CHAPTER IV

GROUP DIGRAPHS

This chapter deals with a characterization of group digraphs.

4.1 Point - Symmetric Digraphs

A digraph (V, E) is said to be a point - symmetric digraph if for every two vertices u, v of (V, E) there exists at least one digraph automorphism α of (V, E) such that $u\alpha = v$.

As an example, consider the digraph (V, E) in Fig. 4.1.1.

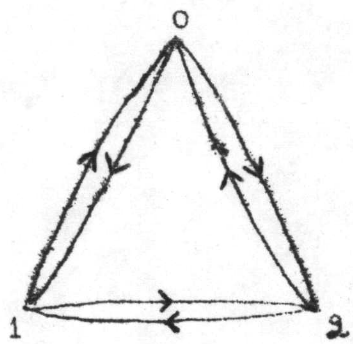

Fig. 4.1.1

Let $\alpha_i : V \longrightarrow V$, $i = 0, 1, 2$ be the following permutations

$$\alpha_0 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\alpha_1 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} .$$

Observe that α_i are digraph automorphisms of (V, E).

To check that for any vertices u, v of (V, E) there is a digraph automorphism α such that $u\alpha = v$, we must find a digraph automorphism α whose permutation representation is of the form

$$\alpha = \begin{pmatrix} \dots u \dots \\ \dots v \dots \end{pmatrix}.$$

Observe that each of the columns $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix},$ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}$ is a column in α_i for some $i = 0, 1, 2$. Hence for any u, v in V, there exists a digraph automorphism α such that $\begin{pmatrix} u \\ v \end{pmatrix}$ is the column in α.

That is, for any u, v in V there exists a digraph automorphism α such that $u\alpha = v$. Hence (V, E) is a point - symmetric digraph.

4.1.1 <u>Theorem</u> Every group digraph is a point - symmetric digraph.

<u>Proof.</u> Let (V,E) be a group digraph. Hence there exists a group G and a subset A of G such that $(V, E) \cong (G, E_A)$.

Let $\varphi : V \longrightarrow G$ be a digraph isomorphism from (V, E) onto (G, E_A).

Let u, w be any two vertices of V.

Let $x = (w\varphi)(u\varphi)^{-1} \in G$.

Define a mapping $\alpha_x : V \longrightarrow V$ as follows.

For each $v \in V$, we put

$$v\alpha_x = [x(v\varphi)]\varphi^{-1}.$$

For every $v_1, v_2 \in V$ such that $v_1\alpha_x = v_2\alpha_x$, we have $[x(v_1\varphi)]\varphi^{-1} = [x(v_2\varphi)]\varphi^{-1}$. Since φ^{-1} is one - to- one, hence $x(v_1\varphi) = x(v_2\varphi)$. Therefore $v_1\varphi = v_2\varphi$. Since φ is one - to- one, Hence $v_1 = v_2$. Therefore α_x is one - to - one.

Let v be any element of V. Hence $v\varphi = y$ for some $y \in G$. Let $v' = (x^{-1}y)\varphi^{-1}$. Then $v' \in V$, and we have

$v'\alpha_x = [x(v'\varphi)]\varphi^{-1} = [x((x^{-1}y)\varphi^{-1}\varphi)]\varphi^{-1} = [x(x^{-1}y)]\varphi^{-1}$

$= y\varphi^{-1} = v$.

Hence α_x is onto.

Next we shall show that α_x is a digraph automorphism of (V, E). For every $v, v' \in V$ we have

$$\begin{aligned}
(v, v') \in E &\iff (v\varphi, v'\varphi) \in E_A \\
&\iff (v\varphi)^{-1} v'\varphi \in A \\
&\iff (v\varphi)^{-1}(x^{-1}x)(v'\varphi) \in A \\
&\iff [x(v\varphi)]^{-1}[x(v'\varphi)] \in A \\
&\iff (x(v\varphi), x(v'\varphi)) \in E_A .
\end{aligned}$$

Since φ^{-1} is a digraph isomorphism from (G, E_A) onto (V, E), hence

$$(x(v\varphi), \ x(v'\varphi)) \in E_A \Longleftrightarrow ([x(v\varphi)]\varphi^{-1}, [x(v'\varphi)]\varphi^{-1}) \in E .$$

Therefore,

$$(v, v') \in E \Longleftrightarrow ([x(v\varphi)]\varphi^{-1}, [x(v'\varphi)]\varphi^{-1}) \in E$$
$$\Longleftrightarrow (v\alpha_x , v'\alpha_x) \in E .$$

Hence α_x is a digraph automorphism of (V, E) .

Also $u\alpha_x = [x(u\varphi)]\varphi^{-1} = [(w\varphi)(u\varphi)^{-1}(u\varphi)]\varphi^{-1}$

$= (w\varphi)\varphi^{-1} = w .$

Hence α_x is a digraph automorphism of (V, E) such that $u\alpha_x = w$.

Therefore (V, E) is a point - symmetric digraph.

<u>Q.E.D.</u>

4.2 G - Group Digraphs

Let (V, E) be a digraph and G be a group. Then (V, E) is said to be a <u>G - group digraph</u> if there is a subset A of the group G such that $(V, E) \cong (G, E_A)$. By definition of a group digraph in section 3.4, we see that (V, E) is a group digraph if it is a G - group digraph for some group G.

4.3 Characterization of a G - Group Digraph

To prove our theorem which give a characterization of a G - group digraph, we need the following lemmas.

4.3.1 Lemma Let (V, E) be a digraph of n vertices. Let Δ be a subgroup of the digraph automorphism group $\Gamma(V, E)$ such that for each pair $v, v' \in V$, there exists $\sigma \in \Delta$ such that $v\sigma = v'$. If $u \in V$ and $\Delta_u = \{\gamma \in \Delta \mid u\gamma = u\}$, then Δ_u is a subgroup of Δ and the index of Δ_u in Δ is n , i.e. $[\Delta : \Delta_u] = n$.

Proof : Let $u \in V$ and $\Delta_u = \{\gamma \in \Delta / u\gamma = u\}$.

Note that $1 \in \Delta_u$, hence $\Delta_u \neq \emptyset$.

Let $\gamma_1, \gamma_2 \in \Delta_u$, hence $u\gamma_1 = u$ and $u\gamma_2 = u$.

Therefore $u(\gamma_1 \gamma_2^{-1}) = (u\gamma_1)\gamma_2^{-1} = u\gamma_2^{-1} = (u\gamma_2)\gamma_2^{-1} = u$.

Thus $\gamma_1 \gamma_2^{-1} \in \Delta_u$. Hence Δ_u is a subgroup of Δ .

Let $\Delta = \Delta_u \alpha_1 \cup \Delta_u \alpha_2 \cup \ldots \cup \Delta_u \alpha_r$ be a decomposition of Δ into cosets relative to Δ_u , i.e. for $i \neq j$ we have $\Delta_u \alpha_i \cap \Delta_u \alpha_j = \emptyset$.

Suppose that $u\alpha_i = u\alpha_j$ for $i \neq j$. Then $u(\alpha_i \alpha_j^{-1}) = u$, which implies that $\alpha_i \alpha_j^{-1} \in \Delta_u$, or equivalently $\Delta_u \alpha_i = \Delta_u \alpha_j$. Hence $\Delta_u \alpha_i \cap \Delta_u \alpha_j = \Delta_u \alpha_i \neq \emptyset$, which is a contradiction. Hence $u\alpha_i \neq u\alpha_j$ when $i \neq j$. Therefore the vertices $u\alpha_1, u\alpha_2, \ldots, u\alpha_r$ are distinct.

Let v be any element of V. Hence there exists $\beta \in \Delta$ such that $u\beta = v$. Since $\beta \in \Delta$, hence $\beta \in \Delta_u \alpha_i$ for some i, $1 \leq i \leq r$, i.e. $\beta = \alpha\alpha_i$ for some $\alpha \in \Delta_u$. Therefore $u\beta = u\alpha\alpha_i = u\alpha_i$. Hence $v = u\alpha_i$ for some i, $1 \leq i \leq r$. Hence $u\alpha_i$ include all vertices of V. Thus r = n, i.e. $[\Delta : \Delta_u] = n$.

<u>Q.E.D.</u>

4.3.2 <u>Lemma</u> Let (V, E) be a digraph of n vertices. Let Δ be a subgroup of $\Pi(V, E)$ of order n such that for each pair $v, v' \in V$, there exists $\delta \in \Delta$ such that $v\delta = v'$. If $\gamma_0 \in \Delta$ is such that $u\gamma_0 = u$ for some $u \in V$, then $\gamma_0 = 1$, the identity of Δ.

<u>Proof</u>: Let $\gamma_0 \in \Delta$, $u \in V$ be such that $u\gamma_0 = u$.

Let $\Delta_u = \{\gamma \in \Delta \mid u\gamma = u\}$. Hence $\gamma_0 \in \Delta_u$.

By lemma 4.3.1, Δ_u is a subgroup of Δ and $[\Delta : \Delta_u] = n$.

Since $|\Delta| = n$, therefore $|\Delta_u| = 1$. Hence $\Delta_u = \{1\}$.

Therefore $\gamma_0 = 1$.

<u>Q.E.D.</u>

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4.3.3 <u>Theorem.</u> Let G be a group. A digraph (V, E) of n vertices is a G - group digraph if and only if (1) G is of order n and (2) the digraph automorphism group $\Pi(V, E)$ of (V, E) contains a subgroup $\Delta \cong G$ such that for each pair $v, v' \in V$ there exists $\sigma \in \Delta$ such that $v\sigma = v'$.

<u>Proof</u> : Let (V, E) be a G - group digraph of n vertices. Hence there exists a subset A of the group G such that $(V, E) \cong (G, E_A)$. By remark 2.4.1, we have $|G| = |V| = n$. Therefore G is a group of order n, i.e. we have (1) .

Let $\varphi : V \to G$ be a digraph isomorphism from (V, E) onto (G, E_A). For each $x \in G$, define a mapping $\alpha_x : V \to V$ as follows. For each $v \in V$, we put

$$v\alpha_x = \{x(v\varphi)\}\varphi^{-1} .$$

By the same argument as given in the proof of theorem 4.1.1, we see that α_x is a digraph automorphism of (V, E).

Let $\Delta = \left\{ \alpha_x \mid x \in G \right\}$.

Clearly Δ is not empty.

Let $\alpha_{x_1}, \alpha_{x_2} \in \Delta$ and $v \in V$. Hence

$$v(\alpha_{x_1}\alpha_{x_2}^{-1}) = (v\alpha_{x_1})\alpha_{x_2}^{-1} = [(x_1(v\varphi)\varphi^{-1}]\alpha_{x_2}^{-1} .$$

Let $[(x_1(v\varphi))\varphi^{-1}]\alpha_{x_2}^{-1} = w$. Hence

$(x_1(v\varphi))\varphi^{-1} = w\alpha_{x_2} = (x_2(w\varphi))\varphi^{-1}$. Hence

$x_1(v\varphi) = x_2(w\varphi)$. Therefore $x_2^{-1}x_1(v\varphi) = w\varphi$.

Hence $[x_2^{-1}x_1(v\varphi)]\varphi^{-1} = w$. Therefore

$v(\alpha_{x_1}\alpha_{x_2}^{-1}) = [x_2^{-1}x_1(v\varphi)]\varphi^{-1} = v\alpha_{x_2^{-1}x_1}$.

Hence $\alpha_{x_1}\alpha_{x_2}^{-1} = \alpha_{x_2^{-1}x_1} \in \Delta$. Therefore Δ is a subgroup of $\Gamma(V, E)$.

Claim that $\Delta \cong G$.

Define a mapping $\theta : G \to \Delta$ as follows.

For each $x \in G$, we put $x\theta = \alpha_{x^{-1}}$.

Let $x_1, x_2 \in G$ be such that $x_1\theta = x_2\theta$. Hence $\alpha_{x_1^{-1}} = \alpha_{x_2^{-1}}$.

Therefore $v\alpha_{x_1^{-1}} = v\alpha_{x_2^{-1}}$ for all $v \in V$. Thus

$[x_1^{-1}(v\varphi)]\varphi^{-1} = [x_2^{-1}(v\varphi)]\varphi^{-1}$.

Hence $x_1^{-1}(v\varphi) = x_2^{-1}(v\varphi)$. It follows that $x_1 = x_2$. Hence θ is one - to - one.

Note that for each $\alpha_x \in \Delta$, we have $x^{-1}\theta = \alpha_x$. Hence θ is onto.

Let $v \in V$ and $x_1, x_2 \in G$. Then we have

$$
\begin{aligned}
v((x_1x_2)\theta) &= v\,\alpha_{(x_1x_2)^{-1}} \\
&= [(x_1x_2)^{-1}(v\varphi)]\varphi^{-1} \\
&= [x_2^{-1}x_1^{-1}(v\varphi)]\varphi^{-1} \\
&= [x_2^{-1}((x_1^{-1}(v\varphi))\varphi^{-1}\varphi)]\varphi^{-1} \\
&= [x_2^{-1}((v\alpha_{x_1^{-1}})\varphi)]\varphi^{-1} \\
&= (v\alpha_{x_1^{-1}})\alpha_{x_2^{-1}} \\
&= v(\alpha_{x_1^{-1}}\alpha_{x_2^{-1}}) \\
&= v(x_1\theta\, x_2\theta).
\end{aligned}
$$

Therefore $(x_1x_2)\theta = x_1\theta\, x_2\theta$. Hence θ is an isomorphism from G onto Δ, i.e. $\Delta \cong G$.

Let v, v' be any two elements of V. Let $x = (v'\varphi)(v\varphi)^{-1}$. Hence $x \in G$ and we have

$$v\,\alpha_x = [x(v\varphi)]\varphi^{-1} = [(v'\varphi)(v\varphi)^{-1}(v\varphi)]\varphi^{-1} = (v'\varphi)\varphi^{-1} = v'$$

Hence there exists $\alpha_x \in \Delta$ such that $v\alpha_x = v'$.

Therefore $\Gamma(V, E)$ has a subgroup Δ such that $\Delta \cong G$ and for each pair $v, v' \in V$ there exists $\delta \in \Delta$ such that $v\delta = v'$.

Hence we have (2).

Conversely, let (V, E) be a digraph of n vertices. Let G be a group such that (1) G is of order n and (2) the digraph automorphism group $\Pi(V, E)$ of (V,E) contains a subgroup $\Delta \cong G$ such that for each pair $v, v' \in V$ there exists $\sigma \in \Delta$ such that $v\sigma = v'$.

<u>Case 1</u> Suppose $E = \emptyset$

Let $V = \{v_1, v_2, \ldots, v_n\}$ and $G = \{x_1, x_2, \ldots, x_n\}$.

Let $A = \emptyset$. Hence

$$E_A = \left\{(x, y) \in G \times G \;\middle|\; x, y \in G,\ x^{-1}y \in A\right\} = \emptyset .$$

Define $\varphi : V \longrightarrow G$ by putting $v_i\varphi = x_i$ for $1 \leq i \leq n$.

Clearly φ is a digraph isomorphism from (V, E) onto (G, E_A). Therefore $(V, E) \cong (G, E_A)$. Hence (V, E) is a G - group digraph.

<u>Case 2</u> Suppose $E \neq \emptyset$. Hence there exists at least one arc in E. Let (w, u) be an arc in E.

Let $A^* = \left\{\sigma \in \Delta \;\middle|\; (u\sigma, u) \in E\right\}$.

Since $w, u \in V$, hence there exists $\sigma_0 \in \Delta$ such that $u\sigma_0 = w$. Therefore $(u\sigma_0, u) = (w, u) \in E$. Hence $A^* \neq \emptyset$.

Let (Δ, E_{A^*}) be the digraph induced by the group Δ and the subset A^*.

We shall show that $(\Delta, E_{A^*}) \cong (V, E)$.

Define a mapping $\varphi^* : \Delta \longrightarrow V$ as follows.

For each $\sigma \in \Delta$, we put $\sigma\varphi^* = u\sigma^{-1}$.

Let σ_1, σ_2 be any element of Δ such that $\sigma_1\varphi^* = \sigma_2\varphi^*$.

Then $u\sigma_1^{-1} = u\sigma_2^{-1}$. Hence $u\sigma_1^{-1}\sigma_2 = u$. Since $\sigma_1, \sigma_2 \in \Delta$, hence $\sigma_1^{-1}\sigma_2 \in \Delta$ which fixes an element u. Hence by lemma 4.3.2, we have $\sigma_1^{-1}\sigma_2 = 1$. Therefore $\sigma_1 = \sigma_2$. Hence φ^* is one - to - one.

Let v be any element of V. Hence there exists some $\sigma' \in \Delta$ such that $v\sigma' = u$. That is $v = u(\sigma')^{-1} = \sigma'\varphi^*$. Hence φ^* is onto.

Finally we shall show that φ^* is a digraph isomorphism from (Δ, E_A^*) onto (V, E).

Let σ_1, σ_2 be any element of Δ.

By definitions of E_A^* and A^* we see that

$$(\sigma_1, \sigma_2) \in E_A^* \iff \sigma_1^{-1}\sigma_2 \in A^*$$

$$\iff (u\sigma_1^{-1}\sigma_2, u) \in E.$$

Since $\sigma_2^{-1} \in \Gamma(V, E)$, hence

$$(u\sigma_1^{-1}\sigma_2, u) \in E \iff (u\sigma_1^{-1}, u\sigma_2^{-1}) \in E$$

$$\iff (\sigma_1\varphi^*, \sigma_2\varphi^*) \in E.$$

Therefore $(\sigma_1, \sigma_2) \in E_A^* \iff (\sigma_1\varphi^*, \sigma_2\varphi^*) \in E$.

Hence φ^* is a digraph isomorphism from (Δ, E_A^*) onto (V, E).

Therefore $(V, E) \cong (\Delta, E_A^*)$.

By (2) we have $\Delta \cong G$. Hence there exists an isomorphism θ from Δ onto G.

Let $A = A^*\theta$ where $A^*\theta = \{ \sigma\theta \,/\, \sigma \in A^* \}$.

By theorem 3.3.3, we have $(\Delta, E_A^*) \cong (G, E_A)$. Hence by remark 2.4.2, we have $(V, E) \cong (G, E_A)$.

Therefore (V, E) is a G - group digraph.

<u>Q.E.D.</u>

4.3.4 <u>Corollary</u> A digraph (V, E) with n vertices is a group digraph if and only if its digraph automorphism group $\Gamma(V,E)$ contains a subgroup Δ of order n such for each pair $v, v' \in V$ there exists $\sigma \in \Delta$ such that $v\sigma = v'$.

<u>Proof.</u> Let (V, E) be a group digraph of n vertices. Hence (V, E) is a G- group digraph for some group G. Hence by theorem 4.3.3, the digraph automorphism group $\Gamma(V, E)$ of (V,E) contains a subgroup Δ of order n such that for each pair $v, v' \in V$ there exists $\sigma \in \Delta$ such that $v\sigma = v'$.

Conversely, let (V, E) be a digraph of n vertices and the digraph automorphism group $\Gamma(V, E)$ of (V, E) contains a subgroup Δ of order n such that for each $v, v' \in V$ there exists $\sigma \in \Delta$ such that $v\sigma = v'$. Then by theorem 4.3.3, (V, E) is a Δ - group digraph. Hence (V, E) is a group digraph

<u>Q.E.D.</u>

4.3.5 Corollary Let (V, E) be a digraph of p vertices, p a prime. Then (V, E) is a cyclic group digraph if and only if (V, E) is a point- symmetric digraph.

Proof: Let (V, E) be a cyclic group digraph of p vertices. Hence, by theorem 4.1.1, (V, E) is point - symmetric.

Conversely, let (V, E) be a point - symmetric digraph of p vertices.

Let $\Pi(V, E)$ be the group of all digraph automorphisms of (V,E). Hence for every v, $v' \in V$, there exists $\sigma \in \Pi(V, E)$ such that $v\sigma = v'$.

Let $u \in V$ and $\Delta_u = \{ \gamma \in \Pi(V, E) / u\gamma = u \}$. By lemma 4.3.1, we have $[\Pi(V, E) : \Delta_u] = p$. Hence p is a divisor of the order of $\Pi(V, E)$. By the first Sylow theorem, $\Pi(V, E)$ contains a subgroup of order p which is cyclic. Let $\Delta = \{ 1, \sigma, \sigma^2, \ldots, \sigma^{p-1} \}$ be this subgroup. Since $\sigma \in \Pi(V, E)$ we may regard it as a permutation of the vertices of (V, E). Hence σ has a unique representation as a product of disjoint cycles. Since σ has order p, hence p is the least common multiple of the orders of its component cycles. Hence each component cycle must have order 1 or p, i.e. each component cycle must have length 1 or p. If some component cycles have length 1, then all cycles must have length 1. In such a case σ must be the identity. Hence the decomposition of σ into cycles must give exactly one cycle of length p. Relabel the vertices so that σ may be represented as the cycle $(v_0\ v_1 \dots v_{p-1})$. Then for any i, $j = 0, 1, \dots, p-1$ we have

$$v_i\, \sigma^j = v_{i+j} ,$$

where $i + j$ is reduced modulo p.

Let v_i, v_k be any pair of vertices of (V, E). Since $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$ forms group under addition, we can find $j \in \mathbb{Z}_p$ such that $i + j = k$. For this choice of j we have $\sigma^j \in \Delta$ and $v_i \sigma^j = v_k$.

Hence $\Gamma(V, E)$ has a subgroup Δ of order p such that for every $v_i, v_k \in V$ there exists $\sigma^j \in \Delta$ such that $v_i \sigma^j = v_k$. Hence by theorem 4.3.3, $(V, E) \cong (\Delta, E_A^*)$. Since Δ is cyclic. Hence (V, E) is a cyclic group digraph.

Q.E.D.