# CHAPTER V

# FOURIER TRANSFORMS AND POTENTIAL THEORY

The materials of this chapter are drawn from references [2], [3], [4], [6], [11], [14], [16].

## 5.1 The Poisson integral

Let $f$ be a measurable function on $R^n$ such that $t \longmapsto |f(t)|/(1+|t|^{n+1})$ is integrable. "The Poisson integral"

$$U(x,y) = \frac{y}{C_n}\int f(t)\frac{dt}{(|x-t|^2+y^2)^{(n+1)/2}}$$

define on $R^{n+1}$ except for $y = 0$. We always suppose that $y > 0$. We choose the constant $C_n$ for which $U(x,y) \equiv 1$, where $f(t) \equiv 1$. We shall verify that

$$C_n = \frac{\pi^{(n+1)/2}}{\Gamma(\frac{n+1}{2})} .$$

Let $U(x,y) \equiv 1$ and $f(t) \equiv 1$, we obtain

$$C_n = \int \frac{y}{(|x-t|^2+y^2)^{(n+1)/2}}\, dt .$$

Since $C_n$ is a constant for all $x$ and $y$. We set $x = 0$ and $y = 1$. Then

$$C_n = \int \frac{dt}{(1+t_1^2+\dots+t_n^2)^{(n+1)/2}}.$$

Let $r^2 = t_1^2+\dots+t_n^2$,

$$C_n = \int_0^\infty \int_{\partial B(0,r)} \frac{1}{(1+r^2)^{(n+1)/2}}\,ds\, dr$$

$$= \int_0^{\infty} \frac{1}{(1+r^2)^{(n+1)/2}} \left[ \int_{\partial B(0,r)} ds \right] dr$$

$$= \int_0^{\infty} \frac{1}{(1+r^2)^{(n+1)/2}} S_n r^{n-1} dr$$

$$= S_n \int_0^{\infty} \frac{r^{n-1}}{(1+r^2)^{(n+1)/2}} dr.$$

Let $r^2 = \frac{x}{1-x}$,

$$C_n = S_n \int_0^1 \frac{\left(\frac{x}{1-x}\right)^{(n-1)/2}}{2\left(1+\frac{x}{1-x}\right)^{(n+1)/2}(1-x)^2\left(\frac{x}{1-x}\right)^{1/2}} dx$$

$$= \frac{S_n}{2} \int_0^1 x^{\left(\frac{n}{2}-1\right)} (1-x)^{\left(\frac{1}{2}-1\right)} dx$$

$$= \frac{S_n}{2} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}.$$

Since $S_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$ for all $n$ [3],

$$C_n = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

5.1.1 Theorem. The Poisson integral $U(x,y)$ tends to $f(x)$ as $y \longrightarrow 0$, a.e. on $R^n$.

Proof; By Lemma (3.14), it suffices to prove convergence at every Lebesgue point of $f$. Let $x$ be a Lebesgue point of $f$, and let

$$E(r) = \int_{|x-t| \leq r} |f(x) - f(t)| dt.$$

Since $x$ is a Lebesgue point of $f$,

$$(5.1.2)\quad \frac{E(r)}{r^n} = \frac{V_n E(r)}{V_n r^n} = \frac{V_n}{|B(x,r)|}\int_{\partial B(x,r)} |f(x)-f(t)|\,dt \to 0, \text{ as } r \to 0.$$

Hence, given an arbitrary $\varepsilon > 0$, there exists a (small) $\delta > 0$ such that $E(r) < \varepsilon r^n$ where $0 < r \leqslant \delta$. Since

$$f(x) - \frac{y}{C_n}\int f(t)\frac{dt}{(|x-t|^2+y^2)^{(n+1)/2}} = \frac{y}{C_n}\int \frac{f(x)-f(t)}{(|x-t|^2+y^2)^{(n+1)/2}}dt.$$

Let $y\int \frac{f(x)-f(t)}{(|x-t|^2+y^2)^{(n+1)/2}}dt = y\int_{|x-t|<y} + y\int_{y\leqslant|x-t|\leqslant\delta} + y\int_{|x-t|>\delta} = I_1+I_2+I_3$.

Now, by (5.1.2) $|I_1| \leqslant \frac{1}{y^n}\int_{|x-t|<y} |f(x)-f(t)|\,dt \longrightarrow 0$, as $y \to 0$.

By (3.17) $|I_2| \leqslant y\int_{y\leqslant|x-t|\leqslant\delta} \frac{|f(x)-f(t)|}{|x-t|^{n+1}}dt = y\int_y^\delta \frac{dE(r)}{r^{n+1}}$.

Hence (as the proof of Theorem 3.15)

$$\limsup_{y\to 0} |I_2| \leqslant (n+1)\varepsilon.$$

Finally,

$$|I_3| \leqslant y\int_{|x-t|>\delta} \frac{|f(x)-f(t)|}{|x-t|^{n+1}}dt.$$

We claim that $\int_{|x-t|>\delta} \frac{|f(x)-f(t)|}{|x-t|^{n+1}}dt$ is finite.

$$\int_{|x-t|>\delta} \frac{|f(x)-f(t)|}{|x-t|^{n+1}}dt \leqslant |f(x)|\int_{|x-t|>\delta}\frac{dt}{|x-t|^{n+1}} + \int_{|x-t|>\delta}\frac{|f(t)|}{|x-t|^{n+1}}dt.$$

By (3.16) the first term is finite. Since $|x-t| > \delta > 0$,

$$\frac{(1+|t|^2)^{1/2}}{|x-t|} \leqslant 1 + \frac{(|x|^2+1)^{1/2}}{|x-t|} \leqslant 1 + \frac{(|x|^2+1)^{1/2}}{\delta} = c_0,$$

therefore

$$\frac{(1+|t|^2)^{(n+1)/2}}{|x-t|^{n+1}} \leqslant c_0^{n+1} = c .$$

$$(5.1.3) \quad \int_{|x-t|>\delta} \frac{|f(t)|}{|x-t|^{n+1}}\,dt = \int_{|x-t|>\delta} \frac{|f(t)|}{(1+|t|^2)^{(n+1)/2}} \frac{(1+|t|^2)^{(n+1)/2}}{|x-t|^{n+1}}\,dt$$

$$\leqslant c \int_{|x-t|>\delta} \frac{|f(t)|}{(1+|t|^2)^{(n+1)/2}}dt \leqslant c \int \frac{|f(t)|}{1+|t|^{n+1}}\,dt < +\infty ,$$

since $t \longmapsto |f(t)|/(1+|t|^{n+1})$ is integrable. Then $I_3 \longrightarrow 0$, as $y \to 0$.

Hence the theorem is completely proved.

For any measurable function $f$ on $R^n$ such that $t \longmapsto |f(t)|/(1+|t|^{n+1})$ is integrable. We introduce "The Conjugate Poisson integral"

$$V(x,y) = \frac{1}{C_n} \int f(t) \frac{x-t}{(|x-t|^2+y^2)^{(n+1)/2}}dt, \text{ where } y>0.$$

$V(x,y)$ is the point in $R^n$ whose ith coordinate is

$$V_i(x,y) = \frac{1}{C_n} \int f(t) \frac{x_i-t_i}{(|x-t|^2+y^2)^{(n+1)/2}}dt.$$

5.1.4 <u>Theorem</u>. For almost every $x \in R^n$,

$$\frac{1}{C_n} \int_{|x-t|>y} f(t) \frac{t-x}{|t-x|^{n+1}}dt + \frac{1}{C_n} \int f(t) \frac{x-t}{(|x-t|^2+y^2)^{(n+1)/2}}dt \longrightarrow 0, \text{ as } y \longrightarrow 0.$$

<u>Proof</u>; The proof is similar to that of Theorem (5.1.1). Let $x$ be a Lebesgue point of $f$.

$$\int_{|x-t|>y} f(t) \frac{t-x}{|t-x|^{n+1}}\,dt + \int f(t) \frac{x-t}{(|x-t|^2+y^2)^{(n+1)/2}}dt =$$

$$= \int_{|x-t|\leqslant y} f(t)\frac{x-t}{(|x-t|^2+y^2)^{(n+1)/2}}dt + \left(\int_{y<|x-t|\leqslant\delta} + \int_{|x-t|>\delta}\right) f(t)(t-x)\left\{\frac{1}{|t-x|^{n+1}}\right.$$

$$\left. - \frac{1}{(|t-x|^2+y^2)^{(n+1)/2}}\right\}dt = I_1 + I_2 + I_3,$$ where $\delta$ as in Theorem(5.1.1). Since $t_i \longmapsto \frac{x_i-t_i}{(|x-t|^2+y^2)^{(n+1)/2}}$ is antisymmetic with respect to $x_i$,

$$\int_{a\leqslant|x-t|\leqslant b} \frac{x-t}{(|x-t|^2+y^2)^{(n+1)/2}}dt = 0, \quad (0\leqslant a\leqslant b\leqslant \infty).$$

Consequently, by (5.1.2)

$$|I_1| = \left|\int_{|t-x|\leqslant y} \{f(t)-f(x)\}\frac{x-t}{(|x-t|^2+y^2)^{(n+1)/2}}dt\right|$$

$$\leqslant \frac{1}{y^n}\int_{|t-x|\leqslant y} |f(t)-f(x)|dt \longrightarrow 0, \text{ as } y \longrightarrow 0.$$

Similarly,

$$\int_{a<|t-x|\leqslant b} \frac{t-x}{|t-x|^{n-1}}\, dt = 0, \quad (0<a\leqslant b\leqslant \infty).$$

We have

$$I_2 = \int_{y<|t-x|\leqslant\delta} \{f(t)-f(x)\}(t-x)\left\{\frac{(|t-x|^2+y^2)^{(n+1)/2} - |t-x|^{n+1}}{|t-x|^{n+1}(|t-x|+y^2)^{(n+1)/2}}\right\} dt.$$

We can show by expansion and comparison term by term that

$$(5.1.5) \quad (|t-x|^2+y^2)^{(n+1)/2} - |t-x|^{n+1} \leqslant \frac{n+1}{2}y^2(|t-x|^2+y^2)^{(n-1)/2}.$$

Then

$$|I_2| \leqslant \frac{n+1}{2}\, y^2 \int_{y<|t-x|\leqslant\delta} \frac{|f(t)-f(x)|}{|t-x|^n(|t-x|^2+y^2)}\, dt$$

$$\leqslant \frac{n+1}{2} y^2 \int\limits_{y<|t-x|\leqslant\delta} \frac{|f(t)-f(x)|}{|t-x|^{n+2}} dt = \frac{n+1}{2} y^2 \int_y^\delta \frac{dE(r)}{r^{n+2}},$$

where $E(r)$ as in Theorem(5.1.1). integrating by part,

$$|I_2| \leqslant \frac{n+1}{2} y^2 \left[\frac{E(r)}{r^{n-2}}\right]_y^\delta + \frac{(n+1)(n+2)}{2} y^2 \int_y^\delta \frac{E(r)}{r^{n+3}} dr ,$$

$$\leqslant \frac{n+1}{2} y^2 \frac{E(\delta)}{\delta^{n+2}} + \frac{(n+1)(n+2)}{2} y^2 \int_y^\delta \frac{E(r)}{r^{n+3}} dr.$$

As $y \to 0$, $y^2 E(\delta)\delta^{-(n+2)}$ tends to zero, because $\delta$ is fixed.

Consider

$$y^2 \int_y^\delta \frac{E(r)}{r^{n+3}} dr < y^2 \int_y^\delta \frac{\varepsilon}{r^3} dr < \varepsilon y^2 \int_y^\infty \frac{1}{r^3} dr = \frac{\varepsilon}{2} ,$$

whence, as $y \to 0$, $\limsup |I_2| \leqslant \frac{(n+1)(n+2)}{4} \varepsilon$ for any arbitrarily small $\varepsilon$. Finally, we obtain by (5.1.3) and (5.1.5) that

$$|I_3| \leqslant \frac{n+1}{2} y^2 \int\limits_{|x-t|>\delta} \frac{|f(t)|}{|t-x|^{n+2}} dt \longrightarrow 0, \text{ as } y \longrightarrow 0.$$

Hence the theorem is completely proved.

## 5.2 The Fourier Transforms of the Poisson Kernals

The Poisson integral $U(x,y) = \frac{y}{C_n}\int f(t)\frac{dt}{(|x-t|^2+y^2)^{(n+1)/2}}$,

and Conjugate Poisson integral $V(x,y) = \frac{1}{C_n}\int f(t)\frac{x-t}{(|x-t|^2+y^2)^{(n+1)/2}} dt$

in section (5.1) are $\frac{(2\pi)^{n/2}}{C_n}(f*a_y)(x)$ and $\frac{(2\pi)^{n/2}}{C_n}(f*b_y)(x)$ where

$a_y$ and $b_y$ are Poisson Kernals of $U$ and $V$ respectively.

In this section we want to show that the Fourier transforms of $a_y$ and $b_y$ are

(5.2.1) $\hat{a}_y(u) = \frac{\sqrt{\pi}}{2^{n/2}\Gamma(\frac{n+1}{2})}e^{-y|u|}$ and $\hat{b}_y(u) = \frac{i\sqrt{\pi}}{2^{n/2}\Gamma(\frac{n+1}{2})}\frac{u}{|u|}e^{-y|u|}$

respectively. $\hat{b}_y$ is the vector valued function which its' components are the Fourier transform of the components of $b_y$ respectively.

Suppose $f$ is continuous function, and $f > 0$, the Riemann-Liouville integral is defined by

$$(-I_x)^a[f(x)] = \frac{1}{\Gamma(a)}\int_x^{\infty} f(t)(t-x)^{a-1}dt \quad \text{if a is rational}$$

number which is not negative integer and $a \neq 0$,

$$I_x^a[f(x)] = f^{(-a)}(x) \quad \text{if a is negative integer,}$$

and $$I_x^0[f(x)] = f(x) .$$

5.2.2 Lemma. $I^a \circ I^b = I^{a+b}$, whenever they converge.

Proof; We may assume that $a \neq 0$ and $b \neq 0$, otherwise there is nothing to be prove.

Case 1. If $a,b$ are rational numbers which are not negative integers, then

$$\begin{aligned} I_x^a \circ I_x^b[f(x)] &= I_x^a\left[\frac{(-1)^b}{\Gamma(b)}\int_x^{\infty} f(t)(t-x)^{b-1}dt\right] \\ &= \lim_{k\to\infty}\frac{(-1)^{a+b}}{\Gamma(a)\Gamma(b)}\int_x^k\int_z^k f(t)(t-z)^{b-1}dt(z-x)^{a-1}dz \\ &= \lim_{k\to\infty}\frac{(-1)^{a+b}}{\Gamma(a)\Gamma(b)}\int_x^k\int_x^t f(t)(t-z)^{b-1}(z-x)^{a-1}dzdt \end{aligned}$$

$$= \frac{(-1)^{a+b}}{\Gamma(a)\Gamma(b)} \int_x^{\infty} f(t) \int_x^{t} (t-z)^{b-1}(z-x)^{a-1} dz dt.$$

Let $s = \frac{z-x}{t-x}$. Then

$$I_x^a o I_x^b [f(x)] = \frac{(-1)^{a+b}}{\Gamma(a)\Gamma(b)} \int_x^{\infty} f(t)(t-x)^{(a+b)-1} dt \int_0^1 (1-s)^{b-1} s^{a-1} ds$$

$$= \frac{(-1)^{a+b}}{\Gamma(a+b)} \int_x^{\infty} f(t)(t-x)^{(a+b)-1} dt = I_x^{(a+b)}[f(x)].$$

<u>Case</u> 2. If a,b are negative integers, then

$$I_x^a o I_x^b [f(x)] = I_x^a[I_x^b[f(x)]] = I_x^a[f^{(-b)}(x)] = f^{(-a-b)}(x)$$
$$= I_x^{a+b}[f(x)] .$$

We can show by induction on $m$ that

$$I_x^{-m} o I_x^m [f(x)] = f(x) \quad \text{and} \quad I_x^m o I_x^{-m} [f(x)] = f(x).$$

<u>Case</u> 3. Suppose $a$ is negative integer and b is rational number which is not negative integer.

If $a > -b$ then $a+b > 0$ and hence

$$I_x^a o I_x^b [f(x)] = I_x^a o I_x^{(-a)+(a+b)} [f(x)]$$

$$= I_x^a o I_x^{-a} o I_x^{(a+b)} [f(x)] = I_x^{(a+b)} [f(x)].$$

If $a < -b$ then $a+b < 0$.

<u>Subcase</u> 3.1 If $a+b$ is integer then b is positive integer and hence

$$I_x^a o I_x^b [f(x)] = I_x^{(a+b)+(-b)} o I_x^b [f(x)]$$

$$= I_x^{(a+b)} o I_x^{(-b)} o I_x^b [f(x)] = I_x^{(a+b)} [f(x)].$$

Subcase 3.2 If $a+b$ is not integer then

$$I_x^a o I_x^b [f(x)] = I_x^a o I_x^{(-a)+(a+b)} [f(x)]$$

$$= I_x^a o I_x^{-a} o I_x^{(a+b)} [f(x)] = I_x^{(a+b)} [f(x)] .$$

Similarly, we can show that $I^a o I^b = I^{a+b}$, whenever $b$ is negative integer and $a$ is rational number which is not negative integer. Hence the **lemma** is completely proved.

Let $|t|^2 = t_1^2+\dots+t_n^2 = r^2$ and $|u|^2 = u_1^2+\dots+u_n^2 = \rho^2$ . Suppose that the function $f$ is a radial function, i.e., $f(t) = f(|t|)$ . We note that by Theorem (4.1.17), the Fourier transform $\hat{f}$ is also a radial function. Suppose further that f is bounded continuous function and $0 < \int_0^\infty f(r) r^{n-1} dr < \infty$ .

Consider the value of $\hat{f}(u)$ where $u = (\rho, 0, \dots, 0)$,

$$\hat{f}(u) = \frac{1}{(2\pi)^{n/2}} \int f(t) e^{i(u.t)} dt$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{i\rho t_1} dt_1 \int_{R^{n-1}} f(r) dt_2 \dots dt_n$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{i\rho t_1} dt_1 \int_0^\infty f(r) [ \int_{\partial B(0,r_1)} ds ] dr_1$$

where $r_1^2 = t_2^2+\dots+t_n^2 = r^2 - t_1^2$ . Consequently,

$$\hat{f}(u) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{i\rho t_1} dt_1 \int_0^\infty f(r) S_{n-1} r_1^{n-2} dr_1$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} e^{i\rho t_1} dt_1 \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \int_0^\infty f(r) r_1^{n-2} dr_1$$

$$= \frac{1}{2^{n/2}\sqrt{\pi}\,\Gamma(\frac{n-1}{2})} \int_{-\infty}^{\infty} e^{i\rho t_1} dt_1 \int_{r=|t_1|}^{r=\infty} f(r)(r^2-t_1^2)^{(n-3)/2} d(r^2).$$

Let $t_1 = r$, $R = r^2$ and $g: x \longmapsto \sqrt{x}$. Consider

$$\frac{1}{\Gamma(\frac{n-1}{2})} \int_{r=|t_1|}^{r=\infty} f(r)(r^2-t_1^2)^{(n-3)/2} d(r^2) = \frac{1}{\Gamma(\frac{n-1}{2})} \int_{R}^{\infty} f\circ g(r^2)(r^2-R)^{(\frac{n-1}{2}-1)} d(r^2)$$

$$= (-I_R)^{(n-1)/2}[f\circ g(R)] = (-I_R)^{(n-1)/2}[f(\sqrt{R})].$$

Then

$$\hat{f}(u) = \frac{1}{2^{n/2}\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\rho r}(-I_R)^{(n-1)/2}[f(r)]\,dr,$$

$$2^{(n-1)/2}\hat{f}(u) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{i\rho r}(-I_R)^{(n-1)/2}[f(r)]\,dr.$$

Since $f$ is continuous, by Corollary (4.2.12)

$$(-I_R)^{(n-1)/2}[f(r)] = \frac{2^{(n-1)/2}}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-i\rho r}\hat{f}(\rho)\,d\rho \text{ . Then}$$

(5.2.3) $$f(r) = 2^{(n-2)/2}\pi^{-1/2}(-I_R)^{-(n-1)/2}\left[\int_{-\infty}^{\infty} e^{-i\rho r}\hat{f}(\rho)\,d\rho\right],$$

which is called the <u>Leray Formula</u>.

Let $f(u) = e^{-y|u|}$. Then

$$\int_{-\infty}^{\infty} e^{-i\rho r}\hat{f}(\rho)\,d\rho = \int_{-\infty}^{\infty} e^{-i\rho r}e^{-y|\rho|}\,d\rho$$

$$= \int_{-\infty}^{0} e^{-i\rho r}e^{y\rho}\,d\rho + \int_{0}^{\infty} e^{-i\rho r}e^{-y\rho}\,d\rho$$

$$= \frac{e^{(y-ir)\rho}}{y-ir}\Bigg|_{-\infty}^{0} + \frac{e^{-(y+ir)}}{-(y+ir)}\Bigg|_{0}^{\infty} = \frac{2y}{y^2+r^2}.$$

Therefore, $$f(r) = 2^{n/2}\pi^{-1/2}y(-I_R)^{-(n-1)/2}\left[\frac{1}{y^2+R}\right].$$

If $n$ is odd, then

$$f(r) = 2^{n/2}\pi^{-1/2}y(-1)^{(n-1)/2}\frac{d^{(n-1)/2}}{dR^{(n-1)/2}}\left[\frac{1}{y^2+R}\right].$$

We can show by induction on $n$ that

$$\frac{d^{(n-1)/2}}{dR^{(n-1)/2}}\left[\frac{1}{y^2+R}\right] = (-1)^{(n-1)/2}\Gamma\left(\frac{n+1}{2}\right)\frac{1}{(y^2+R)^{(n+1)/2}}.$$

Hence $$f(r) = \frac{2^{n/2}\Gamma\left(\frac{n+1}{2}\right) y}{\sqrt{\pi}\,(y^2+r^2)^{(n+1)/2}}.$$

If $n$ is even, then by Lemma (5.2.2)

$$\begin{aligned} f(r) &= 2^{n/2}\pi^{-1/2}y(-I_R)^{-\frac{n}{2}+\frac{1}{2}}\left[\frac{1}{y^2+R}\right] \\ &= 2^{n/2}\pi^{-1/2}y(-I_R)^{-\frac{n}{2}}\circ(-I_R)^{\frac{1}{2}}\left[\frac{1}{y^2+R}\right] \\ &= (-1)^{n/2}2^{n/2}\pi^{-1/2}y\,\frac{d^{n/2}}{dR^{n/2}}\left\{(-I_R)^{1/2}\left[\frac{1}{y^2+R}\right]\right\}. \end{aligned}$$

Since $$(-I_R)^{1/2}\left[\frac{1}{y^2+R}\right] = \frac{1}{\Gamma(\frac{1}{2})}\int_R^{\infty}\frac{1}{y^2+T}(T-R)^{-1/2}\,dT.$$

Let $x^2 = T - R$. Then $2x\,dx = dT$ and

$$\begin{aligned} (-I_R)^{\frac{1}{2}}\left[\frac{1}{y^2+R}\right] &= \frac{1}{\Gamma(\frac{1}{2})}\int_0^{\infty}\frac{2}{x^2+(y^2+R)}\,dx \\ &= \frac{2}{\sqrt{\pi}\sqrt{y^2+R}}\left[\tan^{-1}\frac{x}{\sqrt{y^2+R}}\right]_0^{\infty} = \frac{\sqrt{\pi}}{\sqrt{y^2+R}}. \end{aligned}$$

Then $$f(x) = (-1)^{\frac{n}{2}}2^{\frac{n}{2}}\pi^{-\frac{1}{2}}y\,\frac{d^{n/2}}{dR^{n/2}}\left[\frac{\sqrt{\pi}}{\sqrt{y^2+R}}\right].$$

We can show by induction on $n$ that

$$\frac{d^{\frac{n}{2}}}{dR^{n/2}}\left[\frac{\sqrt{\pi}}{\sqrt{y^2+R}}\right] = \frac{(-1)^{\frac{n}{2}}\Gamma(\frac{n+1}{2})}{(y^2+R)^{(n+1)/2}} .$$

Hence $\quad f(r) = \dfrac{2^{\frac{n}{2}}\Gamma(\frac{n+1}{2})\,y}{\sqrt{\pi}(y^2+r^2)^{(n+1)/2}}.$

Then

$$e^{-y|u|} = \hat{f}(u) = \frac{1}{(2\pi)^{n/2}}\int e^{i(u.t)}\frac{2^{\frac{n}{2}}\Gamma(\frac{n+1}{2})\,y}{\sqrt{\pi}(y^2+t^2)^{(n+1)/2}}dt$$

$$= \frac{2^{\frac{n}{2}}\Gamma(\frac{n+1}{2})}{\sqrt{\pi}}\frac{1}{(2\pi)^{n/2}}\int e^{i(u.t)}\frac{y}{(y^2+t^2)^{(n+1)/2}}dt .$$

So that $\quad \hat{a}_y(u) = \dfrac{\sqrt{\pi}}{2^{n/2}\Gamma(\frac{n+1}{2})}e^{-y|u|} .$

We can not calculate $\hat{b}_y$ directly, because $b_y$ is not radial. Let

$$\phi_1(t) = \frac{1}{(2\pi)^{n/2}}\int\frac{u_1}{|u|}e^{-y|u|}e^{-i(t.u)}du$$

$$= \frac{1}{(2\pi)^{n/2}}\int_{R^{n-1}}\int_{-\infty}^{\infty}\frac{u_1}{|u|}e^{-y|u|}e^{-i(t.u)}du_1du_2\dots du_n$$

$$= \frac{1}{(2\pi)^{n/2}}\int_{-\infty}^{\infty}\int_{R^{n-1}}\frac{u_1}{|u|}e^{-y|u|}e^{-i(t.u)}du_2du_3\dots du_n du_1$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u_1e^{-iu_1t_1}du_1\frac{1}{(2\pi)^{(\frac{n-1}{2})}}\int_{R^{n-1}}\frac{e^{-y(u_1^2+\dots+u_n^2)^{1/2}}}{(u_1^2+\dots+u_n^2)^{1/2}}$$

$$e^{-i(u_2t_2+\dots+u_nt_n)}du_2\dots du_n .$$

Consider

$$f(r) = \frac{1}{(2\pi)^{(\frac{n-1}{2})}}\int_{R^{n-1}}\frac{e^{-y(u_1^2+\dots+u_n^2)^{1/2}}}{(u_1^2+\dots+u_n^2)^{1/2}}e^{-i(u_2t_2+\dots+u_nt_n)}du_2\dots du_n$$

$$= \frac{1}{(2\pi)^{(\frac{n-1}{2})}} \int_{R^{n-1}} \frac{e^{-y(u_1^2+s^2)^{1/2}}}{(u_1^2+s^2)^{1/2}} e^{-i(u_2t_2+\dots+u_nt_n)} du_2 \dots du_n ,$$

where $s^2 = u_2^2+\dots+u_n^2$ , $r^2 = t_2^2+\dots+t_n^2$. Let

$$g(s) = \frac{e^{-y(u_1^2+s^2)^{1/2}}}{(u_1^2+s^2)^{1/2}} .$$ Therefore

$$\int_{-\infty}^{\infty} e^{-isr} g(s) ds = 2K_o(u_1\sqrt{y^2+r^2}),$$ where the equality is obtained from [6;P118] and $x \longmapsto K_o(ax)$ is the function modified by Hankel given by

$$k_o(ax) = \int_0^{\infty} \frac{\cos xt}{\sqrt{t^2+a^2}} dt .$$

Since $g$ is continuous,

$$g(s) = \frac{1}{(2\pi)^{(n-1)/2}} \int_{R^{n-1}} f(r) e^{-i(u_2t_2+\dots+u_nt_n)} dt_2 \dots dt_n .$$

By the Leray formula (5.2.3)

$$f(r) = 2^{(\frac{n-3}{2})} \pi^{-\frac{1}{2}} (-I_R)^{-(\frac{n-2}{2})} \left[ \int_{-\infty}^{\infty} e^{-isr} g(s) ds \right]$$

$$= 2^{(\frac{n-1}{2})} \pi^{-\frac{1}{2}} (-I_R)^{-(\frac{n-2}{2})} \left[ K_o(u_1\sqrt{y^2+R}) \right] ,$$ where $r^2 = R$.

Therefore

$$\emptyset_1(t) = 2^{(\frac{n-2}{2})} \pi^{-1} \int_{-\infty}^{\infty} e^{-iu_1t_1} u_1 (-I_R)^{-(\frac{n-2}{2})} \left[ K_o(u_1\sqrt{y^2+R}) \right] du_1$$

$$= 2^{(\frac{n-2}{2})} \pi^{-1} \int_{-\infty}^{\infty} i\frac{d}{dt_1} (e^{-iu_1t_1}) (-I_R)^{-(\frac{n-2}{2})} \left[ K_o(u_1\sqrt{y^2+R}) \right] du_1$$

$$= i2^{(\frac{n-2}{2})} \pi^{-1} (-I_R)^{-(\frac{n-2}{2})} \left[ \frac{d}{dt_1} \int_{-\infty}^{\infty} e^{-iu_1t_1} K_o(u_1\sqrt{y^2+R}) du_1 \right]$$

Since $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ixy}\frac{1}{\sqrt{x^2+a^2}}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\cos xy}{\sqrt{x^2+a^2}}\,dx$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{\cos|y|x}{\sqrt{x^2+a^2}}\,dx = \sqrt{\frac{2}{\pi}}\,K_o(a|y|)\ ,$$

and hence by Corollary (4.2.12)

$$\int_{-\infty}^{\infty} e^{-ixy}K_o(a|y|)dy = \frac{\pi}{\sqrt{x^2+a^2}}\ .$$

Then $$\phi_1(t) = i2^{(n-2)/2}\pi^{-1}(-I_R)^{-(n-2)/2}\left[\frac{d}{dt_1}\frac{\pi}{\sqrt{x^2+a^2}}\right]$$

$$= -i2^{(n-2)/2}t_1(-I_R)^{-(n-2)/2}\left[\frac{1}{(t_1^2+y^2+R)^{3/2}}\right].$$

If n is even, then

$$\phi_1(t) = -i2^{(n-2)/2}t_1(-1)^{(n-2)/2}\frac{d^{(n-2)/2}}{dR^{(n-2)/2}}\left[\frac{1}{(t_1^2+y^2+R)^{3/2}}\right].$$

We can show by induction on n that

$$\frac{d^{(n-2)/2}}{dR^{(n-2)/2}}\left[\frac{1}{(t_1^2+y^2+R)^{3/2}}\right] = \frac{(-1)^{(n-2)/2}2\Gamma(\frac{n+1}{2})}{\sqrt{\pi}(|t|^2+y^2)^{(n+1)/2}}\ .$$

Then $$\phi_1(t) = \frac{2^{n/2}\Gamma(\frac{n+1}{2})t_1}{i\sqrt{\pi}(|t|^2+y^2)^{(n+1)/2}}\ .$$

If n is odd, then by Lemma (5.2.2)

$$\phi_1(t) = -i2^{(\frac{n-2}{2})}t_1(-1)^{(\frac{n-1}{2})}\frac{d^{(\frac{n-1}{2})}}{dR^{(\frac{n-1}{2})}}\left\{(-I_R)^{\frac{1}{2}}\left[\frac{1}{(t_1^2+y^2+R)^{\frac{3}{2}}}\right]\right\}.$$

Consider $(-I_R)^{\frac{1}{2}}\left[\frac{1}{(t_1^2+y^2+R)^{\frac{3}{2}}}\right] = \frac{1}{\Gamma(\frac{1}{2})}\int_R^\infty \frac{1}{(t_1^2+y^2+T)^{\frac{3}{2}}}(T-R)^{-\frac{1}{2}}\,dT$ .

Let $x^2 = t_1^2+y^2+T$ , $2x\,dx = dT$ ,

$$\int_R^\infty \frac{1}{(t_1^2+y^2+T)^{\frac{3}{2}}(T-R)^{\frac{1}{2}}}\,dT = 2\int_{\sqrt{t_1^2+y^2+R}}^\infty \frac{dx}{x^2\sqrt{x^2-(t_1^2+y^2+R)}}$$

$$= 2\int_c^\infty \frac{dx}{x^2\sqrt{x^2-c^2}}\,, \quad \text{where } c = \sqrt{t_1^2+y^2+R}\ .$$

Let $x = c\sec\theta$, $dx = c\sec\theta\tan\theta\,d\theta$.

$$\int_R^\infty \frac{1}{(t_1^2+y^2+T)^{\frac{3}{2}}(T-R)^{\frac{1}{2}}}\,dT = \frac{2}{c^2}\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta = \frac{2}{c^2}\sin\theta\Big|_0^{\frac{\pi}{2}} = \frac{2}{c^2}\ .$$

Then $(-I_R)^{1/2}\left[\frac{1}{(t_1^2+y^2+R)^{\frac{3}{2}}}\right] = \frac{2}{\sqrt{\pi}(t_1^2+y^2+R)}$ .

We can show by induction on n that

$$\frac{d^{(n-1)/2}}{dR^{(n-1)/2}}\left[\frac{2}{\sqrt{\pi}(t_1^2+y^2+R)}\right] = \frac{-2}{\sqrt{\pi}(t_1^2+y^2+R)^2}\ .$$

Therefore

$$\phi_1(t) = \frac{2^{\frac{n}{2}}\Gamma(\frac{n+1}{2})t_1}{i\sqrt{\pi}(|t|^2+y^2)^{(n+1)/2}}\ .$$

That is

$$\frac{1}{(2\pi)^{\frac{n}{2}}}\int \frac{u_1}{|u|}e^{-y|u|}\,e^{-i(t\cdot u)}\,du = \frac{2^{\frac{n}{2}}\Gamma(\frac{n+1}{2})t_1}{i\sqrt{\pi}(|t|^2+y^2)^{(\frac{n+1}{2})}}\ .$$

Then

$$\frac{u_1}{|u|}e^{-y|u|} = \frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{i(t\cdot u)}\frac{2^{\frac{n}{2}}\Gamma(\frac{n+1}{2})t_1}{i\sqrt{\pi}(|t|^2+y^2)^{(\frac{n+1}{2})}}\,dt, \text{ and hence}$$

$$\hat{b}_y(t) = \frac{1}{(2\pi)^{n/2}}\int e^{i(t.u)} \frac{t_1}{(|t|^2+y^2)^{(\frac{n+1}{2})}} dt = \frac{i\sqrt{\pi}}{2^{\frac{n}{2}}\Gamma(\frac{n+1}{2})} \frac{u_1}{|u|} e^{-y|u|}.$$

## 5.3 The Conjugate Function of Function in $L^2$

5.3.1 Theorem. Let $f \in L^2(R^n)$ ,

(5.3.2) $$f^{\not c}(x) = \lim_{\varepsilon \to 0} \frac{1}{C_n} \int_{|t-x|>\varepsilon} f(t) \frac{t-x}{|t-x|^{n+1}} dt,$$ whenever it exists.

Then a) $f^{\not c}(x)$ exists a.e.,

b) $\| f^{\not c} \|_2 = \| f \|_2$ (and hence $f^{\not c} \in \mathcal{L}^2(R^n)$),

c) $$\lim_{\varepsilon \to 0} \frac{1}{C_n} \int_{|x-t|>\varepsilon} \frac{f^{\not c}(x).(x-t)}{|x-t|^{n-1}} dx = -f(t) \text{ a.e. .}$$

$f^{\not c}$ is said to be the Conjugate function of $f$ .

Proof; Let $F$ be the Fourier transform of $f$ ;

$$F(u) = \lim_{T \to \infty} \text{ in } L^2 \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{|t|<T} e^{i(u.t)} f(t)dt.$$

Let $F^{\not c}(u) = (F_1^{\not c}(u_1), \ldots, F_n^{\not c}(u_n)) = \frac{-iF(u)}{|u|}(u_1, \ldots, u_n) = \frac{-iu}{|u|}F(u),$

and let

$$h(x) = \lim_{U \to \infty} \text{ in } L^2 \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{|u|<U} e^{-i(x.u)} F^{\not c}(u)du.$$

If we can prove that $f^{\not c}(x) = h(x)$ a.e. then by Parseval-Plancherel Theorem (4.3.1)

$$\| f^{\not c} \|_2 = \| F^{\not c} \|_2 = \| F \|_2 = \| f \|_2 ,$$

a) and b) hold. By Theorem (5.1.1)

(5.3.3) $\quad \frac{1}{C_n}\int h(t)\frac{y}{(|x-t|^2+y^2)^{(\frac{n+1}{2})}}\,dt \longrightarrow h(x)$ a.e. as $y \longrightarrow 0$.

By Theorem (5.1.4)

(5.3.4) $\quad -\frac{1}{C_n}\int f(t)\frac{(x-t)}{(|x-t|^2+y^2)^{(\frac{n+1}{2})}}\,dt - \frac{1}{C_n}\int_{|t-x|>y} f(t)\frac{t-x}{|t-x|^{n+1}}\,dt \longrightarrow 0$ a.e.

as $y \longrightarrow 0$.

If we can prove that

(5.3.5) $\quad -\frac{1}{C_n}\int f(t)\frac{x-t}{(|x-t|^2+y^2)^{(\frac{n+1}{2})}}\,dt = \frac{1}{C_n}\int h(t)\frac{y}{(|x-t|^2+y^2)^{(\frac{n+1}{2})}}\,dt$

a.e., then by (5.3.3),(5.3.4) and definition (5.3.2), $h(x) = f^{\phi}(x)$ a.e.. We claim that (5.3.5) holds. We can write (5.3.5) as

$$(-f*b_y)(t) = (h*a_y)(t) .$$

Since the Fourier transform of $(-f*b_y)(t)$ and $(h*a_y)(t)$ are

$$\frac{-i\sqrt{\pi}}{2^{n/2}\Gamma(\frac{n+1}{2})}\,\frac{u}{|u|}\,e^{-y|u|}F(u).$$

Then (5.3.5) holds a.e. .

To prove c). Since

$$\lim_{\varepsilon\to 0}\frac{1}{C_n}\int_{|x-t|>\varepsilon}\frac{f^{\phi}(x).(x-t)}{|x-t|^{n+1}}\,dx = \frac{(2\pi)^{\frac{n}{2}}}{C_n}(f_1^{\phi}*p_1(t) + f_2^{\phi}*p_2(t) + \dots + f_n^{\phi}*p_n(t))$$

where $p_i(t) = \frac{t_i}{|t|^{n+1}}$ . Then its Fourier transform is

$$\frac{(2\pi)^{\frac{n}{2}}}{C_n}\,\frac{i\sqrt{\pi}}{2^{n/2}\Gamma(\frac{n+1}{2})}\left[-i\frac{u_1}{|u|}F_1^{\phi}(u)-\dots-i\frac{u_n}{|u|}F_n^{\phi}(u)\right]$$

$$= -i\frac{u_1}{|u|}F^{\phi}(u)-\dots-i\frac{u_n}{|u|}F_n^{\phi}(u)$$

$$= \left\{(-i\frac{u_1}{|u|})(-i\frac{u_1}{|u|}F(u))\right\} + \dots + \left\{(-i\frac{u_n}{|u|})(-i\frac{u_n}{|u|}F(u))\right\} = -F(u) .$$

Therefore $\lim_{\varepsilon \to 0} \frac{1}{C_n} \int_{|x-t|>\varepsilon} \frac{f^{\varphi}(x).(x-t)}{|x-t|^{n+1}} dx = -f(t)$ a.e..

Hence the theorem is completely proved.

## 5.4 The Relation Between the Conjugate Function and the Generalized Potentials.

5.4.1 Definition; A real-valued function u defined on $R^n$ and having continuous second partial derivatives thereon is called a harmonic function if $\Delta u = 0$ on $R^n$, where $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ which is called the Laplacian of u .

5.4.2 Definition; The problem of determining the harmonic function in a region which assumes preseribed value on the boundary is called a Dirichlet problem. If instead the normal derivative of the harmonic function is prescribed on the boundary, the problem is called a Neumann problem.

Let f be a continuous function on $R^n$ such that $t \longmapsto |f(t)|/(1+|t|^{n+1})$ is integrable. M. Riesz defined the Riemann-Liouville or Generalized Potential by

$$J^a[f(x)] = \frac{1}{M_n(a)} \int f(t) \frac{dt}{|x-t|^{n-a}} ,$$

where

$$M_n(a) = \frac{\pi^{n/2} 2^a \Gamma(\frac{a}{2})}{\Gamma(\frac{n-a}{2})} .$$

In particular for $a = 1$ we obtain,

(5.4.3) $$J\left[f(x)\right] = \frac{1}{M_n(1)}\int f(t)\frac{dt}{|x-t|^{n-1}} .$$

If we interpret $x$ as a point $(x,0) = (x_1,\dots x_n,0) \in R^{n+1}$, (5.4.3) gives the value of

$$P^f(x,y) = \frac{1}{M_n(1)}\int f(t)\frac{dt}{(|x-t|^2+y^2)^{(n-1)/2}}$$

on the hyperplane $y = 0$ .

5.4.4 <u>Theorem</u>. If $f$ is a continuous function on $R^n$ such that $t \longmapsto |f(t)|/(1+|t|^{n+1})$ is integrable, then the function $-P^f$ is a solution of Neumann problem on half-space $y > 0$ of $R^{n+1}$ for the function f.

<u>Proof</u>; Since $\frac{(n-1)}{M_n(1)} = \frac{(n-1)\Gamma(\frac{n-1}{2})}{2\pi^{n/2}\Gamma(\frac{1}{2})} = \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} = \frac{1}{C_n}$,

$$\frac{\partial}{\partial y}\left[-P^f(x,y)\right] = \frac{-1}{(n-1)C_n}\int f(t)\frac{\partial}{\partial y}\left\{\frac{1}{(|x-t|^2+y^2)^{(n-1)/2}}\right\}dt$$

$$= \frac{y}{C_n}\int f(t)\frac{dt}{(|x-t|^2+y^2)^{(n+1)/2}} = U(x,y).$$

By Theorem (5.1.1) $\left[\frac{\partial}{\partial y}\left[-P^f(x,y)\right]\right]_{y=0} = \lim_{y\to 0} U(x,y) = f(x).$

Next, we must show that $-P^f$ is harmonic function.

$$\frac{\partial^2}{\partial y^2}\left[-P^f(x,y)\right] = \frac{\partial}{\partial y}U(x,y) = \frac{1}{C_n}\int f(t)\frac{\partial}{\partial y}\frac{y}{(|x-t|^2+y^2)^{(n+1)/2}}dt$$

$$= \frac{1}{C_n}\int f(t)\frac{|x-t|^2-ny^2}{(|x-t|^2+y^2)^{(n+3)/2}}dt,$$

$$\frac{\partial}{\partial x_i}\left[-P^f(x,y)\right] = \frac{-1}{(n-1)C_n}\int f(t)\frac{\partial}{\partial x_i}\left\{\frac{1}{(|x-t|^2+y^2)^{(n-1)/2}}\right\}dt$$

$$= \frac{1}{C_n}\int f(t)\frac{x_i-t_i}{(|x-t|^2+y^2)^{(n+1)/2}}dt = V_i(x,y)$$

$$\frac{\partial^2}{\partial x_i^2}\left[-P^f(x,y)\right] = \frac{\partial}{\partial x_i}V_i(x,y) = \frac{1}{C_n}\int f(t)\frac{\partial}{\partial x_i}\left\{\frac{x_i-t_i}{(|x-t|^2+y^2)^{(n+1)/2}}\right\}dt$$

$$= \frac{1}{C_n}\int f(t)\frac{|x-t|^2+y^2-(x_i-t_i)^2-n(x_i-t_i)^2}{(|x-t|^2+y^2)^{(n+3)/2}}dt$$

$$= \frac{1}{C_n}\int f(t)\frac{|x-t|^2}{(|x-t|^2+y^2)^{(\frac{n+3}{2})}}dt + \frac{y^2}{C_n}\int f(t)\frac{dt}{(|x-t|^2+y^2)^{(\frac{n+3}{2})}}$$

$$-\frac{1}{C_n}\int f(t)\frac{(x_i-t_i)^2}{(|x-t|^2+y^2)^{(\frac{n+3}{2})}}dt - \frac{n}{C_n}\int f(t)\frac{(x_i-t_i)^2}{(|x-t|^2+y^2)^{(\frac{n+3}{2})}}dt.$$

Consider

$$\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\left[-P^f(x,y)\right] = \frac{n}{C_n}\int f(t)\frac{|x-t|^2}{(|x-t|^2+y^2)^{(\frac{n+3}{2})}}dt$$

$$+\frac{ny^2}{C_n}\int f(t)\frac{dt}{(|x-t|^2+y^2)^{(\frac{n+3}{2})}}$$

$$-\frac{1}{C_n}\int f(t)\frac{|x-t|^2}{(|x-t|^2+y^2)^{(\frac{n+3}{2})}}dt$$

$$-\frac{n}{C_n}\int f(t)\frac{|x-t|^2}{(|x-t|^2+y^2)^{(\frac{n+3}{2})}}dt$$

$$= -\frac{1}{C_n}\int f(t)\frac{|x-t|^2-ny^2}{(|x-t|^2+y^2)^{(\frac{n+3}{2})}}dt.$$

Then $\Delta\left[-P^f(x,y)\right] = 0$. Hence the theorem is completely proved.

If $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$, then $V(x,y) = \nabla[-P^f(x,y)]$.

5.4.5 Theorem. If f is a continuous function on $R^n$ such that $t \longmapsto |f(t)|/(1+|t|^{n+1})$ is integrable, then for almost every $x \in R^n$, $\nabla J[f(x)] = f^{\phi}(x)$.

Proof; By Theorem (5.1.4) and (5.3.1),

$$\lim_{y \to 0} [-V(x,y)] = f^{\phi}(x) \quad \text{a.e.}\ .$$

since

$$-V(x,y) = \nabla P^f(x,y), \text{ and}$$

$$\lim_{y \to 0} [-V(x,y)] = \nabla J\,|f(x)|\ .$$

Then

$$\nabla J\,|f(x)| = f^{\phi}(x) \quad \text{a.e.}\ .$$