CHAPTER III

ANALYTIC INVARIANTS OF A SEMIGROUP ACTION AND ANALYTIC HOMOMORPHISMS

The purpose of this chapter is to study some of the properties and the applications of the analytic invariants of a semigroup action and the analytic homomorphisms on the semigroup of real numbers. The first part of this section discuss group invariants. We then extend this concept to invariants of a semigroup action, especially, the analytic invariants. The remainder of this section studies analytic homomorphisms on the semigroup of real numbers.

Let G be a group, X be a set. Suppose that G acts on X on the left. Then there exists a $\psi : G \times X \to X$ such that for any g, h belonging to G and x belonging to X:

$$\psi(gh,x) = \psi(g,\psi(h,x)) \quad \text{and}$$

$$\psi(e,x) = x \quad \text{where } e \text{ is the identity in } G.$$

Let $\mathcal{M}(X,Y)$ = set of all maps from X into Y.

$$= \{f \mid f: X \to Y\}.$$

<u>Proposition 3.1.</u> If ψ is a left action of G on X, then ψ induces a right action of G on $\mathcal{M}(X,Y)$.

<u>Proof</u>: Let $f \in \mathcal{M}(X,Y)$, $g \in G$ and $x \in X$.

Define $\Phi : \mathcal{M}(X,Y) \times G \to \mathcal{M}(X,Y)$ by

$$[\Phi(f,g)](x) = f(\psi(g,x)).$$

We shall show that Φ is a right action of G on $\mathcal{M}(X,Y)$. Let $g, h \in G$ and $x \in X$. Then,

$$[\Phi(f,gh)](x) = f(\psi(gh,x))$$
$$= f(\psi(g,\psi(h,x)))$$

and

$$[\Phi(\Phi(f,g),h)](x) = (\Phi(f,g))(\psi(h,x))$$
$$= f(\psi(g,\psi(h,x))).$$

Therefore, $[\Phi(f,gh)](x) = [\Phi(\Phi(f,g),h)](x)$.

Thus, $\Phi(f,gh) = \Phi(\Phi(f,g),h)$ since $x \in X$ is arbitrary.

$$[\Phi(f,e)](x) = f(\psi(e,x))$$
$$= f(x).$$

Therefore, $\Phi(f,e) = f$.

This proves that Φ is right action of G on $\mathcal{M}(X,Y)$. #

Now, suppose G acts on a set X on the left. That is, there exists $\psi : G \times X \to X$ such that $\psi(gh,x) = \psi(g,\psi(h,x)), \psi(e,x)=x$ $\forall g,h \in G$ and $\forall x \in X$. Let $\mathcal{M}(Y,X) = \{f | f: Y \to X\}$.

<u>Proposition 3.2</u> If ψ is a left action on X then ψ induces a left action of G on $\mathcal{M}(Y,X)$

<u>Proof</u>: Let $f \in \mathcal{M}(Y,X)$, $g \in G$ and $y \in Y$.

Define $\Phi : G \times \mathcal{M}(Y,X) \to \mathcal{M}(Y,X)$ by

$$[\Phi(g,f)](y) = \psi(g,f(y)).$$

We want to show Φ is a left action of G on $\mathcal{M}(Y,X)$.

Let $g, h \in G$ and $y \in Y$, then

$$[\Phi(gh,f)](y) = \psi(gh,f(y))$$
$$= \psi(g,\psi(h,f(y))),$$

and

$$[\Phi(g,\Phi(h,f))](y) = \psi(g,\Phi(h,f)(y))$$
$$= \psi(g,\psi(h,f(y))).$$

Therefore, $[\Phi(gh,f)](y) = [\Phi(g,\Phi(h,f))](y)$

Thus, $\Phi(gh,f) = \Phi(g,\Phi(h,f))$ since y is arbitrary.

Now, $[\Phi(e,f)](y) = \psi(e,f(y))$

$= f(y)$, this implies that

$\Phi(e,f) = f$. Therefore Φ is a left action of G on $\mathcal{M}(Y,X)$. #

Proposition 3.3 If ψ is a left action of G on X then ψ induces a left action of G on $\mathcal{M}(X,X)$, where $\mathcal{M}(X,X) = \{f | f: X \to X\}$.

Proof: Let $f \in \mathcal{M}(X,X)$, $g \in G$ and $x \in X$.

Define $\Phi : G \times \mathcal{M}(X,X) \to \mathcal{M}(X,X)$ by

$[\Phi(g,f)](x) = \psi(g,f(\psi(g^{-1},x)))$. We want to show that Φ is a left action of G on $\mathcal{M}(X,X)$. Let $g, h \in G$, and $x \in X$, then

$$\begin{aligned}[\Phi(gh,f)](x) &= \psi(gh,f(\psi((gh)^{-1},x))) \\ &= \psi(gh,f(\psi(h^{-1}g^{-1},x))) \\ &= \psi(gh,f(\psi(h^{-1},\psi(g^{-1},x))) \\ &= \psi(g,\psi(h,f(\psi(h^{-1},\psi(g^{-1},x))))),\end{aligned}$$

and

$$\begin{aligned}[\Phi(g,\Phi(h,f)](x) &= \psi(g,\Phi(h,f)(\psi(g^{-1},x))) \\ &= \psi(g,\psi(h,f(\psi(h^{-1},\psi(g^{-1},x))))).\end{aligned}$$

Therefore, $\Phi(gh,f) = \Phi(g,\Phi(h,f))$. Next, $[\Phi(e,f)](x) = \psi(e,f(\psi(e^{-1},x)))$ $= \psi(e,f(\psi(e,x))) = \psi(e,f(x)) = f(x)$, this implies that $\psi(e,f) = f$. Therefore, Φ is a left action of G on $\mathcal{M}(X,X)$. #

Let $\psi : G \times X \to X$ be a left action, $x \in X$ is an invariant of ψ if $\psi(g,x) = x \quad \forall g \in G$. We sometimes write $g \cdot x$ instead of $\psi(g,x)$. By proposition (3.1) ψ induces a right action $\Phi : \mathcal{M}(X,Y) \times G \to \mathcal{M}(X,Y)$ defined by $[\Phi(f,g)](x) = f(\psi(g,x))$. Therefore,

$f \in \mathcal{M}(X,Y)$ is an invariant of Φ if $\Phi(f,g) = f \ \forall g \in G$ or $[\Phi(f,g)](x) = f(x) \ \forall x \in X, \forall g \in G$. Thus we see that $f \in \mathcal{M}(X,Y)$ is an invariant of Φ if $f(\psi(g,x)) = f(x) \ \forall x \in X, \forall g \in G$.

Define a relation $\sim$ on X by $x \sim y \Longleftrightarrow \exists g \in G$ such that $y = \psi(g,x)$. We see that $\sim$ is an equivalence relation. To prove this, let $x, y, z \in X$, $x \sim x$ since $\psi(e,x) = x$. Assume $x \sim y$ then there exists $g \in G$ such that $y = \psi(g,x)$. Since $g^{-1} \in G$, it follows that $\psi(g^{-1},y) = \psi(g^{-1},\psi(g,x)) = \psi(g^{-1}g,x) = \psi(e,x) = x$. This implies $y \sim x$. Next, assume $x \sim y$ and $y \sim z$, we see that there exist g_1, g_2 such that $y = \psi(g_1,x)$, $z = \psi(g_2,y)$. Since $g_2g_1 \in G$, we get that $z = \psi(g_2,y) = \psi(g_2,\psi(g_1,x)) = \psi(g_2g_1,x)$. Therefore $x \sim z$.

Therefore f is an invariant of Φ and $x,y \in X$, $x \sim y$ implies that $f(x) = f(y)$ i.e. f has the same value for all elements of an equivalence class.

Proposition 3.4 f is an invariant of Φ iff f has the same value for all elements of an equivalence class.

Proof: Assume f is an invariant of Φ. Then $\Phi(f,g) = f \ \forall g \in G$. Let $x \in X$ then $[\Phi(f,g)](x) = f(x)$. Therefore $f(\psi(g,x)) = f(x)$. Let $y = \psi(g,x)$ i.e. $x \sim y$. Thus, $f(x) = f(y)$. Therefore f has the same value for all elts of an equivalence class.

Next, assume f has the same value for all elts of an equivalence class. Let $x \in X$, $g \in G$. Therefore $\psi(g,x) \sim x$ and implies that $f(\psi(g,x)) = f(x)$. That is $\Phi(g,f) = f$. Hence f is an invariant of Φ.

Next, by proposition (3.2) ψ induces a left action Φ on $\mathcal{M}(Y,X)$ where $\Phi : G \times \mathcal{M}(Y,X) \to \mathcal{M}(Y,X)$ defined by $[\Phi(g,f)](y) = \psi(g,f(y))$. We see that $f \in \mathcal{M}(Y,X)$ is an invariant of Φ if $\Phi(g,f) = f \ \forall g \in G$ or $[\Phi(g,f)](y) = f(y) \ \forall g \in G \ \forall y \in Y$. Thus,

$f \in \mathcal{M}(Y,X)$ is an invariant of Φ if $\psi(g,f(y)) = f(y) \ \forall g \in G$. and $\forall y \in Y$.

<u>Proposition 3.5</u> Let ψ be a left action of G on X and $f \in \mathcal{M}(Y,X)$. Then f is an invariant of Φ iff f(y) is an invariant of $\psi \ \forall y \in Y$.

<u>Proof</u>: Assume f is an invariant of Φ. Then $\Phi(g,f) = f \ \forall g \in G$. Then $[\Phi(g,f)](y) = f(y) \ \forall y \in Y \ \forall g \in G$. Therefore $\psi(g,f(y)) = f(y) \ \forall y \in Y \ \forall g \in G$. Hence f(y) is an invariant of ψ.

Now, assume f(y) is an invariant of $\psi \ \forall y \in Y$. Therefore $\psi(g,f(y)) = f(y) \ \forall g \in G \ \forall y \in Y$. Therefore by definition, $[\Phi(g,f)](y) = f(y) \ \forall g \in G \ \forall y \in Y$. Thus, $\Phi(g,f) = f \ \forall g \in G$. This proves that f is an invariant of Φ.

By proposition (3.3) ψ induces a left action Φ on $\mathcal{M}(X,X)$ where $\Phi: G \times \mathcal{M}(X,X) \to \mathcal{M}(X,X)$ is defined by $[\Phi(g,f)](x) = \psi(g,f(\psi(g^{-1},x)))$. Therefore, $f \in \mathcal{M}(X,X)$ is an invariant of Φ iff $\Phi(g,f) = f \ \forall g \in G$ or $[\Phi(g,f)](x) = f(x) \ \forall g \in G \ \forall x \in X$. So we see that $f \in \mathcal{M}(X,X)$ is an invariant of Φ iff $\psi(g,f(\psi(g^{-1},x))) = f(x) \ \forall x \in X \ \forall g \in G$. Let $h = g^{-1}$, therefore $h \in G$. Therefore, $\psi(g,f(\psi(g^{-1},x))) = \psi(h^{-1},f(\psi(h,x)))$. Now, $f \in \mathcal{M}(X,X)$ is an invariant of Φ iff $\psi(h^{-1},f(\psi(h,x))) = f(x) \ \forall h \in G, \forall x \in X$.

<u>Proposition 3.6</u> If ψ is a left action of G on X then $\psi(g,x) = y$ iff $\psi(g^{-1},y) = x$.

<u>Proof:</u> If $\psi(g,x) = y$ then $\psi(g^{-1},y) = \psi(g^{-1},\psi(g,x)) =$

$\psi(g^{-1}g,x) = \psi(e,x) = x$.

If $\psi(g^{-1},y) = x$ then $\psi(g,x) = \psi(g,\psi(g^{-1},y) =$

$\psi(gg^{-1},y) = \psi(e,y) = y$.

Therefore by proposition(3.6) $\psi(h^{-1},f(\psi(h,x))) = f(x)$ iff $\psi(h,f(x)) = f(\psi(h,x))$, hence $f \in \mathcal{M}(X,X)$ is an invariant of Φ iff $f(\psi(h,x)) = \psi(h,f(x))$. Therefore f is an invariant of Φ if and only if f is a G-homomorphism.

Now, let $\mathcal{M}(X \times X,Y) = \{f \mid f: X \times X \to Y\}$. Define $\Phi : \mathcal{M}(X \times X) \times G \to \mathcal{M}(X \times X,Y)$ by

$$[\Phi(f,g)](x_1,x_2) = f(\psi(g,x_1),\psi(g,x_2)) \quad \forall\, g \in G,\ x_1,x_2 \in X.$$

and

$$\Phi : G \times \mathcal{M}(X \times X,X) \to \mathcal{M}(X \times X,X) \text{ by}$$

$$[\Phi(g,f)](x_1,x_2) = \psi(g,f(\psi(g^{-1},x_1),\ \psi(g^{-1},x_2)))$$ then we have the same results as above.

We now extend these concepts to the semigroup with zero case. Let S be a semigroup with zero and ψ be a left semigroup action of S on X. Therefore this left semigroup action of S on X induces a right semigroup action Φ on $\mathcal{M}(X,Y)$ and induces a left action on $\mathcal{M}(Y,X)$ by defining as above. Moreover, ψ induces a right semigroup action Φ on $\mathcal{M}(X \times X,Y)$. But for $\mathcal{M}(X,X)$ and $\mathcal{M}(X \times X,X)$ we can't define an action as above because given $s \in S$, s^{-1} mightnot exist. We can now give an important definition.

Definition: Let S be a semigroup acting on a set X and let $\psi : S \times X \to X$ be the semigroup action. $f \in \mathcal{M}(X,X)$ is said to be a generalized invariant of S if $f(\psi(s,x)) = \psi(s,f(x))$, $\forall s \in S$ $\forall x \in X$.

$f \in \mathcal{M}(X \times X, X)$ is said to be a generalized invariant of S if $f(\psi(s,x_1), \psi(s,x_2)) = \psi(s,f(x_1,x_2))$. $\forall s \in S$ $\forall x_1,x_2 \in X$.

Fix $n_o \in \mathbb{N}$.

Define $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\psi(t,x) = t^{n_o}x$. From previous chapter ψ is a semigroup action of a semigroup $\mathbb{R}$ on $\mathbb{R}$. We want to find analytic functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that f and g are generalized invariants.

We first find an analytic function $f: \mathbb{R} \to \mathbb{R}$ such that $f(\psi(t,x)) = \psi(t,f(x))$. We suppose that $f(x) = C_0+C_1x+C_2x^2+\dots$ .... We need to find C_i such that the conditions $f(\psi(t,x)) = \psi(t,f(x))$ holds. We see that

$$f(\psi(t,x)) = f(t^{n_o}x)$$

(3.1) $$= C_0+C_1t^{n_o}x+C_2(t^{n_o}x)^2+C_3(t^{n_o}x)^3+ \dots\dots$$

and

$$\psi(t,f(x)) = t^{n_o}f(x)$$

(3.2) $$= C_0t^{n_o}+C_1t^{n_o}x+C_2t^{n_o}x^2+ \dots\dots\dots\dots\dots$$

Therefore, we want to find C_i such that (3.1) and (3.2) are equal. Consider the term t^mx^n ($\forall m$, $\forall n \in N$), we see that all the C_i (except C_1) are zero.

Hence $f \in \mathcal{M}(\mathbb{R},\mathbb{R})$ is a generalized invariant iff $f(x) = Cx$ where $C \in \mathbb{R}$, i.e. f is a linear function.

Next, we find analytic functions $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $g(\psi(t,x_1), \psi(t,x_2)) = \psi(t,g(x_1,x_2))$. We suppose that

$$g(x_1,x_2) = C_{00}+C_{10}x_1+C_{01}x_2+ C_{20}x_1^2+C_{11}x_1x_2+C_{02}x_2^2+\cdots\cdots$$

We need to find C_{ij} such that the condition $g(\psi(t,x_1),\psi(t,x_2)) = \psi(t,g(x_1,x_2))$ holds. We have that

$$\begin{aligned} g(\psi(t,x_1),\psi(t,x_2)) &= g(t^{n_o}x_1,t^{n_o}x_2) \\ &= C_{00}+C_{10}t^{n_o}x_1+C_{01}t^{n_o}x_2+C_{20}(t^{n_o}x_1)^2 \\ &\quad +C_{11}t^{n_o}x_1t^{n_o}x_2+C_{02}(t^{n_o}x_2)^2+ \cdots\cdots \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \psi(t,g(x_1,x_2)) &= t^{n_o}g(x_1,x_2). \\ &= C_{00}t^{n_o}+C_{10}t^{n_o}x_1+C_{01}t^{n_o}x_2+C_{20}t^{n_o}x_1^2 \\ &\quad +C_{11}t^{n_o}x_1x_2+C_{02}t^{n_o}x_2^2+\cdots\cdots \end{aligned} \tag{3.4}$$

Therefore, we want to find C_{ij} such that (3.3) and (3.4) are equal. We get that $C_{ij}= 0 \ \forall i,j$ except for C_{10}, C_{01}. Therefore $g \in \mathcal{M}(\mathbb{R} \times \mathbb{R},\mathbb{R})$ is a generalized invariant iff g is in the form:

$$g(x_1,x_2) = Cx_1+ dx_2 \quad \text{where } C,\ d \in \mathbb{R}.$$

That is, g is a linear function.

Hence the analytic generalized invariants $f : \mathbb{R} \to \mathbb{R}$ are written in the form: $f(x) = Cx$, $C \in \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is written in the form $g(x_1,x_2) = Cx_1+ dx_2$, $C, d \in \mathbb{R}$. #

Analytic homomorphism:

Definition. Let $(S,\circ,0)$ and $(S',\circ',0')$ be semigroups with zero. A mapping f of S into S' is said to be homomorphism

if 1) $f(x\circ y) = f(x)\circ' f(y) \quad \forall\, x, y \in S.$

2) $f(0) = 0'.$

We first find analytic homomorphism $\psi : \mathbb{R} \to \mathbb{R}$ where $\mathbb{R}$ has the usual multiplication i.e. We want to find $\psi : \mathbb{R} \to \mathbb{R}$ such that $\psi(xy) = \psi(x)\,\psi(y) \quad \forall\, x, y \in \mathbb{R}$. Suppose $\psi : \mathbb{R} \to \mathbb{R}$ is analytic homomorphism such that $\psi(x) = C_1x+C_2x^2+C_3x^3+\dots\dots+C_nx^n+\dots\dots\dots$. It suffices to find C_i such that $\psi(x)\psi(y) = \psi(xy)$.

We have that

$$(3.5) \qquad \psi(xy) = C_1xy+C_2x^2y^2+C_3x^3y^3+C_4x^4y^4+ \dots\dots\dots\dots$$

and

$$(3.6) \qquad \psi(x)\psi(y) = (C_1x+C_2x^2+C_3x^3+\dots\dots)(C_1y+C_2y^2+C_3y^3+\dots\dots)$$

$$= C_1C_1xy+C_1C_2xy^2+C_2C_1x^2y+C_3C_1x^3y+C_2C_2x^2y^2+ C_1C_3xy^3+ \dots\dots\dots\dots\dots\dots\dots$$

If $C_i = 0 \ \forall\, i$, then $\psi \equiv 0$. Now assume there exists k such that $C_k \neq 0$. Let n be the smallest natural number such that $C_n \neq 0$. We claim that $C_m = 0 \quad \forall\, m \neq n$. We prove this by comparing the coefficient of the term x^my^n $(m \neq n)$ in (3.5) and (3.6), respectively. Then,

$$0 = C_mC_n .$$

But $C_n \neq 0$ implying that $C_m = 0$.

Now, we consider the coefficient of the term x^ny^n in (3.5) and (3.6), respectively. Then we get that

$C_n = C_n C_n$ which implies that $C_n = 1$ since $C_n \neq 0$.

Hence the analytic homomorphisms $\psi : \mathbb{R} \to \mathbb{R}$ are the function $\psi(x) = x^n$ for some $n \in \mathbb{N}$ and the 0 function.

Next, let $M(2,\mathbb{R})$ = {two by two matrices with entries in $\mathbb{R}$}. We want to find all $\psi : \mathbb{R} \to M(2,\mathbb{R})$ such that ψ is an analytic homomorphism. Let $\psi(x) = C_1x+C_2x^2+C_3x^3+\ldots\ldots+C_nx^n+\ldots\ldots$ where $C_i \in M(2,\mathbb{R})$. We want to find C_i such that $\psi(xy) = \psi(x)\psi(y)$. As above, since $\psi(x)\psi(y) = \psi(xy)$, we get two conditions

(3.7) $$C_nC_n = C_n$$

(3.8) $$C_nC_m = C_mC_n = 0 \quad \text{if} \quad m \neq n.$$

If $C_i = \bar{0}$ $\forall i$ then $\psi(x) \equiv \bar{0}$ $\forall x \in \mathbb{R}$.

Assume there exist k such that $C_k \neq \bar{0}$. Let m be the smallest natural number k such that $C_m \neq \bar{0}$. We now have two cases to consider:

<u>Case 1</u>. We assume that $\det C_m \neq 0$. Then C_m^{-1} exists. Since we have condition $C_mC_m = C_m$, then we get that

$$C_mC_m = C_m$$

$$C_mC_mC_m^{-1} = C_mC_m^{-1}$$

$$C_mI = I$$

$$C_m = I \quad \text{where I is the identity matrix.}$$

Next, we claim that $C_k = \bar{0}$, $k \neq m$. By using (3.8), therefore $C_kC_m = \bar{0}$, $C_kI = \bar{0}$. So, $C_k = \bar{0}$. Then

$$\psi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x^m.$$

Case 2 We assume that in this case det $C_m = 0$ and $C_m \neq \bar{0}$.

We let $C_m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d \in \mathbb{R}$. We have to find a,b,c,d such that (3.7) holds. Condition:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{implies that}$$

(3.9) $$a^2 + bc = a$$

(3.10) $$ab + bd = b$$

(3.11) $$ac + cd = c$$

(3.12) $$bc + d^2 = d$$

From (3.9) and (3.12) we get that $a^2 - d^2 - a + d = 0$, and hence $(a-d)(a+d-1) = 0$ and from (3.10) and (3.11) we also have that $b(a+d-1) = 0$, $c(a+d-1) = 0$.

We can assume that $a+d = 1$. Since $a+d \neq 1$, it follows that $a = d$, $b = 0$, $c = 0$ and hence det $C_m \neq 0$ or $C_m = \bar{0}$, a contradiction.

Since det $C_m = 0$, it follows that $ad-bc = 0$. Therefore, $ad = bc$ and hence $a(1-a) = bc$. Now consider the following:

If $b = 0$ and $c = 0$ then $a(1-a) = 0$. Therefore,

$$C_m = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

(3.13) or

$$C_m = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

If $b = 0$ but $c \neq 0$ then $a(1-a) = 0$. Therefore,

$$C_m = \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}$$

(3.14) or

$$C_m = \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix}$$

If $b \neq 0$ and $c = 0$ then we also have that

$$C_m = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$$

(3.15) or

$$C_m = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$$

Now, assume that $b \neq 0$, $c \neq 0$ then $a \neq 0$, $a \neq 1$ therefore $b = \frac{a(1-a)}{c}$. Then

(3.16) $$C_m = \begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & 1-a \end{pmatrix}, \ a \neq 0, \ a \neq 1, \ c \neq 0.$$

If $k > m$, $C_k = \bar{0}$ then we get that

$$\psi(x) = C_m x^m$$

It is easy to verify that $\psi(x)\psi(y) = \psi(xy)$ in this case. Now, assume that there exists an $\ell > m$ such that $C_\ell \neq \bar{0}$. Let n be the smallest ℓ such that $C_n \neq \bar{0}$, $n > m$. We want to find C_n, using the conditions in (3.7) and (3.8).

If $C_m = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then we let $C_n = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$.

Therefore, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} s & t \\ u & v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ which implies that $s=0, t=0$ and $\begin{pmatrix} s & t \\ u & v \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ which implies that $s = 0$, $u = 0$.

So, we can get that $C_n = \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix}$. Using the fact that $C_n C_n = C_n$ we get that $\begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & v \end{pmatrix}$. This implies that $C_n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

In this case we claim that $C_k = 0 \ \forall k > n$. We prove this by letting $C_k = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$ and using the fact that $C_k C_m = \bar{0} = C_k C_n$.

That is, $\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Then, $m_1, m_3 = 0$ and $m_2, m_4 = 0$ and hence $C_k = 0 \ \forall k > n$.

Thus,

$$(3.17) \qquad \psi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x^n.$$

If $C_m = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ then by the same arguement as above we have that

$$C_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } C_k = \bar{0} \ \forall k > n.$$

Hence,

$$(3.18) \qquad \psi(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x^m + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^n.$$

If $C_m = \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix}$ then by the same proof as above we get that $C_n = \begin{pmatrix} 0 & 0 \\ -c & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ -c & 0 \end{pmatrix}$ and we can also prove that $C_k = \bar{0} \ \forall k > n$. Hence,

$$(3.19) \qquad \psi(x) = \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & 0 \\ -c & 1 \end{pmatrix} x^n \text{ or}$$

$$(3.20) \qquad \psi(x) = \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix} x^m + \begin{pmatrix} 1 & 0 \\ -c & 0 \end{pmatrix} x^n, \ m < n.$$

If $C_m = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$ then using the same arguments as before we can show that $C_n = \begin{pmatrix} 0 & -b \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & -b \\ 0 & 0 \end{pmatrix}$ and we can prove that $\forall\, k > n$, $C_k = \bar{0}$. Then,

$$(3.21) \qquad \psi(x) = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & -b \\ 0 & 1 \end{pmatrix} x^n \quad \text{or}$$

$$(3.22) \qquad \psi(x) \quad \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} x^m + \begin{pmatrix} 1 & -b \\ 0 & 0 \end{pmatrix} x^n, \ m < n.$$

In any of the above cases it is easy to verify that ψ is a homomorphism.

Now, we have that $C_m = \begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & 1-a \end{pmatrix}$, $a \neq 0$, $a \neq 1$, $c \neq 0$.

Using (3.7) and (3.8) we get that

$C_n = \begin{pmatrix} 1-a & \frac{-a(1-a)}{c} \\ -c & a \end{pmatrix}$ and we also prove that $\forall\, k > n$

$C_k = \bar{0}$ by using the fact that $C_k C_n = \bar{0} = C_k C_m$. This gives us:

$$(3.23) \qquad \psi(x) = \begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & 1-a \end{pmatrix} x^m + \begin{pmatrix} 1-a & -\frac{a(1-a)}{c} \\ -c & a \end{pmatrix} x^n.$$

Therefore, $\psi(xy) = \begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & 1-a \end{pmatrix} x^m y^m + \begin{pmatrix} 1-a & -\frac{a(1-a)}{c} \\ -c & a \end{pmatrix} x^n y^n$,

and

$$\psi(x)\psi(y) = \left[\begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & 1-a \end{pmatrix} x^m + \begin{pmatrix} 1-a & -\frac{a(1-a)}{c} \\ -c & a \end{pmatrix} x^n\right]\left[\begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & 1-a \end{pmatrix} y^m + \begin{pmatrix} 1-a & -\frac{a(1-a)}{c} \\ -c & a \end{pmatrix} y^n\right]$$

$$= \begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & 1-a \end{pmatrix}\begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & 1-a \end{pmatrix} x^m y^m + \begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & 1-a \end{pmatrix}\begin{pmatrix} 1-a & -\frac{a(1-a)}{c} \\ -c & a \end{pmatrix}$$

$$x^m y^n + \begin{pmatrix} 1-a & -\frac{a(1-a)}{c} \\ -c & a \end{pmatrix}\begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & 1-a \end{pmatrix} x^n y^m +$$

$$\begin{pmatrix} 1-a & -\frac{a(1-a)}{c} \\ -c & a \end{pmatrix}\begin{pmatrix} 1-a & -\frac{a(1-a)}{c} \\ -c & a \end{pmatrix} x^n y^n.$$

$$= \begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & 1-a \end{pmatrix} x^m y^m + \begin{pmatrix} 1-a & -\frac{a(1-a)}{c} \\ -c & a \end{pmatrix} x^n y^n.$$

$= \psi(xy)$, hence ψ is a homomorphism **and obviously $\psi(0) = 0$.**

<u>Theorem</u> Let ψ be analytic homomorphism of $\mathbb{R}$ into $M(2,\mathbb{R})$. If $\psi \not\equiv 0$ and $\psi \not\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x^n$ then ψ is equivalent to the homomorphisms ψ_1 or ψ_2 where $\psi_1(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^m$, $\psi_2(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x^n$, $m \neq n$.

<u>Proof</u>: We recall that two representations ψ and ψ' are said to be equivalent if there exist a non-singular matrix A such that

$$\psi'(x) = A\psi(x)A^{-1} \quad \forall x \in \mathbb{R}.$$

We claim that $\psi(x) = C_m x^m$ where C_m are in (3.13, 3.14, 3.15, 3.16) is equivalent to $\psi_1(x)$. and $\psi(x)$ in (3.17, 3.18, 3.19, 3.20, 3.21, 3.22, 3.23) is equivalent to $\psi_2(x)$.

It is enough to prove that $\psi(x)$ in (3.17, 3.18, 3.19, 3.20, 3.21, 3.22, 3.23) are equivalent to $\psi_2(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x^n$.

If $\psi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x^n$ then it is obvious by choosing

$A = I$.

If $\psi(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x^m + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^n$, then choose $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Therefore $A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We see that $A\psi(x)A^{-1} =$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x^n.$$

If $\psi(x) = \begin{pmatrix} 1 & k \\ 0 & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & -k \\ 0 & 1 \end{pmatrix} x^n$, then choose $A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

therefore $A^{-1} = \begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix}$. We see that $A\psi(x)A^{-1} =$

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} x^m + \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 1 \end{pmatrix} x^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x^n.$$

If $\psi(x) = \begin{pmatrix} 0 & k \\ 0 & 1 \end{pmatrix} x^m + \begin{pmatrix} 1 & -k \\ 0 & 0 \end{pmatrix} x^n$, then choose $A = \begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix}$

therefore $A^{-1} = \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix}$. We see that $A\psi(x)A^{-1} =$

$$\begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} 0 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix}\begin{pmatrix} 1 & -k \\ 0 & 0 \end{pmatrix}\begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix} x^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x^n.$$

If $\psi(x) = \begin{pmatrix} 1 & 0 \\ k & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & 0 \\ -k & 1 \end{pmatrix} x^n$, then choose $A = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}$

therefore $A^{-1} = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$. We see that $A\psi(x)A^{-1} =$

$$\begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} x^m + \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -k & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix} x^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x^n.$$

If $\psi(x) = \begin{pmatrix} 0 & 0 \\ k & 1 \end{pmatrix} x^m + \begin{pmatrix} 1 & 0 \\ -k & 0 \end{pmatrix} x^n$ then choose $A = \begin{pmatrix} -k & 1 \\ 1 & 0 \end{pmatrix}$

therefore $A^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}$. Then we see that $A\psi(x)A^{-1} =$

$$\begin{pmatrix} -k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ k & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} x^m + \begin{pmatrix} k & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -k & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} x^n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x^n.$$

If $\psi(x) = \begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & 1-a \end{pmatrix} x^m + \begin{pmatrix} 1-a & -\frac{a(1-a)}{c} \\ -c & a \end{pmatrix} x^n$. We choose

$$A = \begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & -a \end{pmatrix} \text{ then } A^{-1} = \frac{1}{-a}\begin{pmatrix} -a & -\frac{a(1-a)}{c} \\ -c & a \end{pmatrix}.$$

We see that

$$\frac{1}{-a}\begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & -a \end{pmatrix}\begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & 1-a \end{pmatrix}\begin{pmatrix} -a & -\frac{a(1-a)}{c} \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and}$$

$$\frac{1}{-a}\begin{pmatrix} a & \frac{a(1-a)}{c} \\ c & -a \end{pmatrix}\begin{pmatrix} 1-a & -\frac{a(1-a)}{c} \\ -c & a \end{pmatrix}\begin{pmatrix} -a & -\frac{a(1-a)}{c} \\ -c & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ which}$$

implies that $A\psi(x)A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} x^m + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x^n$. #

Remark Every details we have discussed are also true in complex numbers.