CHAPTER II

METHOD OF ANALYSIS

2.1 Proposed Deflection Function for Case 1

Taking the co-ordinate axes as shown in Fig.1 and in view of symmetry with respect to x and y axis, an approximate solution can be constructed by assuming the deflection function in the form of polynomials of even degree as follows.

$$
\begin{aligned}
w &= C_1 + C_2x^2 + C_3y^2 + C_4x^4 + C_5x^2y^2 + C_6y^4 + C_7x^6 + C_8x^4y^2 \\
&\quad + C_9x^2y^4 + C_{10}y^6 + C_{11}x^4y^4 \qquad (1)
\end{aligned}
$$

Rotate the co-ordinate axes by 60°, w becomes, in terms of ξ and η,

$$
\begin{aligned}
w &= C_1 + \frac{1}{4}(C_2 + 3C_3)\,\xi^2 + \frac{1}{4}(3C_2 + C_3)\,\eta^2 - \frac{\sqrt{3}}{2}(C_2 - C_3)\,\xi\eta \\
&\quad + \frac{1}{16}(C_4 + 3C_5 + 9C_6)\,\xi^4 - \frac{\sqrt{3}}{4}(C_4 + C_5 - 3C_6)\,\xi^3\eta \\
&\quad + \frac{1}{16}(9C_4 + 3C_5 + C_6)\,\eta^4 + \frac{1}{64}(C_7 + 3C_8 + 9C_9 + 27C_{10})\,\xi^6 \\
&\quad - \frac{\sqrt{3}}{32}(3C_7 + 5C_8 + 3C_9 - 27C_{10})\,\xi^5\eta + \frac{1}{64}(45C_7 + 31C_8 - 27C_9 \\
&\quad + 135C_{10})\,\xi^4\eta^2 - \frac{\sqrt{3}}{16}(15C_7 + C_8 - C_9 - 15C_{10})\,\xi^3\eta^3 + \frac{1}{64}(135C_7 \\
&\quad - 27C_8 + 31C_9 + 45C_{10})\,\xi^2\eta^4 - \frac{\sqrt{3}}{32}(27C_7 - 3C_8 - 5C_9 - 3C_{10})\,\xi\eta^5 \\
&\quad + \frac{1}{64}(27C_7 + 9C_8 + 3C_9 + C_{10})\,\eta^6 + \frac{1}{256}(9\xi^8 + 36\xi^6\eta^2 - 24\sqrt{3}\xi^7\eta \\
&\quad + 40\sqrt{3}\,\xi^5\eta^3 - 74\,\xi^4\eta^4 - 40\sqrt{3}\,\xi^3\eta^5 + 36\,\xi^2\eta^6 + 24\sqrt{3}\xi\eta^7 \\
&\quad + 9\eta^8)\,C_{11} + \frac{1}{8}(9C_4 - C_5 + 9C_6)\,\xi^2\eta^2 - \frac{\sqrt{3}}{4}(3C_4 - C_5 - C_6)\,\xi\eta^3 \qquad (2)
\end{aligned}
$$

in which

$$x = \frac{1}{2}(\xi - \sqrt{3}\eta) \tag{3a}$$

$$y = \frac{1}{2}(\sqrt{3}\,\xi + \eta) \tag{3b}$$

and ξ, η denote the rotated co-ordinate axes. From Fig. 1, it is seen that the deflection surface of the plate is also symmetrical with respect to the ξ- η axes, hence only even functions of ξ and η in the expression (2) are kept. Therefore, the following results can be obtained.

$$C_2 = C_3,\ C_4 = \frac{C_5}{2} = C_6,\ C_7 = -\frac{1}{15}C_8 = \frac{1}{15}C_9 = -C_{10},\ C_{11} = 0 \tag{4}$$

Substituting eq. (4) into the expression (1) and rearranging, one has

$$w = C_1 + C_2(x^2 + y^2) + C_4(x^2+y^2)^2 + C_7\,(x^6 - 15x^4y^2 + 15x^2y^4 - y^6) \tag{5}$$

The remaining four constants in eq. (5) must be so chosen that the deflection function satisfy the boundary conditions and the equilibrium equation. Consider one of the edges of the hexagonal plate, the exact boundary conditions are

$$w\Big|_{a,\frac{a}{\sqrt{3}}} = 0 \tag{6}$$

$$M_x\Big|_{a,y} = 0\ ,\quad 0 < y < \frac{a}{\sqrt{3}} \tag{7}$$

$$V_x\Big|_{a,y} = 0\ ,\quad 0 < y < \frac{a}{\sqrt{3}} \tag{8}$$

To facilitate the solution, the boundary conditions (7) and (8) are replaced by the vanishing of the total bending moment and the total effective transverse shear force. That is

$$\int_0^{\frac{a}{\sqrt{3}}} \left\{M_x\right\}_{a,y} dy = \int_0^{\frac{a}{\sqrt{3}}} D\left[\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right]_{a,y} dy = 0 \qquad (9)$$

$$\int_0^{\frac{a}{\sqrt{3}}} \left[V_x\right]_{a,y} dy = \int_0^{\frac{a}{\sqrt{3}}} D\left[\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]_{a,y} dy = 0 \qquad (10)$$

Now, three of the remaining constants can be found by forcing the deflection function to satisfy the boundary conditions (6), (9) and (10). Substituting eq. (5) into eqs. (6), (9) and (10) yields three simultaneous algebraic equations as follows :

$$C_1 + \frac{4}{3}a^2C_2 + \frac{16}{9}a^2C_4 - \frac{64}{27}a^6C_7 = 0 \qquad (11)$$

$$2(1+\nu)C_2 + \frac{16}{9}a^2(7+3\nu)C_4 + \frac{32}{3}a^4(1-\nu)C_7 = 0 \qquad (12)$$

$$(5-\nu)C_4 - 10(1-\nu)a^2C_7 = 0 \qquad (13)$$

Solving for C_2, C_4 and C_7 in term of C_1, one gets

$$C_2 = -\frac{3(25+6\nu)}{(95-68\nu-11\nu^2)}\frac{1}{a^2}C_1 \qquad (14)$$

$$C_4 = \frac{135}{16}\frac{1-\nu^2}{(95-68\nu-11\nu^2)}\frac{1}{a^4}C_1 \qquad (15)$$

$$C_7 = \frac{27}{32}\frac{(5-\nu)(1+\nu)}{(95-68\nu-11\nu^2)}\frac{1}{a^6}C_1 \qquad (16)$$

Rewriting eqs. (12), (13) and (14) as

$$C_2 = kc_1 \qquad (17)$$

$$C_4 = m\, C_1 \tag{18}$$

$$C_7 = n\, C_1 \tag{19}$$

where

$$k = -\frac{3(25 + 6\nu)(1 - \nu)}{(95 - 68\nu - 11\nu^2)} \frac{1}{a^2}$$

$$m = \frac{135}{16} \frac{1 - \nu^2}{(95 - 68\nu - 11\nu^2)} \frac{1}{a^4}$$

$$n = \frac{27}{32} \frac{(5 - \nu)(1 + \nu)}{(95 - 68\nu - 11\nu^2)} \frac{1}{a^6}$$

Then the proposed deflection function becomes

$$w = C_1 \left[1 + k(x^2 + y^2) + m(x^2 + y^2)^2 + n(x^6 - 15x^4y^2) + 15x^2y^4 - y^6 \right] \tag{20}$$

2.2 Solution for Case 1.

In terms of cartesian co-ordinates, the well-known differential equation of bending of plates is[2]

$$\nabla^4 w = \frac{q}{D} \tag{21}$$

where w = small deflection of the plate in the z-direction

$$\nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \tag{22}$$

q = uniformly distributed load

D = the flexural rigidity of the plate

Substituting eq. (20) in eq. (21), the coefficient C_1 can now be determined as

$$C_1 = \frac{1}{540} \frac{(95 - 68\nu - 11\nu^2)}{1 - \nu^2} \frac{qa^4}{D} \tag{23}$$

Upon the substitution of C_1, k, m and n into eq. (20), one obtains the approximate solution of uniformly loaded hexagonal plates.

$$w = \frac{1}{540} \{\frac{qa^4}{D} (\frac{95 - 68\nu - 11\nu^2}{1 - \nu^2} - 3 \frac{25 + 6\nu}{1 + \nu} (\frac{x^2}{a^2} + \frac{y^2}{a^2}) + \frac{135}{16} (\frac{x^2}{a^2} + \frac{y^2}{a^2})^2 + \frac{27}{32} \frac{5 - \nu}{1 - \nu} (\frac{x^6}{a^6} - 15 \frac{x^4 y^2}{a^6} + 15 \frac{x^2 y^4}{a^6} + \frac{y^6}{a^6})\} \tag{24}$$

The expression for the bending moment, twisting moment, shearing forces and effective transverse shear force are[2]

$$M_x = - D(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2})$$

$$M_x = \frac{qa^4}{540} \{6(25 + 6\nu) - \frac{135}{4} \left[(3 + \nu)\frac{x^2}{a^2} + (1 + 3\nu)\frac{y^2}{a^2}\right] - \frac{405}{16}(5 - \nu) \left[\frac{x^4}{a^4} - \frac{6x^2 y^2}{a^4} + \frac{y^4}{a^4}\right]\} \tag{25}$$

$$M_y = - D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)$$

$$M_y = \frac{qa^4}{540}\{6(25 + 6\nu) - \frac{135}{4}\left[(1+3\nu)\frac{x^2}{a^2} + (3 + \nu)\frac{y^2}{a^2}\right] + \frac{405}{16}(5 - \nu)$$
$$\left[\frac{x^4}{a^4} - \frac{6x^2y^2}{a^4} + \frac{y^4}{a^4}\right]\} \qquad (26)$$

$$M_{xy} = D(1 - \nu)\frac{\partial^2 w}{\partial x \partial y}$$

$$M_{xy} = - \frac{qa^2}{16}\frac{xy}{a^2}\left[3(5 - \nu)\left(\frac{x^2}{a^2} - \frac{y^2}{a^2}\right) - 2(1 - \nu)\right] \qquad (27)$$

$$Q_x = - D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$Q_x = - \frac{qx}{2} \qquad (28)$$

$$Q_y = - D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$Q_y = - \frac{qy}{2} \qquad (29)$$

$$V_x = - D\left[\frac{\partial^3 w}{\partial x^3} + (2 - \nu)\frac{\partial^3 w}{\partial x \partial y^2}\right]$$

$$= \frac{qx}{16}\left[3(5-\nu)\left(\frac{x^2}{a^2} - 3\frac{y^2}{a^2}\right) - 2(5-\nu)\right] \qquad (30)$$

The corner force is

$$R = 2\left[M_{xy}\right]_{a, \frac{a}{\sqrt{3}}}$$

$$= \frac{1}{\sqrt{3}} qa^2 \qquad (31)$$

as should be expected from equilibrium

2.3 Proposed Deflection Function for Case 2.

The proposed deflection function of the hexagonal plates supported at the corners and loaded at the center will now be constructed by assuming the deflection function in the form

$$w = C_1 + C_2x^2 + C_3y^2 + C_4x^6 + C_5x^4y^2 + C_6x^2y^4 + C_7y^6$$
$$+ C_8x^2 \ln \frac{\sqrt{x^2 + y^2}}{a} + C_9y^2 \ln \frac{\sqrt{x^2 + y^2}}{a} \tag{32}$$

The co-ordinate axes are located as shown in Fig. 1. The logarithmic terms have been included in view of the singularity at the center of the plate. By rotating the co-ordinates axes through an angle of 60° eq. (31) becomes :

$$w = C_1 + \frac{1}{4}(C_2 + 3C_3)\xi^2 + \frac{1}{4}(3C_2 + C_3)\eta^2 - \frac{\sqrt{3}}{2}(C_2 - C_3)\xi\eta$$
$$+ \frac{1}{64}(C_4 + 3C_5 + 9C_6 + 27C_7)\xi^6 - \frac{\sqrt{3}}{32}(3C_4 + 5C_5 + 3C_6 - 27C_7)\xi^5\eta$$
$$+ \frac{1}{64}(45C_4 + 31C_5 - 27C_6 + 135C_7)\xi^4\eta^2$$
$$- \frac{\sqrt{3}}{32}(15C_4 + C_5 - C_6 - 15C_7)\xi^3\eta^3$$
$$+ \frac{1}{64}(135C_4 - 27C_5 + 31C_6 + 45C_7)\xi^2\eta^4$$
$$- \frac{\sqrt{3}}{32}(27C_4 - 3C_5 - 5C_9 - 3C_7)\xi\eta^5$$
$$+ \frac{1}{64}(27C_4 + 9C_5 + 3C_6 + C_{10})\eta^6 + \frac{1}{4}(C_8 + 3C_9)\xi^2 \ln \frac{\sqrt{\xi^2 + \eta^2}}{a}$$
$$- \frac{\sqrt{3}}{2}(C_8 - C_9)\xi\eta \ln \frac{\sqrt{\xi^2 + \eta^2}}{a} + \frac{1}{4}(3C_8 + C_9)\eta^2 \ln \frac{\sqrt{\xi^2 + \eta^2}}{a} \tag{33}$$

From Fig. 1, it is seen that the deflection surface of the plate is also symmetrical with respect to the ξ-η axis, hence only even functions of ξ and η in the expression (32) are kept. Therefore, the following results can be obtained

$$C_2 = C_3,\ C_4 = -\frac{1}{15} C_5 = \frac{1}{15} C_6 = -C_7,\ C_8 = C_9 \tag{34}$$

Substituting eq. (33) into the expression (31) and rearranging, one has

$$\begin{aligned} w &= C_1 + C_2(x^2 + y^2) + C_4\,(x^6 - 15x^4y^2 + 15x^2y^4 - y^6) \\ &\quad + C_8\,(x^2 + y^2)\ \ln \frac{\sqrt{x^2 + y^2}}{a} \end{aligned} \tag{35}$$

By employing the boundary conditions in eqs. (6), (9) and (10) the following three simultaneous algebraic equations are obtained as:

$$C_1 + \frac{4}{3} a^2 C_2 - \frac{64}{27} a^2 C_4 + \frac{4}{3} a^2 C_2 \ \ln \frac{2}{\sqrt{3}} = 0 \tag{36}$$

$$\begin{aligned} &2(1 + \nu)\ C_2 + \frac{32}{3}\ a^4\ (1 - \nu)\ C_4 \\ &+ \left[\frac{2\pi}{\sqrt{3}} - (1 - \nu) + 2\ (1+\nu)\ \ln \frac{2\pi}{\sqrt{3}}\right] C_8 = 0 \end{aligned} \tag{37}$$

$$-\,80\ (1-\nu)\ a^2 C_4 + \left[\frac{2\pi}{\sqrt{3}} + \frac{3}{2}\,(1-\nu)\right] C_8 = 0 \tag{38}$$

Solving for C_2, C_4 and C_8 in term of C_1 one gets

$$C_2 = -\frac{\frac{160}{3}\left[\frac{7\pi}{5\sqrt{3}} - \frac{6}{5} + 3\ln\frac{2}{\sqrt{3}} - \left(\frac{17\pi}{5\sqrt{3}} - 3\right)\nu - \left(\frac{9}{5} + 3\ln\frac{2}{\sqrt{3}}\right)\nu^2\right]}{\frac{128}{27}\left[\frac{53\pi}{\sqrt{3}} - \frac{33}{2} - \left(\frac{49\pi}{\sqrt{3}} - 45\right)\nu - \frac{57}{2}\nu^2\right]}\ \frac{C_1}{a^2} \tag{39}$$

$$C_4 = \frac{2\left[\frac{2\pi}{\sqrt{3}} + \frac{3}{2} + \frac{2\pi}{\sqrt{3}}\nu - \frac{3}{2}\nu^2\right]}{\frac{128}{27}\left[\frac{53\pi}{\sqrt{3}} - \frac{33}{2} - \left(\frac{49\pi}{\sqrt{3}} - 45\right)\nu - \frac{57}{2}\nu^2\right]}\ \frac{C_1}{a^6} \tag{40}$$

$$C_8 = - \frac{16\left[5 + 4\nu - \nu^2\right]}{\frac{128}{27}\left[\frac{53\pi}{\sqrt{3}} - \frac{33}{2}\left(\frac{49\pi}{\sqrt{3}} - 45\right)\nu - \frac{57}{2}\nu^2\right]} \frac{C_1}{a^2} \tag{41}$$

or in the forms of eqs. (17), (18) and (19), the constants k, m and n are

$$k = - \frac{\frac{160}{3}\left[\frac{7\pi}{5\sqrt{3}} - \frac{6}{5} + 3\ln\left(\frac{17\pi}{5\sqrt{3}} - 3\right)\nu - \left(\frac{9}{5} + 3\ln\frac{2}{\sqrt{3}}\right)\nu^2\right]}{\frac{128}{27}\left[\frac{53\pi}{\sqrt{3}} - \frac{33}{2} - \left(\frac{49\pi}{\sqrt{3}} - 45\right)\nu - \frac{57}{2}\nu^2\right]} \frac{1}{a^2} \tag{42}$$

$$m = \frac{2\left[\frac{2\pi}{\sqrt{3}} + \frac{3}{2} + \frac{2\pi}{\sqrt{3}}\nu - \frac{3}{2}\nu^2\right]}{\frac{128}{27}\left[\frac{53\pi}{\sqrt{3}} - \frac{33}{2} - \left(\frac{49\pi}{\sqrt{3}} - 45\right)\nu - \frac{57}{2}\nu^2\right]} \frac{1}{a^6} . \tag{43}$$

$$n = - \frac{16\left[5 + 4\nu - \nu^2\right]}{\frac{128}{27}\left[\frac{53\pi}{\sqrt{3}} - \frac{33}{2} - \left(\frac{49\pi}{\sqrt{3}} - 45\right)\nu - \frac{57}{2}\nu^2\right]} \frac{1}{a^2} \tag{44}$$

Finally the proposed deflection function becomes

$$w = C_1\left[1 + k(x^2 + y^2) + m\,(x^6 - 15x^4y^2 + 15x^2y^4 - y^6) + n\,(x^2 + y^2)\ln\frac{\sqrt{x^2 + y^2}}{a}\right] \tag{45}$$

2.4 Solution for Case 2.

The energy method will be used to find the solution for the case of a concentrated load applied at the center of the plate. The strain energy in pure bending of the plate is

$$V = \frac{D}{2}\iint\left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2\,(1 - \nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right]\right\} dA \tag{46}$$

Substituting eq. (44) into eq. (45), one gets

$$
\begin{aligned}
V = \frac{D}{2} C_1^2 \int_0^a \int_0^{x/\sqrt{3}} \{ & 8(1 + \nu)\ (k^2 + kn) + 2\ (5 + 3\nu)\ n^2 \\
& + 16n\ (k + n)\ (1 + \nu)\ \ln \frac{\sqrt{x^2 + y^2}}{a} + 8\ (1 + \nu)\ n^2\ \left(\ln \frac{\sqrt{x^2 + y^2}}{a}\right)^2 \\
& + 1800\ (1 - \nu)\ m^2\ (x^8 + 4x^6 y^2 + 6x^4 y^4 + 4x^2 y^6 + y^8) \\
& + 120\ (1 - \nu)\ mn\ (x^4 - y^4) \\
& - 1920\ (1 - \nu)\ mnx^2 y^2\ \frac{x^2 - y^2}{x^2 + y^2} \}\ dydx
\end{aligned} \tag{47}
$$

To facilitate the solution, the integration limits of the logarithmic terms shall be replaced by those of a circular plate of equal area, i.e. $\sqrt{\frac{2\sqrt{3}}{\pi}}$ a and $\frac{\pi}{6}$. Transformation from cartesian co-ordinates to polar co-ordinates is accomplished by the following relationships.

$$r^2 = x^2 + y^2$$

and
$$dA = rdrd\theta \tag{48}$$

Thus, eq. (46) becomes,

$$
\begin{aligned}
V = 6DC_1^2 \{ & \int_0^a \int_0^{x/\sqrt{3}} \Big[8(1 + \nu)\ (k^2 + kn) + 2\ (5 + 3\nu)n^2 \\
& + 1800\ (1 - \nu)m^2\ (x^8 + 4x^6 y^2 + 6x^4 y^4 + 4x^2 y^6 + y^8) \\
& + 120\ (1 - \nu)mn\ (x^4 - y^4) - 1920\ (1 - \nu)\ mnx^2 y^2\ \frac{x^2 - y^2}{x^2 + y^2} \Big]\ dydx \\
& + \int_0^{\frac{\pi}{6}} \int_0^{\sqrt{\frac{2\sqrt{3}}{\pi}}a} \Big[16n(k + n)\ (1 + \nu)\ln\frac{r}{a} + 8(1 + \nu)n\left(\ln\frac{r}{a}\right)^2 \Big]\ rdrd\theta \} \\
= 6DC_1^2 \{ & \frac{4a^2}{\sqrt{3}}\ (1 + \nu)\ \left(k^2 + nk - \frac{n^2}{2}\right) + \frac{a^2}{\sqrt{3}}\ (5 + 3\nu)\ n^2 \\
& + \frac{163328}{567\sqrt{3}}\ a^{10}\ (1 - \nu)\ m^2 + \frac{184}{9\sqrt{3}}\ a^6\ (1 - \nu)\ mn - \frac{360}{\pi^3}\ a^6\ (1 - \nu)mn \\
& + \frac{4a^2}{\sqrt{3}}\ (1 + \nu)\ln\sqrt{\frac{2\sqrt{3}}{\pi}} \left[n\ (2k + n) + n^2 \ln\sqrt{\frac{2\sqrt{3}}{\pi}} \right] \}
\end{aligned} \tag{49}
$$

The potential energy due to the concentrated load P is

$$PC_1 = Pw \Big|_{x = 0,\, y = 0} \tag{50}$$

therefore the total energy of the system for concentrated load P applied at center is

$$\begin{aligned} I = 6DC_1^2 \Big\{ & \frac{4a^2}{\sqrt{3}} (1 + \nu)\left(k^2 + nk - \frac{n^2}{2}\right) + \frac{a^2}{\sqrt{3}} (5 + 3\nu)\, n^2 \\ & + \frac{163328}{567\sqrt{3}}\, a^{10} (1-\nu) m^2 + \frac{184}{9\sqrt{3}}\, a^6 (1-\nu)\, mn - \frac{360}{\pi^3}\, a^6 (1-\nu)\, mn \\ & + \frac{4a^2}{\sqrt{3}} (1+\nu) \ln\sqrt{\frac{2\sqrt{3}}{\pi}} \Big\{ n(2k+n) + n^2 \ln\sqrt{\frac{2\sqrt{3}}{\pi}} \Big\} - PC_1 \end{aligned} \tag{51}$$

The coefficient C_1 can now be determined by taking the derivative of eq. (51) with respect to C_1 and equating the result to zero. Hence one obtains

$$C_1 = \frac{P}{120} \; \frac{1}{\begin{aligned} & \frac{4a^2}{\sqrt{3}} (1-\nu) \left[\left(k^2 + nk - \frac{n^2}{2}\right) + n\,(2k+n) \ln\sqrt{\frac{2\sqrt{3}}{\pi}} + n^2 \left(\ln\sqrt{\frac{2\sqrt{3}}{\pi}}\right)^2 \right] \\ & + \frac{a^2}{\sqrt{3}} (5+3\nu) n^2 + 4a^6 (1-\nu) \left[\frac{40832}{567\sqrt{3}}\, a^4 m^2 + \frac{46}{9\sqrt{3}}\, mn - \frac{90}{\pi^3}\, mn \right] \end{aligned}} \tag{52}$$

The expressions for the bending moments, twisting moment, shearing forces and effective transverse shear force are

$$\begin{aligned} M_x = - DC_1 \Big[& 2k(1+\nu) + 30m\,(1-\nu)\,(x^4 - 6x^2y^2 + y^2) \\ & + \frac{2n}{x^2 + y^2} (x^2 + \nu y^2) + n\,(1+\nu) \left(2\ln\sqrt{\frac{x^2}{a^2} + \frac{y^2}{a^2}} + 1\right) \Big] \end{aligned} \tag{53}$$

$$\begin{aligned} M_y = - DC_1 \Big[& 2k(1+\nu) - 30m\,(1-\nu)\,(x^4 - 6x^2y^2 + y^4) \\ & + \frac{2n}{x^2 + y^2} (\nu x^2 + y^2) + n\,(1+\nu) \left(2\ln\sqrt{\frac{x^2}{a^2} + \frac{y^2}{a^2}} + 1\right) \Big] \end{aligned} \tag{54}$$

$$M_{xy} = -2D(1 - \nu)\, C_1 xy \left[60m\,(x^2 - y^2) - \frac{n}{x^2 + y^2}\right] \tag{55}$$

$$Q_x = -4D\, C_1 n \frac{x}{x^2 + y^2} \tag{56}$$

$$Q_y = -4D\, C_1 n \frac{y}{x^2 + y^2} \tag{57}$$

$$V_x = 2D(1 - \nu) C_1\, x \left[60m(x^2 - 3y^2) + \frac{2ny^2}{(x^2 + y^2)^2} - \frac{1 + \nu}{1 - \nu}\, \frac{n}{x^2 + y^2}\right] \tag{58}$$

The corner force is

$$R = 2(M_{xy})_{a,\, \frac{a}{\sqrt{3}}}$$

For $\nu = 0.3$

$$R = 0.1447\ P \tag{59}$$

which does not agree with the equilibrium requirement, $R = \frac{P}{6}$, because of the approximate boundary conditions on bending moment and effective transverse shear force. The other reasons are probably due to the energy method and the selected deflection function which can not describe the behavior of the plate under the concentrated load accurately. The total potential energy is not minimized exactly. However, it will be shown later that the proposed solution yields reasonable agreement with the experimental results, even though the proposed approximate method of solution does not satisfy equilibrium exactly.