CHAPTER III

THE LIMIT DISTRIBUTION OF SUMS OF MULTIDIMENSIONAL RANDOM VECTORS

3.1 Introduction

Let $X_i = (X_i^1, \ldots, X_i^m)$, $i = 1,\ldots,n$ be random vectors .

Let

(3.1.1) $$S_n = X_1 + \ldots + X_n .$$

Hence

$$S_n = (S_n^1 , \ldots , S_n^m) ,$$

where

$$S_n^j = \sum_{i=1}^{n} X_i^j , \; j = 1,\ldots,m .$$

In this chapter we shall give conditions under which the distribution of S_n tends to a multivariate Poisson distribution . We shall discuss the case $m = 2$ in detail . The case $m > 2$ will be discussed briefly .

First , we give some motivation by considering a certain factorial moment of sums of 2-dimensional random vectors . Let $X_i = (X_i^1 , X_i^2)$, $i=1,\ldots,n$ be 2-dimensional random vectors such that their components can take only the values 0 or 1 . By Theorem 2.4.1 we know that

$$\mu'_{[3][2]}(S_n) = \sum_{\substack{i_{11}=1 \\ i_{11}\neq i_{12}\neq i_{13},\; i_{21}\neq i_{22} \\ i_{11}\neq i_{13}}}^{n} \sum_{i_{12}=1}^{n} \sum_{i_{13}=1}^{n} \sum_{i_{21}=1}^{n} \sum_{i_{22}=1}^{n} P(X_{i_{11}}^1 = X_{i_{12}}^1 = X_{i_{13}}^1 = X_{i_{21}}^2 = X_{i_{22}}^2 = 1) .$$

Define

$$G(X^1_{i_{11}}, X^1_{i_{12}}, X^1_{i_{13}}, X^2_{i_{21}}, X^2_{i_{22}})$$

$$= P(X^1_{i_{11}}=1)P(X^1_{i_{12}}=1)P(X^1_{i_{13}}=1)P(X^2_{i_{21}}=1)P(X^2_{i_{22}}=1)$$

$$+ P(X^1_{i_{11}}=1)P(X^1_{i_{12}}=1)P(X^2_{i_{21}}=1)\left[P(X^1_{i_{13}}=X^2_{i_{22}}=1) - P(X^1_{i_{13}}=1)P(X^2_{i_{22}}=1)\right]$$

$$+P(X^1_{i_{11}}=1)P(X^1_{i_{12}}=1)P(X^2_{i_{22}}=1)\left[P(X^1_{i_{13}}=X^2_{i_{21}}=1)-P(X^1_{i_{13}}=1)P(X^2_{i_{21}}=1)\right]$$

$$+P(X^1_{i_{11}}=1)P(X^1_{i_{13}}=1)P(X^2_{i_{21}}=1)\left[P(X^1_{i_{12}}=X^2_{i_{22}}=1)-P(X^1_{i_{12}}=1)P(X^2_{i_{22}}=1)\right]$$

$$+P(X^1_{i_{11}}=1)P(X^1_{i_{13}}=1)P(X^2_{i_{22}}=1)\left[P(X^1_{i_{12}}=X^2_{i_{21}}=1)-P(X^1_{i_{12}}=1)P(X^2_{i_{21}}=1)\right]$$

$$+P(X^1_{i_{12}}=1)P(X^1_{i_{13}}=1)P(X^2_{i_{21}}=1)\left[P(X^1_{i_{11}}=X^2_{i_{22}}=1)-P(X^1_{i_{11}}=1)P(X^2_{i_{22}}=1)\right]$$

$$+p(X^1_{i_{12}}=1)P(X^1_{i_{13}}=1)P(X^2_{i_{22}}=1)\left[P(X^1_{i_{11}}=X^2_{i_{21}}=1)-P(X^1_{i_{11}}=1)P(X^2_{i_{21}}=1)\right]$$

$$+P(X^1_{i_{11}}=1)\left[P(X^1_{i_{12}}=X^2_{i_{21}}=1)-P(X^1_{i_{12}}=1)P(X^2_{i_{21}}=1)\right]\cdot$$

$$\left[P(X^1_{i_{13}}=X^2_{i_{22}}=1)-P(X^1_{i_{13}}=1)P(X^2_{i_{22}}=1)\right]$$

$$+P(X^1_{i_{11}}=1)\left[P(X^1_{i_{12}}=X^2_{i_{22}}=1)-P(X^1_{i_{12}}=1)P(X^2_{i_{22}}=1)\right]\cdot$$

$$\left[P(X^1_{i_{13}}=X^2_{i_{21}}=1)-P(X^1_{i_{13}}=1)P(X^2_{i_{21}}=1)\right]$$

$$+P(X^1_{i_{12}}=1)\left[P(X^1_{i_{11}}=X^2_{i_{21}}=1)-P(X^1_{i_{11}}=1)P(X^2_{i_{21}}=1)\right]\cdot$$

$$\left[P(X^1_{i_{13}}=X^2_{i_{22}}=1)-P(X^1_{i_{13}}=1)P(X^2_{i_{22}}=1)\right]$$

$$+P(X^1_{i_{12}}=1)\left[P(X^1_{i_{11}}=X^2_{i_{22}}=1)-P(X^1_{i_{11}}=1)P(X^2_{i_{22}}=1)\right]\cdot$$
$$\left[P(X^1_{i_{13}}=X^2_{i_{21}}=1)-P(X^1_{i_{13}}=1)P(X^2_{i_{21}}=1)\right]$$
$$+P(X^1_{i_{13}}=1)\left[P(X^1_{i_{11}}=X^2_{i_{21}}=1)-P(X^1_{i_{11}}=1)P(X^2_{i_{21}}=1)\right]\cdot$$
$$\left[P(X^1_{i_{12}}=X^2_{i_{22}}=1)-P(X^1_{i_{12}}=1)P(X^2_{i_{22}}=1)\right]$$
$$+P(X^1_{i_{13}}=1)\left[P(X^1_{i_{11}}=X^2_{i_{22}}=1)-P(X^1_{i_{11}}=1)P(X^2_{i_{22}}=1)\right]$$
$$\left[P(X^1_{i_{12}}=X^2_{i_{21}}=1)-P(X^1_{i_{12}}=1)P(X^2_{i_{21}}=1)\right] .$$

Hence , we have

$$(*)\left|\mu_{[3][2]}(S_n)-\sum_{i_{11}=1}^{n}\sum_{i_{12}=1}^{n}\sum_{i_{13}=1}^{n}\sum_{i_{21}=1}^{n}\sum_{i_{22}=1}^{n}G(X^1_{i_{11}},X^1_{i_{12}},X^1_{i_{13}},X^2_{i_{21}},X^2_{i_{22}})\right|$$

$$=\Bigg|\sum_{\substack{i_{11}=1\\ i_{11}\neq i_{12}\neq i_{13},\, i_{21}\neq i_{22}\\ i_{11}\neq i_{13}}}^{n}\sum_{i_{12}=1}^{n}\sum_{i_{13}=1}^{n}\sum_{i_{21}=1}^{n}\sum_{i_{22}=1}^{n}P(X^1_{i_{11}}=X^1_{i_{12}}=X^1_{i_{13}}=X^2_{i_{21}}=X^2_{i_{22}}=1)$$

$$-\sum_{i_{11}=1}^{n}\sum_{i_{12}=1}^{n}\sum_{i_{13}=1}^{n}\sum_{i_{21}=1}^{n}\sum_{i_{22}=1}^{n}G(X^1_{i_{11}},X^1_{i_{12}},X^1_{i_{13}},X^2_{i_{21}},X^2_{i_{22}})\Bigg|$$

$$=\Bigg|\sum_{\substack{i_{11}=1\\ i_{11}\neq i_{12}\neq i_{13}\neq i_{11},\, i_{21}\neq i_{22}}}^{n}\sum_{i_{12}=1}^{n}\sum_{i_{13}=1}^{n}\sum_{i_{21}=1}^{n}\sum_{i_{22}=1}^{n}P(X^1_{i_{11}}=X^1_{i_{12}}=X^1_{i_{13}}=X^2_{i_{21}}=X^2_{i_{22}}=1)$$

$$-\sum_{\substack{i_{11}=1\\ i_{11}\neq i_{12}\neq i_{13},\ i_{21}\neq i_{22}\\ i_{11}\neq i_{13}}}^{n}\sum_{i_{12}=1}^{n}\sum_{i_{13}=1}^{n}\sum_{i_{21}=1}^{n}\sum_{i_{22}=1}^{n}G(X^1_{i_{11}},X^1_{i_{12}},X^1_{i_{13}},X^2_{i_{21}},X^2_{i_{22}})$$

$$- \sum_{i_{11}=1}^{n} \sum_{i_{12}=1}^{n} \sum_{i_{13}=1}^{n} \sum_{i_{21}=1}^{n} \sum_{\substack{i_{22}=1 \\ i_{1k_1}=i_{1k_2} \text{ or } i_{21}=i_{22} \text{ for some } k_1,k_2}}^{n} G(X^1_{i_{11}}, X^1_{i_{12}}, X^1_{i_{13}}, X^2_{i_{21}}, X^2_{i_{22}}) \Bigg|$$

$$\leq \Bigg| \sum_{\substack{i_{11}=1 \\ i_{11}\neq i_{12}\neq i_{13},\ i_{21}\neq i_{22} \\ i_{11}\neq i_{13}}}^{n} \sum_{i_{12}=1}^{n} \sum_{i_{13}=1}^{n} \sum_{i_{21}=1}^{n} \sum_{i_{22}=1}^{n} \Big\{ P(X^1_{i_{11}} = X^1_{i_{12}} = X^1_{i_{13}} = X^2_{i_{21}} = X^2_{i_{22}} = 1) - G(X^1_{i_{11}}, X^1_{i_{12}}, X^1_{i_{13}}, X^2_{i_{21}}, X^2_{i_{22}}) \Big\} \Bigg|$$

$$+ \Bigg| \sum_{i_{11}=1}^{n} \sum_{i_{12}=1}^{n} \sum_{i_{13}=1}^{n} \sum_{i_{21}=1}^{n} \sum_{\substack{i_{22}=1 \\ i_{1k_1}=i_{1k_2} \text{ or } i_{21}=i_{22} \text{ for some } k_1,k_2}}^{n} G(X^1_{i_{11}}, X^1_{i_{12}}, X^1_{i_{13}}, X^2_{i_{21}}, X^2_{i_{22}}) \Bigg| .$$

Observe that if both terms on the right tend to zero i.e. if

$$(1)\ \lim_{n\to\infty} \Bigg| \sum_{\substack{i_{11}=1 \\ i_{11}\neq i_{12}\neq i_{13},\ i_{21}\neq i_{22} \\ i_{11}\neq i_{13}}}^{n} \sum_{i_{12}=1}^{n} \sum_{i_{13}=1}^{n} \sum_{i_{21}=1}^{n} \sum_{i_{22}=1}^{n} \Big\{ P(X^1_{i_{11}} = X^1_{i_{12}} = X^1_{i_{13}} = X^2_{i_{21}} = X^2_{i_{22}} = 1) - G(X^1_{i_{11}}, X^1_{i_{12}}, X^1_{i_{13}}, X^2_{i_{21}}, X^2_{i_{22}}) \Big\} \Bigg| = 0 ,$$

$$(2)\ \lim_{n\to\infty} \Bigg| \sum_{i_{11}=1}^{n} \sum_{i_{12}=1}^{n} \sum_{i_{13}=1}^{n} \sum_{i_{21}=1}^{n} \sum_{\substack{i_{22}=1 \\ i_{1k_1}=i_{1k_2} \text{ or } i_{21}=i_{22} \text{ for some } k_1,k_2}}^{n} G(X^1_{i_{11}}, X^1_{i_{12}}, X^1_{i_{13}}, X^2_{i_{21}}, X^2_{i_{22}}) \Bigg| = 0 ,$$

then the quantity on the left hand side of (*) will tend to zero. This would imply that $\lim_{n\to\infty} \mu'_{[3][2]}(S_n)$ exists if and only if

$$\lim_{n\to\infty} \sum_{i_{11}=1}^{n} \sum_{i_{12}=1}^{n} \sum_{i_{13}=1}^{n} \sum_{i_{21}=1}^{n} \sum_{i_{22}=1}^{n} G(X^1_{i_{11}}, X^1_{i_{12}}, X^1_{i_{13}}, X^2_{i_{21}}, X^2_{i_{22}}) \text{ exists .}$$

Observe that

$$\sum_{i_{11}=1}^{n}\sum_{i_{12}=1}^{n}\sum_{i_{13}=1}^{n}\sum_{i_{21}=1}^{n}\sum_{i_{22}=1}^{n} G(X_{i_{11}}^1, X_{i_{12}}^1, X_{i_{13}}^1, X_{i_{21}}^2, X_{i_{22}}^2)$$

$$= \left[\sum_{i=1}^{n} P(X_i^1=1)\right]^3 \left[\sum_{i=1}^{n} P(X_i^2=1)\right]^2$$

$$+ 6\left[\sum_{i=1}^{n} P(X_i^1=1)\right]^2 \left[\sum_{i=1}^{n} P(X_i^2=1)\right]\left[\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left\{P(X_{i_1}^1=X_{i_2}^2=1)-P(X_{i_1}^1=1)P(X_{i_2}^2=1)\right\}\right]$$

$$+ 6\left[\sum_{i=1}^{n} P(X_i^1=1)\right]\left[\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left\{P(X_{i_1}^1=X_{i_2}^2=1)-P(X_{i_1}^1=1)P(X_{i_2}^2=1)\right\}\right]^2$$

$$= \left[E(\sum_{i=1}^{n} X_i^1)\right]^3 \left[E(\sum_{i=1}^{n} X_i^2)\right]^2$$

$$+ 6\left[E(\sum_{i=1}^{n} X_i^1)\right]^2 \left[E(\sum_{i=1}^{n} X_i^2)\right]\left[\text{cov}(\sum_{i=1}^{n} X_i^1, \sum_{i=1}^{n} X_i^2)\right]$$

$$+ 6\left[E(\sum_{i=1}^{n} X_i^1)\right]\left[\text{cov}(\sum_{i=1}^{n} X_i^1, \sum_{i=1}^{n} X_i^2)\right]^2 .$$

If we assume that

$$\lim_{n\to\infty} E(\sum_{i=1}^{n} X_i^1) = \lambda_1 ,$$

$$\lim_{n\to\infty} E(\sum_{i=1}^{n} X_i^2) = \lambda_2 ,$$

$$\lim_{n\to\infty} \text{cov}(\sum_{i=1}^{n} X_i^1, \sum_{i=1}^{n} X_i^2) = \lambda_{12} .$$

We would have

$$\lim_{n\to\infty} \sum_{i_{11}=1}^{n} \sum_{i_{12}=1}^{n} \sum_{i_{13}=1}^{n} \sum_{i_{21}=1}^{n} \sum_{i_{22}=1}^{n} G(X^1_{i_{11}}, X^1_{i_{12}}, X^1_{i_{13}}, X^2_{i_{21}}, X^2_{i_{22}})$$

$$= \lambda_1^3 \lambda_2^2 + 6\lambda_1^2 \lambda_2 \lambda_{12} + 6 \lambda_1 \lambda_{12}^2 .$$

This would imply that

$$\lim_{n\to\infty} \mu'_{[3][2]}(S_n) = \lambda_1^3 \lambda_2^2 + 6 \lambda_1^2 \lambda_2 \lambda_{12} + 6 \lambda_1 \lambda_{12}^2 ,$$

which is the factorial moment of a bivariate Poisson of order 3,2 .

From the above consideration we see that under certain regularity conditions the factorial moments of S_n can be shown to converge to the corresponding factorial moments of a certain multivariate Poisson distribution . However , there is a difficulty in stating these regularity conditions . The difficulty lies in the definition of expression like $G(X^1_{i_{11}}, X^1_{i_{12}}, X^1_{i_{13}}, X^2_{i_{21}}, X^2_{i_{22}})$ used above . To overcome this difficulty we shall introduce the following notations .

For any two distinct random variables X^1 , X^2 define

(3.1.2) $$b^1(X^1) = P(X^1=1) ,$$

$$b^1(X^2) = P(X^2=1) ,$$

$$b^2(X^1,X^2) = P(X^1=X^2=1) - P(X^1=1)P(X^2=1) .$$

Let (X^1_i, X^2_i) $i=1,\ldots,n$ be any 2-dimensional random vectors .

Define

(3.1.3) $$A^j = \left\{ X^j_i \,/\, i=1,\ldots,n \right\} , \quad j=1,2 ,$$

(3.1.4) $$C^j_r = \left\{ c^j \,/\, c^j : \{1,\ldots,r\} \longrightarrow A^j \right\} , \quad r=0,1,\ldots ,$$

(3.1.5) $D_r^j = \left\{ c^j \,/\, c^j \in C_r^j \text{ and } c^j(k_1) \neq c^j(k_2) \text{ , if } k_1 \neq k_2 \right\}$,

(3.1.6) $F_r = \left\{ f_r \,/\, f_r : \{1,\dots,r\} \to \{1,\dots,r\} \text{ , } f_r \text{ is 1-1} \right\}$.

For any non-negative integers r_1, r_2 and any $c^1 \in C^1_{r_1}$, $c^2 \in C^2_{r_2}$ define

(3.1.7) $b(c^1,c^2) = P(c^1(1)= \dots =c^1(r_1)=c^2(1)= \dots =c^2(r_2)=1)$,

(3.1.8) $G(c^1,c^2)$

$$= \sum_{f_{r_1} \in F_{r_1}} \sum_{f_{r_2} \in F_{r_2}} \sum_{p=0}^{\min\{r_1,r_2\}} \frac{\prod_{k=1}^{p} b^2(c^1 f_{r_1}(k), c^2 f_{r_2}(k)) \prod_{k=p+1}^{r_1} b^1(c^1 f_{r_1}(k)) \prod_{k=p+1}^{r_2} b^1(c^2 f_{r_2}(k))}{(r_1-p)!(r_2-p)!p!}$$

3.2 Main Theorem

Theorem 3.2.1 Let $X_i = (X_i^1, X_i^2)$, $i=1,\dots,n$ be 2-dimensional random vectors where the components can take only the values 0 or 1. Let $S_n = X_1 + \dots + X_n$. If the probabilities $b^1(X_i^1)$, $b^1(X_i^2)$, $b^2(X_{i_1}^1, X_{i_2}^2)$, $b(c^1,c^2)$ satisfy the following conditions

(3.2.1) $\lim\limits_{n\to\infty} \max\limits_{1\leq i\leq n} b^1(X_i^1) = 0$,

(3.2.2) $\lim\limits_{n\to\infty} \max\limits_{1\leq i\leq n} b^1(X_i^2) = 0$,

(3.2.3) $\lim\limits_{n\to\infty} \sum_{i=1}^{n} b^1(X_i^1) = \lambda_1$,

$$(3.2.4)\quad \lim_{n\to\infty} \sum_{i=1}^{n} b^1(X_i^2) = \lambda_2 ,$$

$$(3.2.5)\quad \lim_{n\to\infty} n \max_{\substack{1\le i_1\le n \\ 1\le i_2\le n}} \left| b^2(X_{i_1}^1, X_{i_2}^2) \right| = 0 ,$$

$$(3.2.6)\quad \lim_{n\to\infty} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \left| b^2(X_{i_1}^1, X_{i_2}^2) \right| < M ,$$

$$(3.2.7)\quad \lim_{n\to\infty} \sum_{i_1=1}^{n} \sum_{i_2=1}^{n} b^2(X_{i_1}^1, X_{i_2}^2) = \lambda_{12} ,$$

where λ_1 , λ_2 , λ_{12} , M are some non-negative real numbers , and if for any $n > 0$ and any non-negative integers r_1, r_2 such that $r_1 + r_2 \geq 2$ there exist subset $I_{r_1,r_2}(n)$ of $D_{r_1}^1 \times D_{r_2}^2$ such that

$$(3.2.8)\quad \lim_{n\to\infty} \sum_{(c^1,c^2)\in I_{r_1,r_2}(n)} b(c^1,c^2) = 0 ,$$

$$(3.2.9)\quad \lim_{n\to\infty} \sum_{(c^1,c^2)\in I_{r_1,r_2}(n)} G(c^1,c^2) = 0 ,$$

and

$$(3.2.10)\quad \lim_{n\to\infty} \frac{b(c^1,c^2)}{G(c^1,c^2)} = 1$$

uniformly on $J_{r_1,r_2}(n) = D_{r_1}^1 \times D_{r_2}^2 - I_{r_1,r_2}(n)$, then as n tends to infinity , S_n tends to a bivariate Poisson distribution $X = (X^1, X^2)$ with marginal means λ_1 , λ_2 and $\text{cov}(X^1, X^2) = \lambda_{12}$.

First we prove a lemma .

Lemma 3.2.1 If X_i , i=1,...,n satisfy (3.2.1) , (3.2.2) , (3.2.3) , (3.2.4) , (3.2.5) , (3.2.6) of the above theorem , then for any non-negative integers r_1, r_2 such that $r_1 \geqq 2$ or $r_2 \geqq 2$, we have

$$(3.2.11)\quad \lim_{n\to\infty} \sum_{(c^1,c^2)\notin D^1_{r_1}\times D^2_{r_2}} G(c^1,c^2) = 0 .$$

Proof Observe that for any non-negative integers r_1, r_2

$$(3.2.12)\quad \left|\sum_{(c^1,c^2)\notin D^1_{r_1}\times D^2_{r_2}} G(c^1,c^2)\right| \leqq \left|\sum_{\substack{(c^1,c^2)\notin D^1_{r_1}\times D^2_{r_2}\\ c^1\notin D^1_{r_1}}} G(c^1,c^2)\right| + \left|\sum_{\substack{(c^1,c^2)\notin D^1_{r_1}\times D^2_{r_2}\\ c^2\notin D^2_{r_2}}} G(c^1,c^2)\right| ,$$

and

$$(3.2.13)\quad \left|\sum_{\substack{(c^1,c^2)\notin D^1_{r_1}\times D^2_{r_2}\\ c^1\notin D^1_{r_1}}} G(c^1,c^2)\right|$$

$$\leqq \sum_{f_{r_1}\in F_{r_1}} \sum_{f_{r_2}\in F_{r_2}} \sum_{p=0}^{\min\{r_1,r_2\}} \left| \sum_{\substack{(c^1,c^2)\notin D^1_{r_1}\times D^2_{r_2}\\ c^1\notin D^1_{r_1}}} \left[\prod_{k=1}^{p} b^2(c^1 f_{r_1}(k), c^2 f_{r_2}(k)) \cdot \prod_{k=p+1}^{r_1} b^1(c^1 f_{r_1}(k)) \prod_{k=p+1}^{r_2} b^1(c^2 f_{r_2}(k)) \right] \Big/ (r_1-p)!(r_2-p)!p! \right| .$$

For any $f_{r_1}\in F_{r_1}$, $f_{r_2}\in F_{r_2}$ and any $p=0,1,\ldots,\min\{r_1,r_2\}$,

$$\left| \sum_{\substack{(c^1,c^2)\notin D^1_{r_1}\times D^2_{r_2}\\ c^1\notin D^1_{r_1}}} \prod_{k=1}^{p} b^2(c^1 f_{r_1}(k), c^2 f_{r_2}(k)) \prod_{k=p+1}^{r_1} b^1(c^1 f_{r_1}(k)) \prod_{k=p+1}^{r_2} b^1(c^2 f_{r_2}(k)) \right|$$

$$= \left| \sum_{\substack{i_{11}=1 \\ i_{1k_1}=i_{1k_2} \text{ for some } k_1,k_2}}^{n} \cdots \sum_{i_{1r_1}=1}^{n} \sum_{i_{21}=1}^{n} \cdots \sum_{i_{2r_2}=1}^{n} \prod_{k=1}^{p} b^2(X^1_{i_{1k}}, X^2_{i_{2k}}) \prod_{k=p+1}^{r_1} b^1(X^1_{i_{1k}}) \prod_{k=p+1}^{r_2} b^1(X^2_{i_{2k}}) \right|$$

$$\leq \sum_{\substack{i_{11}=1 \\ i_{1k_1}=i_{1k_2} \text{ for some } k_1,k_2}}^{n} \cdots \sum_{i_{1r_1}=1}^{n} \sum_{i_{21}=1}^{n} \cdots \sum_{i_{2r_2}=1}^{n} \prod_{k=1}^{p} \left| b^2(X^1_{i_{1k}}, X^2_{i_{2k}}) \right| \left| \prod_{k=p+1}^{r_1} b^1(X^1_{i_{1k}}) \prod_{k=p+1}^{r_2} b^1(X^2_{i_{2k}}) \right|$$

$$\leq \sum_{\substack{i_{11}=1 \\ i_{1k_1}=i_{1k_2} \text{ for some } k_1,k_2 \leq p}}^{n} \cdots \sum_{i_{1r_1}=1}^{n} \sum_{i_{21}=1}^{n} \cdots \sum_{i_{2r_2}=1}^{n} \prod_{k=1}^{p} \left| b^2(X^1_{i_{1k}}, X^2_{i_{2k}}) \right| \left| \prod_{k=p+1}^{r_1} b^1(X^1_{i_{1k}}) \right| \prod_{k=p+1}^{r_2} b^1(X^2_{i_{2k}})$$

$$+ \sum_{\substack{i_{11}=1 \\ i_{1k_1}=i_{1k_2} \text{ for some } k_1,k_2 \text{ with } k_1 \leq p,\, p<k_2 \leq r_1}}^{n} \cdots \sum_{i_{1r_1}=1}^{n} \sum_{i_{21}=1}^{n} \cdots \sum_{i_{2r_2}=1}^{n} \prod_{k=1}^{p} \left| b^2(X^1_{i_{1k}}, X^2_{i_{2k}}) \right| \prod_{k=p+1}^{r_1} b^1(X^1_{i_{1k}}) \prod_{k=p+1}^{r_2} b^1(X^2_{i_{2k}})$$

$$+ \sum_{\substack{i_{11}=1 \\ i_{1k_1}=i_{1k_2} \text{ for some } k_1,k_2 \text{ with } p<k_1,k_2 \leq r_1}}^{n} \cdots \sum_{i_{1r_1}=1}^{n} \sum_{i_{21}=1}^{n} \cdots \sum_{i_{2r_2}=1}^{n} \prod_{k=1}^{p} \left| b^2(X^1_{i_{1k}}, X^2_{i_{2k}}) \right| \prod_{k=p+1}^{r_1} b^1(X^1_{i_{1k}}) \prod_{k=p+1}^{r_2} b^1(X^2_{i_{2k}})$$

$$= \left\{ \left[\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \sum_{i_3=1}^{n} \left| b^2(X^1_{i_1}, X^2_{i_2}) \right| \left| b^2(X^1_{i_1}, X^2_{i_3}) \right| \right] \left[\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \left| b^2(X^1_{i_1}, X^2_{i_2}) \right| \right]^{p-2} \left[\sum_{i=1}^{n} b^1(X^1_i) \right]^{r_1-p} \left[\sum_{i=1}^{n} b^1(X^2_i) \right]^{r_2-p} \right\}$$

$$+ \left\{ \left[\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} b^1(X^1_{i_1}) \left| b^2(X^1_{i_1}, X^2_{i_2}) \right| \right] \left[\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} \left| b^2(X^1_{i_1}, X^2_{i_2}) \right| \right]^{p-1} \left[\sum_{i=1}^{n} b^1(X^1_i) \right]^{r_1-p-1} \left[\sum_{i=1}^{n} b^1(X^2_i) \right]^{r_2-p} \right\}$$

$$+\left\{\left[\left\{\sum_{i=1}^{n} b^1(X_i^1)\right\}^2\right]\left[\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left|b^2(X_{i_1}^1,X_{i_2}^2)\right|\right]^p\left[\sum_{i=1}^{n} b^1(X_i^1)\right]^{r_1-p-2}\right.$$

$$\left.\left[\sum_{i=1}^{n} b^1(X_i^2)\right]^{r_2-p}\right\}$$

$$\leq\left\{\left[n\max_{\substack{1\leq i_1\leq n\\1\leq i_2\leq n}}\left|b^2(X_{i_1}^1,X_{i_2}^2)\right|\right]\left[\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left|b^2(X_{i_1}^1,X_{i_2}^2)\right|\right]^{p-1}\cdot\right.$$

$$\left.\left[\sum_{i=1}^{n} b^1(X_i^1)\right]^{r_1-p}\left[\sum_{i=1}^{n} b^1(X_i^2)\right]^{r_2-p}\right\}$$

$$+\left\{2\left[\max_{1\leq i\leq n} b^1(X_i^1)\right]\left[\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left|b^2(X_{i_1}^1,X_{i_2}^2)\right|\right]^p\right.$$

$$\left.\left[\sum_{i=1}^{n} b^1(X_i^1)\right]^{r_1-p-1}\left[\sum_{i=1}^{n} b^1(X_i^2)\right]^{r_2-p}\right\}.$$

By using (3.2.1),(3.2.3),(3.2.4),(3.2.5),(3.2.6) it follows that

$$(3.2.14)\quad \lim_{n\to\infty}\left|\sum_{\substack{(c^1,c^2)\notin D_{r_1}^1\times D_{r_2}^2\\ c^1\notin D_{r_1}^1}}\prod_{k=1}^{p} b^2(c^1 f_{r_1}(k),c^2 f_{r_2}(k))\prod_{k=p+1}^{r_1} b^1(c^1 f_{r_1}(k))\cdot\prod_{k=p+1}^{r_2} b^1(c^2 f_{r_2}(k))\right| = 0 .$$

By using (3.2.14) it follows from (3.2.13) that

$$(3.2.15)\quad \lim_{n\to\infty}\left|\sum_{\substack{(c^1,c^2)\notin D_{r_1}^1\times D_{r_2}^2\\ c^1\notin D_{r_1}^1}} G(c^1,c^2)\right| = 0 .$$

By using the same argument it can be shown that

$$(3.2.16)\quad \lim_{n\to\infty} \left| \sum_{\substack{(c^1,c^2)\notin D^1_{r_1}\times D^2_{r_2} \\ c^2\notin D^2_{r_2}}} G(c^1,c^2) \right| = 0 \ .$$

By virtue of (3.2.15) and (3.2.16) , (3.2.11) follows from (3.2.12) .

We now prove the theorem .

<u>Proof</u> (Theorem 3.2.1)

Observe that

$$(3.2.17)\quad \mu'_{[0][0]}(S_n) = 1 \ .$$

Let r_1, r_2 be non-negative integers such that $r_1+r_2 \geq 1$.

By Theorem 2.4.1, we have

$$(3.2.18)\quad \mu'_{[r_1][r_2]}(S_n) = \sum_{(c^1,c^2)\in D^1_{r_1}\times D^2_{r_2}} b(c^1,c^2) \ .$$

When $r_1+r_2 = 1$, we have $r_1=1$, $r_2=0$ or $r_1=0$, $r_2=1$.

If $r_1=1$, $r_2=0$, then by (3.2.18) , we have

$$\mu'_{[1][0]}(S_n) = \sum_{i=1}^{n} b^1(X^1_i) \ .$$

Hence by using (3.2.3) , we have

$$(3.2.19)\quad \lim_{n\to\infty} \mu'_{[1][0]}(S_n) = \lambda_1 \ .$$

Similary , if $r_1=0$, $r_2=1$, we have

$$(3.2.20)\quad \lim_{n\to\infty} \mu'_{[0][1]}(S_n) = \lambda_2 \ .$$

Now we assume that $r_1+r_2 \geq 2$; We break up the right hand side of

(3.2.18) into two sums

$$(3.2.21)\quad \sum_{(c^1,c^2)\in D^1_{r_1}\times D^2_{r_2}} b(c^1,c^2) = \sum_{(c^1,c^2)\in I^{(n)}_{r_1,r_2}} b(c^1,c^2) + \sum_{(c^1,c^2)\in J^{(n)}_{r_1,r_2}} b(c^1,c^2) .$$

From (3.2.10) it follows that for any $\epsilon > 0$, we can find N such that for all $n \geqq N$ and all $(c^1,c^2)\in J^{(n)}_{r_1,r_2}$

$$\left| \frac{b(c^1,c^2)}{G(c^1,c^2)} - 1 \right| < \epsilon .$$

So for all $(c^1,c^2)\in J^{(n)}_{r_1,r_2}$, we have

$$(1-\epsilon)G(c^1,c^2) < b(c^1,c^2) < (1+\epsilon)G(c^1,c^2) .$$

By summing up over all $(c^1,c^2)\in J^{(n)}_{r_1,r_2}$, we have

$$(1-\epsilon)\sum_{(c^1,c^2)\in J^{(n)}_{r_1,r_2}} G(c^1,c^2) < \sum_{(c^1,c^2)\in J^{(n)}_{r_1,r_2}} b(c^1,c^2) < (1+\epsilon)\sum_{(c^1,c^2)\in J^{(n)}_{r_1,r_2}} G(c^1,c^2) .$$

Hence

$$(3.2.22)\quad \left| \sum_{(c^1,c^2)\in J^{(n)}_{r_1,r_2}} b(c^1,c^2) - \sum_{(c^1,c^2)\in J^{(n)}_{r_1,r_2}} G(c^1,c^2) \right| < \epsilon \sum_{(c^1,c^2)\in J^{(n)}_{r_1,r_2}} G(c^1,c^2) .$$

Observe that for any $f_{r_1}\in F_{r_1}$, $f_{r_2}\in F_{r_2}$ and any $p=0,1,\dots,\min\{r_1,r_2\}$

$$(3.2.23)\quad \left[\sum_{i_1=1}^{n}\sum_{i_2=1}^{n} b^2(X^1_{i_1},X^2_{i_2})\right]^p \left[\sum_{i=1}^{n} b^1(X^1_i)\right]^{r_1-p} \left[\sum_{i=1}^{n} b^1(X^2_i)\right]^{r_2-p}$$

$$= \sum_{i_{11}=1}^{n}\cdots\sum_{i_{1r_1}=1}^{n}\sum_{i_{21}=1}^{n}\cdots\sum_{i_{2r_2}=1}^{n} \prod_{k=1}^{p} b^2(X^1_{i_{1k}},X^2_{i_{2k}}) \prod_{k=p+1}^{r_1} b^1(X^1_{i_{1k}}) \prod_{k=p+1}^{r_2} b^1(X^2_{i_{2k}})$$

$$= \sum_{(c^1,c^2)\in C^1_{r_1}\times C^2_{r_2}} \prod_{k=1}^{p} b^2(c^1 f_{r_1}(k), c^2 f_{r_2}(k)) \prod_{k=p+1}^{r_1} b^1(c^1 f_{r_1}(k)) \prod_{k=p+1}^{r_2} b^1(c^2 f_{r_2}(k)) .$$

$$(3.2.24) \quad \sum_{(c^1,c^2)\in C^1_{r_1}\times C^2_{r_2}} G(c^1,c^2)$$

$$= \sum_{(c^1,c^2)\in C^1_{r_1}\times C^2_{r_2}} \sum_{f\in F_{r_1}} \sum_{f\in F_{r_2}} \sum_{p=0}^{\min\{r_1,r_2\}} \Big[\prod_{k=1}^{p} b^2(c^1 f_{r_1}(k), c^2 f_{r_2}(k))$$

$$\prod_{k=p+1}^{r_1} b^1(c^1 f_{r_1}(k)) \prod_{k=p+1}^{r_2} b^1(c^2 f_{r_2}(k))\Big] \Big/ (r_1-p)!(r_2-p)!p!$$

$$= \sum_{p=0}^{\min\{r_1,r_2\}} \sum_{f\in F_{r_1}} \sum_{f\in F_{r_2}} \sum_{(c^1,c^2)\in C^1_{r_1}\times C^2_{r_2}} \Big[\prod_{k=1}^{p} b^2(c^1 f_{r_1}(k), c^2 f_{r_2}(k))$$

$$\prod_{k=p+1}^{r_1} b^1(c^1 f_{r_1}(k)) \prod_{k=p+1}^{r_2} b^1(c^2 f_{r_2}(k))\Big] \Big/ (r_1-p)!(r_2-p)!p!$$

$$= \sum_{p=0}^{\min\{r_1,r_2\}} \frac{r_1!r_2!}{(r_1-p)!(r_2-p)!p!} \Big[\sum_{i_1=1}^{n} \sum_{i_2=1}^{n} b^2(X^1_{i_1}, X^2_{i_2})\Big]^p \Big[\sum_{i=1}^{n} b^1(X^1_i)\Big]^{r_1-p} \Big[\sum_{i=1}^{n} b^1(X^2_i)\Big]^{r_2-p}$$

the last equality follows from an application of (3.2.23) .

By using (3.2.3),(3.2.4),(3.2.7) it follows from (3.2.24) that

$$(3.2.25) \quad \lim_{n\to\infty} \sum_{(c^1,c^2)\in C^1_{r_1}\times C^2_{r_2}} G(c^1,c^2) = \sum_{p=0}^{\min\{r_1,r_2\}} \frac{r_1!r_2!}{(r_1-p)!(r_2-p)!p!} \lambda_1^{r_1-p} \lambda_2^{r_2-p} \lambda_{12}^{p} .$$

On the other hand , we have

$$(3.2.26) \quad \sum_{(c^1,c^2)\in C^1_{r_1}\times C^2_{r_2}} G(c^1,c^2) = \sum_{(c^1,c^2)\in D^1_{r_1}\times D^2_{r_2}} G(c^1,c^2) + \sum_{(c^1,c^2)\notin D^1_{r_1}\times D^2_{r_2}} G(c^1,c^2)$$

$$= \sum_{(c^1,c^2)\in I_{r_1,r_2}(n)} G(c^1,c^2) + \sum_{(c^1,c^2)\in J_{r_1,r_2}(n)} G(c^1,c^2) + \sum_{(c^1,c^2)\notin D^1_{r_2}\times D^2_{r_2}} G(c^1,c^2) .$$

Observe that if $r_1=1$, $r_2=1$, then $C^1_{r_1}\times C^2_{r_2} = D^1_{r_1}\times D^2_{r_2}$.

Hence by using (3.2.9) it follows from (3.2.26) that

$$(3.2.27) \quad \lim_{n\to\infty} \sum_{(c^1,c^2)\in C^1_{r_1}\times C^2_{r_2}} G(c^1,c^2) = \lim_{n\to\infty} \sum_{(c^1,c^2)\in J_{r_1,r_2}(n)} G(c^1,c^2) .$$

If $r_1\geq 2$ or $r_2\geq 2$ by using (3.2.9) and Lemma 3.2.1 it follows from (3.2.26) that

$$(3.2.28) \quad \lim_{n\to\infty} \sum_{(c^1,c^2)\in C^1_{r_1}\times C^2_{r_2}} G(c^1,c^2) = \lim_{n\to\infty} \sum_{(c^1,c^2)\in J_{r_1,r_2}(n)} G(c^1,c^2) .$$

Hence if $r_1+r_2\geq 2$, then we have

$$(3.2.29) \quad \lim_{n\to\infty} \sum_{(c^1,c^2)\in C^1_{r_1}\times C^2_{r_2}} G(c^1,c^2) = \lim_{n\to\infty} \sum_{(c^1,c^2)\in J_{r_1,r_2}(n)} G(c^1,c^2) .$$

By comparing (3.2.25) and (3.2.29) , we have

$$(3.2.30) \quad \lim_{n\to\infty} \sum_{(c^1,c^2)\in J_{r_1,r_2}(n)} G(c^1,c^2) = \sum_{p=0}^{\min\{r_1,r_2\}} \frac{r_1!r_2!}{(r_1-p)!(r_2-p)!p!} \lambda_1^{r_1-p} \lambda_2^{r_2-p} \lambda_{12}^{p} .$$

By using (3.2.30) it follows from (3.2.22) that

$$\lim_{n\to\infty} \left| \sum_{(c^1,c^2)\in J_{r_1,r_2}(n)} b(c^1,c^2) - \sum_{(c^1,c^2)\in J_{r_1,r_2}(n)} G(c^1,c^2) \right|$$

$$< \epsilon \sum_{p=0}^{\min\{r_1,r_2\}} \frac{r_1!r_2!}{(r_1-p)!(r_2-p)!p!} \lambda_1^{r_1-p} \lambda_2^{r_2-p} \lambda_{12}^{p} .$$

We can choose ϵ arbitrary small so that

$$\lim_{n\to\infty}\left|\sum_{(c^1,c^2)\in J^{(n)}_{r_1,r_2}} b(c^1,c^2) - \sum_{(c^1,c^2)\in J^{(n)}_{r_1,r_2}} G(c^1,c^2)\right| = 0 .$$

Hence

$$(3.2.31)\quad \lim_{n\to\infty}\sum_{(c^1,c^2)\in J^{(n)}_{r_1,r_2}} b(c^1,c^2) = \sum_{p=0}^{\min\{r_1,r_2\}} \frac{r_1!r_2!}{(r_1-p)!(r_2-p)!p!}\lambda_1^{r_1-p}\lambda_2^{r_2-p}\lambda_{12}^{p}.$$

By using (3.2.8),(3.2.31) it follows from (3.2.18),(3.2.21) that

$$(3.2.32)\quad \lim_{n\to\infty}\mu'_{[r_1][r_2]}(S_n) = \sum_{p=0}^{\min\{r_1,r_2\}} \frac{r_1!r_2!}{(r_1-p)!(r_2-p)!p!}\lambda_1^{r_1-p}\lambda_2^{r_2-p}\lambda_{12}^{p}.$$

By (3.2.17),(3.2.19),(3.2.20) and (3.2.32) it follows that for any non-negative integers r_1,r_2

$$\lim_{n\to\infty}\mu'_{[r_1][r_2]}(S_n) = \sum_{p=0}^{\min\{r_1,r_2\}} \frac{r_1!r_2!}{(r_1-p)!(r_2-p)!p!}\lambda_1^{r_1-p}\lambda_2^{r_2-p}\lambda_{12}^{p} .$$

By Theorem 2.3.3 , we have

$$\lim_{n\to\infty}\mu'_{[r_1][r_2]}(S_n) = \mu'_{[r_1][r_2]}(X) .$$

From this , it also follows that there exists $K(r_1,r_2)$ such that

$$\left|\mu'_{[r_1][r_2]}(S_n)\right| \leqq K(r_1,r_2)$$

for all n .

Hence , by Theorem 2.3.2 , S_n tends to X as n tends to infinity .

3.4 Application

In this section we give an example where Theorem 3.2.1 is applicable . Consider three urns A_1, A_2, A_3. For i=1,2,3 , let A_i contain R_i red balls and B_i black balls. Let $N_i = R_i + B_i$. We draw 2n balls , one at a time without replacement , from these urns as follows.

First draw a ball from A_1 at random . If the first ball drawn is red , we draw the second ball from A_2 , if the first ball drawn is black , we draw the second ball from A_3 .

After $2k^{th}$ ball is drawn , we draw the $(2k+1)^{st}$ ball from A_1 at random . If it is red , we draw the $(2k+2)^{nd}$ ball from A_2 , if it is black , we draw the $(2k+2)^{nd}$ ball from A_3 . The process stops when the $2n^{th}$ ball have been drawn . Define

$$X_i^j = \begin{cases} 0 \text{ , if } [2(i-1)+j]^{th} \text{ ball drawn is black .} \\ \\ 1 \text{ , if } [2(i-1)+j]^{th} \text{ ball drawn is red .} \end{cases}$$

Let $S_n = (X_1^1, X_1^2) + (X_2^1, X_2^2) + \ldots + (X_n^1, X_n^2)$. By straight forward computation , we have

$$P(X_i^1=1) = \frac{R_1}{N_1} ,$$

$$P(X_i^2=1) = \frac{R_1}{N_1}\frac{R_2}{N_2} + \frac{B_1}{N_1}\frac{R_3}{N_3} ,$$

$$P(X_{i_1}^1 = X_{i_2}^2 = 1) = \begin{cases} \frac{R_1}{N_1}\frac{R_2}{N_2} , & \text{if } i_1 = i_2 \text{ .} \\ \\ \frac{R_1}{N_1}\frac{R_1-1}{N_1-1}\frac{R_2}{N_2} + \frac{R_1}{N_1}\frac{B_1}{N_1-1}\frac{R_3}{N_3}, & \text{if } i_1 \neq i_2 \text{ .} \end{cases}$$

Hence

$$P(X^1_{i_1} = X^2_{i_2} = 1) - P(X^1_{i_1} = 1)P(X^2_{i_2} = 1)$$

$$= \begin{cases} \frac{R_1}{N_1}\frac{B_1}{N_1}\left(\frac{R_2}{N_2} - \frac{R_3}{N_3}\right) & , \text{ if } i_1 = i_2 \text{ .} \\ \frac{R_1}{N_1}\frac{B_1}{N_1(N_1-1)}\left(\frac{R_3}{N_3} - \frac{R_2}{N_2}\right) & , \text{ if } i_1 \neq i_2 \text{ .} \end{cases}$$

For any $n > 0$

(1) $$\max_{1 \le i \le n} b^1(X^1_i) = \frac{R_1}{N_1} \text{ ,}$$

(2) $$\max_{1 \le i \le n} b^1(X^2_i) = \frac{R_1}{N_1}\frac{R_2}{N_2} + \frac{B_1}{N_1}\frac{R_3}{N_3} \text{ ,}$$

(3) $$\sum_{i=1}^{n} b^1(X^1_i) = n\frac{R_1}{N_1} \text{ ,}$$

(4) $$\sum_{i=1}^{n} b^1(X^2_i) = n\left[\frac{R_1}{N_1}\frac{R_2}{N_2} + \frac{B_1}{N_1}\frac{R_3}{N_3}\right] \text{ ,}$$

(5) $$n \max_{\substack{1 \le i_1 \le n \\ 1 \le i_2 \le n}} \left|b^2(X^1_{i_1}, X^2_{i_2})\right| = n\left[\frac{R_1}{N_1}\frac{B_1}{N_1}\left|\frac{R_2}{N_2} - \frac{R_3}{N_3}\right|\right] \text{ ,}$$

(6) $$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\left|b^2(X^1_{i_1}, X^2_{i_2})\right|$$

$$= n\left[\frac{R_1}{N_1}\frac{B_1}{N_1}\left|\frac{R_2}{N_2} - \frac{R_3}{N_3}\right|\right] + (n^2-n)\left[\frac{R_1}{N_1}\frac{B_1}{N_1(N_1-1)}\left|\frac{R_2}{N_2} - \frac{R_3}{N_3}\right|\right]$$

(7) $$\sum_{i_1=1}^{n}\sum_{i_2=1}^{n} b^2(X^1_{i_1}, X^2_{i_2})$$

$$= n\left\{\frac{R_1}{N_1}\frac{B_1}{N_1}\left[\frac{R_2}{N_2} - \frac{R_3}{N_3}\right]\right\} + (n^2-n)\left\{\frac{R_1}{N_1}\frac{B_1}{N_1(N_1-1)}\left[\frac{R_3}{N_3} - \frac{R_2}{N_2}\right]\right\} \text{ .}$$

For any $n > 0$ and any non-negative integers r_1, r_2 such that $r_1 + r_2 \geq 2$, we choose

$$I_{r_1,r_2}(n) = \left\{ (c^1,c^2) \,/\, (c^1,c^2) \in D^1_{r_1} \times D^2_{r_2} \text{ and } c^1(k_1) = c^2(k_2), \text{ for some } k_1,k_2 \right\}.$$

Hence

$$(8) \quad \sum_{(c^1,c^2)\in I(n)_{r_1,r_2}} b\,(c^1,c^2) = \sum_{i=1}^{\min\{r_1,r_2\}} \Bigg[\binom{n}{i}\binom{n-i}{r_1-i}\binom{n-r_1}{r_2-i}$$

$$\left(\frac{R_1}{N_1}\frac{R_2}{N_2}\cdots\frac{R_1-i+1}{N_1-i+1}\,\frac{R_2-i+1}{N_2-i+1}\right)\left(\frac{R_1-i}{N_1-i}\cdots\frac{R_1-r_1+1}{N_1-r_1+1}\right)$$

$$\left(\frac{R_1-r_1}{N_1-r_1}\,\frac{R_2-i}{N_2-i}\cdots\frac{R_1-r_1-r_2+i+1}{N_1-r_1-r_2+i+1}\,\frac{R_2-r_2+1}{N_2-r_2+1}\right.$$

$$+ \cdots$$

$$\left. + \frac{B_1}{N_1-r_1}\,\frac{R_3}{N_3}\cdots\frac{B_1-r_2+i+1}{N_1-r_1-r_2+i+1}\,\frac{R_3-r_2+i+1}{N_3-r_2+i+1}\right)\Bigg].$$

$$(9) \quad \sum_{(c^1,c^2)\in I(n)_{r_1,r_2}} G(c^1,c^2) \leq \sum_{i=1}^{\min\{r_1,r_2\}} \sum_{p=0}^{\min\{r_1,r_2\}} \Bigg\{ \binom{n}{i}\binom{n-i}{r_1-i}\binom{n-r_1}{r_2-i}$$

$$\frac{r_1!\,r_2!}{(r_1-p)!\,(r_2-p)!\,p!}\left(\frac{R_1}{N_1}\right)^{r_1-p}$$

$$\left(\frac{R_1}{N_1}\frac{R_2}{N_2} + \frac{B_1}{N_1}\frac{R_3}{N_3}\right)^{r_2-p}\left[\frac{R_1}{N_1}\frac{B_1}{N_1}\left|\frac{R_2}{N_2} - \frac{R_3}{N_3}\right|\right]^{p}\Bigg\}.$$

For any $(c^1,c^2) \in J_{r_1,r_2}(n)$

$$(10)\quad \frac{b(c^1,c^2)}{G(c^1,c^2)} = \Bigg[\left(\frac{R_1}{N_1}\frac{R_1-1}{N_1-1}\cdots\frac{R_1-r_1+1}{N_1-r_1+1}\right)$$

$$\Bigg(\frac{R_1-r_1}{N_1-r_1}\,\frac{R_2}{N_2}\cdots\frac{R_1-r_1-r_2+1}{N_1-r_1-r_2+1}\,\frac{R_2-r_2+1}{N_2-r_2+1}$$

$$+\frac{R_1-r_1}{N_1-r_1}\,\frac{R_2}{N_2}\cdots\frac{R_1-r_1-r_2+2}{N_1-r_1-r_2+2}\,\frac{R_2-r_2+2}{N_2-r_2+2}\,\frac{B_1}{N_1-r_1-r_2+1}\,\frac{R_3}{N_3}$$

$$+\cdots$$

$$+\frac{B_1}{N_1-r_1}\,\frac{R_3}{N_3}\cdots\frac{B_1-r_2+1}{N_1-r_1-r_2+1}\,\frac{R_3-r_2+1}{N_3-r_2+1}\Bigg)\Bigg] \Bigg/$$

$$\Bigg[\sum_{p=0}^{\min\{r_1,r_2\}}\Bigg\{\left(\frac{R_1}{N_1}\right)^{r_1-p}\left(\frac{R_1}{N_1}\frac{R_2}{N_2}+\frac{B_1}{N_1}\frac{R_3}{N_3}\right)^{r_2-p}$$

$$\left[\left(\frac{R_1}{N_1}\,\frac{B_1}{N_1(N_1-1)}\right)\left(\frac{R_3}{N_3}-\frac{R_2}{N_2}\right)\right]^{p}\Bigg\}\Bigg] .$$

The above quantities given in (1) to (10) are respectively those used in (3.2.1) to (3.2.10) of Theorem 3.2.1. Under appropriate assumptions, (1) to (10) imply (3.2.1) to (3.2.10). For example, if we assume that N_i, R_i, B_i depend on n in such away that

(i) $\lim_{n\to\infty} R_i = +\infty$, $i=1,2,3$,

(ii) $\lim_{n\to\infty} \frac{R_i}{N_i} = 0$, $i=1,2,3$,

(iii) $\lim_{n\to\infty} n\frac{R_i}{N_i} = a_i$, $i=1,2,3$,

then we also have

(iv) $\lim_{n\to\infty} \frac{B_i}{N_i} = 1$, $i=1,2,3$.

By using (i) to (iv), it can be seen that (1) to (10) imply respectively, (3.2.1) to (3.2.10). Here the values of λ_1, λ_2 and λ_{12} are a_1, a_3 and 0 respectively. Hence by Theorem 3.2.1, S_n tends to a bivariate Poisson $X = (X^1, X^2)$ with $E(X^1) = a_1$, $E(X^2) = a_3$, and $\mathrm{cov}(X^1, X^2) = 0$.

3.5 Remarks on Extension to Higher Dimensions

It appears that Theorem 3.2.1 can be extended to the case where X_i, $i=1,\ldots,n$ are m-dimensional random vectors. However, the author is not yet able to find a systematic way of expressing the formula for the quantities similar to $G(X^1_{i_{11}},\ldots,X^1_{i_{1r_1}},X^2_{i_{21}},\ldots,X^2_{i_{2r_2}})$. For the case $m=3$, $r_1=2$, $r_2=2$, $r_3=1$ such a quantity is given by

$$
\begin{aligned}
G(X^1_{i_{11}},X^1_{i_{12}},X^2_{i_{21}},X^2_{i_{22}},X^3_{i_{31}}) \\
&= b^1(X^1_{i_{11}})b^1(X^1_{i_{12}})b^1(X^2_{i_{21}})b^1(X^2_{i_{22}})b^1(X^3_{i_{31}}) \\
&+ b^1(X^1_{i_{11}})b^1(X^1_{i_{12}})b^1(X^2_{i_{21}})b^2(X^2_{i_{22}},X^3_{i_{31}}) \\
&+ b^1(X^1_{i_{11}})b^1(X^1_{i_{12}})b^1(X^2_{i_{22}})b^2(X^2_{i_{21}},X^3_{i_{31}}) \\
&+ b^1(X^1_{i_{11}})b^1(X^2_{i_{21}})b^1(X^2_{i_{22}})b^2(X^1_{i_{12}},X^3_{i_{31}})
\end{aligned}
$$

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$$+ b^1(X^1_{i_{12}})b^1(X^2_{i_{21}})b^1(X^2_{i_{22}})b^2(X^1_{i_{11}},X^3_{i_{31}}) \Big\}$$

$$+\Big\{b^1(X^1_{i_{11}})b^1(X^2_{i_{21}})b^1(X^3_{i_{31}})b^2(X^1_{i_{12}},X^2_{i_{22}})$$

$$+ b^1(X^1_{i_{11}})b^1(X^2_{i_{22}})b^1(X^3_{i_{31}})b^2(X^1_{i_{12}},X^2_{i_{21}})$$

$$+ b^1(X^1_{i_{12}})b^1(X^2_{i_{21}})b^1(X^3_{i_{31}})b^2(X^1_{i_{11}},X^2_{i_{22}})$$

$$+ b^1(X^1_{i_{12}})b^1(X^2_{i_{22}})b^1(X^3_{i_{31}})b^2(X^1_{i_{11}},X^2_{i_{21}}) \Big\}$$

$$+\Big\{b^1(X^1_{i_{11}})b^1(X^2_{i_{21}})b^3(X^1_{i_{12}},X^2_{i_{22}},X^3_{i_{31}})$$

$$+ b^1(X^1_{i_{11}})b^1(X^2_{i_{22}})b^3(X^1_{i_{12}},X^2_{i_{21}},X^3_{i_{31}})$$

$$+ b^1(X^1_{i_{12}})b^1(X^2_{i_{21}})b^3(X^1_{i_{11}},X^2_{i_{22}},X^3_{i_{31}})$$

$$+ b^1(X^1_{i_{12}})b^1(X^2_{i_{22}})b^3(X^1_{i_{11}},X^2_{i_{21}},X^3_{i_{31}}) \Big\}$$

$$+\Big\{b^1(X^1_{i_{11}})b^2(X^1_{i_{12}},X^2_{i_{21}})b^2(X^2_{i_{22}},X^3_{i_{31}})$$

$$+ b^1(X^1_{i_{11}})b^2(X^1_{i_{12}},X^2_{i_{22}})b^2(X^2_{i_{21}},X^3_{i_{31}})$$

$$+ b^1(X^1_{i_{12}})b^2(X^1_{i_{11}},X^2_{i_{21}})b^2(X^2_{i_{22}},X^3_{i_{31}})$$

$$+ b^1(X^1_{i_{12}})b^2(X^1_{i_{11}},X^2_{i_{22}})b^2(X^2_{i_{21}},X^3_{i_{31}}) \Big\}$$

$$+\Big\{b^1(X^2_{i_{21}})b^2(X^1_{i_{11}},X^2_{i_{22}})b^2(X^1_{i_{12}},X^3_{i_{31}})$$

$$+ b^1(X^2_{i_{21}})b^2(X^1_{i_{12}},X^2_{i_{22}})b^2(X^1_{i_{11}},X^3_{i_{31}})$$

$$+ b^1(X^2_{i_{22}})b^2(X^1_{i_{11}},X^2_{i_{21}})b^2(X^1_{i_{12}},X^3_{i_{31}})$$

$$+ b^1(X^2_{i_{22}})b^2(X^1_{i_{12}},X^2_{i_{21}})b^2(X^1_{i_{11}},X^3_{i_{31}})\Big\}$$

$$+\Big\{b^1(X^3_{i_{31}})b^2(X^1_{i_{11}},X^2_{i_{21}})b^2(X^1_{i_{12}},X^2_{i_{22}})$$

$$+ b^1(X^3_{i_{31}})b^2(X^1_{i_{11}},X^2_{i_{22}})b^2(X^1_{i_{12}},X^2_{i_{21}})\Big\}$$

$$+\Big\{b^2(X^1_{i_{11}},X^2_{i_{21}})b^3(X^1_{i_{12}},X^2_{i_{22}},X^3_{i_{31}})$$

$$+ b^2(X^1_{i_{11}},X^2_{i_{22}})b^3(X^1_{i_{12}},X^2_{i_{21}},X^3_{i_{31}})$$

$$+ b^2(X^1_{i_{12}},X^2_{i_{21}})b^3(X^1_{i_{11}},X^2_{i_{22}},X^3_{i_{31}})$$

$$+ b^2(X^1_{i_{12}},X^2_{i_{22}})b^3(X^1_{i_{11}},X^2_{i_{21}},X^3_{i_{31}})\Big\},$$

where $b^3(X^1,X^2,X^3) = P(X^1=X^2=X^3=1) - P(X^1=1)P(X^2=1)P(X^3=1)$.

In the above expression for $G(X^1_{i_{11}},X^1_{i_{12}},X^2_{i_{21}},X^2_{i_{22}},X^3_{i_{31}})$, brackets are inserted to indicate grouping of similar terms.

To state and prove a theorem for the m-dimensional case, we probably need to develop a better system of notations.