3.3 from this formula.

Appendix A

To show that $\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$.

By the symmetry of the function e^{-ax^2}, we have

$$\int_0^{\infty} e^{-ax^2} dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{-ax^2} dx \qquad \ldots\ldots\ (A-1)$$

Now to find the value of $\int_{-\infty}^{\infty} e^{-x^2} dx$,

we let $$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-y^2} dy .$$

Since a definite integral is a function of its limits only, not of the variable of integration.

So $$I^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dxdy.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2 + y^2)} dxdy.$$

The limits on the double integral indicate that the integration is over the whole (x,y) plane. Now we change to polar coordinates, writing

$$x = r\cos\theta ,$$

$y = r\sin\theta$, and the element of area in polar (r, θ) coordinates is $r\,d\,r\,d\,\theta$.

Hence $$I^2 = \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\, r\, d\, r\, d\theta$$

$$= [\theta]_0^{2\pi} \int_0^{\infty} e^{-r^2}\, r\, d\, r.$$

$$= 2\pi \left[-\frac{1}{2}\, e^{-r^2} \right]_0^{\infty}$$

$$= 2\pi \cdot \frac{1}{2} \quad = \pi .$$

So $$I = \int_{-\infty}^{\infty} e^{-x^2} dx \quad = \sqrt{\pi} .$$

Writing $a^{1/2}x$ for x, we have

$$\int_{-\infty}^{\infty} e^{-ax^2} dx \quad = \sqrt{\frac{\pi}{a}} \qquad \ldots\ldots\text{(A-2)}$$

Substituting (A-2) into (A-1), we obtain

$$\int_0^{\infty} e^{-ax^2} dx \quad = \frac{1}{2}\sqrt{\frac{\pi}{a}} .$$

Appendix B

To show that $\int_0^\infty x^m e^{-ax^2} dx = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}$.

Let $I_m = \int_0^\infty x^m e^{-ax^2} dx$, ... (B-1)

where m is a positive integer.

We shall separate m into odd and even numbers. For m odd, (B-1) becomes

$$I_1 = \int_0^\infty x e^{-ax^2} dx = \left[-\frac{1}{2a} e^{-ax^2}\right]_0^\infty = \frac{1}{2a} , \quad \text{(B-2)}$$

by using the method of differentiation under the integral sign,

$$I_3 = \int_0^\infty x^3 e^{-ax^2} dx = -\frac{\partial}{\partial a} \int_0^\infty x e^{-ax^2} dx$$

$$= -\frac{\partial}{\partial a}\left(\frac{1}{2a}\right) = \frac{1}{2a^2} , \quad \text{.......... (B-3)}$$

$$I_5 = \int_0^\infty x^5 e^{-ax^2} dx = -\frac{\partial}{\partial a} \int_0^\infty x^3 e^{-ax^2} dx$$

$$= -\frac{\partial}{\partial a}\left(\frac{1}{2a^2}\right) = \frac{1}{a^3} , \quad \text{.......... (B-4)}$$

and so on.

For m even, (B-1) becomes

$$I_0 = \int_0^\infty e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} , \quad \text{.......... (B-5).}$$

then using the technique of differentiation under the integral,

$$I_2 = \int_0^\infty x^2 e^{-ax^2} dx = -\frac{\partial}{\partial a} \int_0^\infty e^{-ax^2} dx$$

$$= -\frac{\partial}{\partial a}\left(\frac{1}{2}\sqrt{\frac{\pi}{a}}\right) = \frac{1}{4}\frac{\pi^{1/2}}{a^{3/2}} \quad , \text{.........} \quad (B-6)$$

$$I_4 = \int_0^\infty x^4 e^{-ax^2} dx = -\frac{\partial}{\partial a} \int_0^\infty x^2 e^{-ax^2} dx$$

$$= -\frac{\partial}{\partial a}\left(\frac{1}{4}\frac{\pi^{1/2}}{a^{3/2}}\right) = \frac{3}{8}\frac{\pi^{1/2}}{a^{5/2}} \quad , \text{.........} \quad (B-7)$$

and so on.

Thus from the recursion formulas of the Gamma Function that

$$\Gamma(n + 1) = n\,\Gamma(n) ,$$

$$\Gamma(n + 1) = n! \qquad \text{if } n = 0, 1, 2, \ldots \text{ where } 0! = 1,$$

and the value of $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we can see that (B-2), (B-3),(B-4), (B-5), (B-6) and (B-7) are in the form

$$\int_0^\infty x^m e^{-ax^2} dx = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}} \quad , \text{ for both odd and even}$$

values of m.

Therefore $\displaystyle\int_0^\infty x^m e^{-ax^2} dx = \frac{\Gamma[(m+1)/2]}{2a^{(m+1)/2}}$, where m = 0, 1, 2,....

Appendix C

Find $\mathcal{L}^{-1}\left\{e^{-a\sqrt{s}}\right\}$, where $a > 0$.

I) If $\mathcal{L}\left\{f(t)\right\} = F(s)$, then we have

$$\mathcal{L}\left\{f'(t)\right\} = sF(s)-f(0) = sF(s) \quad \text{if } f(0) = 0.$$

Thus if $\mathcal{L}^{-1}\left\{F(s)\right\} = f(t)$ and $f(0) = 0$, then

$$\mathcal{L}^{-1}\left\{sF(s)\right\} = f'(t).$$

II) By using the complex inversion formula we can show that

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = 1 - \frac{2}{\sqrt{\pi}}\int_0^{a/2\sqrt{t}} e^{-u^2}\,du$$

$$= 1 - \text{erf}\,(a/2\sqrt{t})\,, \text{ where } a > 0.$$

Hence $F(s) = \dfrac{e^{-a\sqrt{s}}}{s}$ and $f(t) = 1 - \text{erf}\,(a/2\sqrt{t})$,

with $f(0) = 1 - \text{erf}\,(\infty) = 1 - 1 = 0.$

Then by I, it follows that

$$\mathcal{L}^{-1}\left\{e^{-a\sqrt{s}}\right\} = f'(t) = \frac{d}{dt}\left\{1-\text{erf}(a/2\sqrt{t})\right\}$$

$$\text{erf}\left(\frac{a}{2\sqrt{t}}\right) = \frac{2}{\sqrt{\pi}}\left\{\frac{a}{2\sqrt{t}} - \frac{\left(\frac{a}{2\sqrt{t}}\right)^3}{3.1!} + \frac{\left(\frac{a}{2\sqrt{t}}\right)^5}{5.2!} - \frac{\left(\frac{a}{2\sqrt{t}}\right)^7}{7.3!} + \cdots\right\}$$

$$= \frac{1}{\sqrt{\pi}}\left\{at^{-1/2} - \frac{a^3t^{-3/2}}{3.2^2.1!} + \frac{a^5t^{-5/2}}{5.2^4.2!} - \frac{a^7t^{-7/2}}{7.2^6.3!} + \cdots\right\}$$

$$\frac{d}{dt}\left\{1-\mathrm{erf}(a/2\sqrt{t})\right\} = -\frac{1}{\sqrt{\pi}}\left\{-\frac{1}{2}at^{-3/2} + \frac{3}{2}\cdot\frac{a^3t^{-5/2}}{3.2^2.1!} - \frac{5}{2}\,\frac{a^5t^{-7/2}}{5.2^4.2!} + \frac{7}{2}\,\frac{a^7t^{-9/2}}{7.2^6.3!} - \ldots\right\}$$

$$= \frac{at^{-3/2}}{2\sqrt{\pi}}\left\{1 - \frac{a^2t^{-1}}{2^2.1!} + \frac{a^4t^{-2}}{2^4.2!} - \frac{a^6t^{-3}}{2^6.3!} + \ldots\right\}$$

$$= \frac{at^{-3/2}}{2\sqrt{\pi}}\left\{1 - \frac{a^2t^{-1}}{4} + \frac{\left(\frac{a^2t^{-1}}{4}\right)^2}{2!} - \frac{\left(\frac{a^2t^{-1}}{4}\right)^3}{3!} + \ldots\right\}$$

$$= \frac{ae^{-a^2/4t}}{2\sqrt{\pi t^3}} .$$

Therefore $\mathcal{L}^{-1}\left\{e^{-a\sqrt{s}}\right\} = \dfrac{ae^{-a^2/4t}}{2\sqrt{\pi t^3}}$.