CHAPTER IV

EXISTENCE AND UNIGUENESS THEOREM FOR SOLUTIONS OF DIFFERENTIAL AND INTEGRAL EQUATIONS

In this chapter we show some applications of Banach's contraction theorem. By using this theorem we prove the existence and uniqueness for differential and integral equations.

The materials of this chapter are drawn from reference [4].

1. Existence and uniqueness theorem for solutions of differential equations

4.1.1 **Theorem.** Given a function $f(x, y)$ defined and continuous on a plane domain G containing the point (x_o, y_o). Suppose f satisfies a Lipschitz condition, i.e. there exists a real number M such that

$$\left| f(x, y) - f(x, \tilde{y}) \right| \leq M \left| y - \tilde{y} \right|$$

for all $x, y, \tilde{y}$ such that $(x, y) \in G$ and $(x, \tilde{y}) \in G$. Then there is $\left[x_o - \delta , x_o + \delta \right]$ in which the differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \ldots\ldots\ldots(1)$$

has the unique solution

$$y = \varphi(x)$$

satisfying the initial condition

$$\varphi(x_o) = y_o \qquad \ldots\ldots\ldots\ldots(2)$$

Proof. The differential equation (1) and the initial condition (2) are equivalent to the integral equation

$$\varphi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))\, dt \quad \ldots\ldots (3)$$

Let $\varepsilon > 0$ be given. By continuity of f, there exists $\delta_0 > 0$ such that for any $(x, y) \in G$. $|(x, y) - (x_0, y_0)| < \delta_0$ implies $|f(x, y) - f(x_0, y_0)| < \varepsilon$. But $|f(x,y)| - |f(x_0,y_0)| \leq |f(x, y) - f(x_0, y_0)|$, therefore $|f(x,y)| < |f(x_0,y_0)| + \varepsilon$, so that we can find a real number K such that $|f(x,y)| \leq K$ for any $(x, y) \in G' = S((x_0,y_0), \delta_0) \subset G$.

Choose $\delta = \min\left\{\dfrac{\delta_0}{2(1 + K)}, \dfrac{1}{M + 1}\right\} > 0$.

We claim that

1) $(x, y) \in G'$ if $|x - x_0| < \delta$ and $|y - y_0| \leq K\delta$

2) $M\delta < 1$.

If $|x - x_0| \leq \delta$ and $|y - y_0| \leq K\delta$, then

$$\begin{aligned}|(x, y) - (x_0, y_0)| &\leq |x - x_0| + |y - y_0| \\ &\leq \delta(1 + K) \\ &\leq \frac{\delta_0}{2} \\ &< \delta_0\end{aligned}$$

and hence $(x, y) \in G'$. Since $\delta \leq \dfrac{1}{M + 1}$, $M\delta < 1$.

Let C^* be the space of continuous function φ defined on the closed interval $[x_0 - \delta, x_0 + \delta]$ such that $|\varphi(x) - y_0| \leq K\delta$ for all $x \in [x_0 - \delta, x_0 + \delta]$ with a metric

$$d(\varphi, \tilde{\varphi}) = \sup_x |\varphi(x) - \tilde{\varphi}(x)|.$$

We claim that the space C^* is complete.

Let Y_0 be defined by $Y_0(x) = y_0$ for all $x \in [x_0-\delta, x_0+\delta]$. Then $Y_0 \in C_{[x_0-\delta, x_0+\delta]}$. It follows that

$$\begin{aligned} C^* &= \{\varphi / |\varphi(x) - y_0| \leqslant K\delta \text{ for all } x \in [x_0-\delta, x_0+\delta]\} \\ &= \{\varphi / \sup_x |\varphi(x) - Y_0(x)| \leqslant K\delta\} \\ &= \{\varphi / d(\varphi, Y_0) \leqslant K\delta\}, \end{aligned}$$

i.e. C^* is a closed ball and hence is a closed subspace of the complete space $C_{[x_0-\delta, x_0+\delta]}$. By theorem 2.2.12, C^* is complete.

We now defined a mapping A as follows :

For $\varphi \in C^*$ define $A\varphi = \Psi$, where

$$\Psi(x) = y_0 + \int_{x_0}^{x} f(t, \varphi(t))dt \quad (|x - x_0| \leqslant \delta).$$

We claim that A is a contraction mapping carry C^* into itself. Since, if $\varphi \in C^*$ and $x \in [x_0-\delta, x_0+\delta]$ then

$$\begin{aligned} |\Psi(x) - y_0| &= \left|\int_{x_0}^{x} f(t, \varphi(t))\, dt\right| \\ &\leqslant \int_{x_0}^{x} |f(t, \varphi(t))||dt| \\ &\leqslant K \int_{x_0}^{x} |dt| \\ &\leqslant K|x - x_0| \\ &\leqslant K\delta \end{aligned}$$

so that $\psi \in C^*$. A is a mapping carry C^* into itself. Moreover,

$$
\begin{aligned}
|\psi(x) - \tilde{\psi}(x)| &= \left| \int_{x_0}^{x} [f(t, \varphi(t)) - f(t, \tilde{\varphi}(t))]\, dt \right| \\
&\leqslant \int_{x_0}^{x} | f(t, \varphi(t)) - f(t, \tilde{\varphi}(t)) |\, |dt| \\
&\leqslant M \int_{x_0}^{x} | \varphi(t) - \tilde{\varphi}(t) |\, |dt| \\
&\leqslant M \sup_x | \varphi(x) - \tilde{\varphi}(x) | \int_{x_0}^{x} |dt| \\
&\leqslant M\, d(\varphi, \tilde{\varphi})\, |x - x_0| \\
&\leqslant M \delta\, d(\varphi, \tilde{\varphi})
\end{aligned}
$$

for all $x \in [x_0 - \delta , x_0 + \delta]$. It follows that

$$\sup_x | \psi(x) - \tilde{\psi}(x) | \leqslant M\delta\, d(\varphi, \tilde{\varphi}) .$$

Since $M\delta < 1$, hence A is a contraction mapping carry C^* into itself. By Banach's contraction theorem, there exists a unique function $\varphi \in C^*$ such that $\varphi = A\varphi$. Observe that φ such that $\varphi = A\varphi$ satisfies (3).

Hence φ is the unique solution of (1) satisfying (2).

This theorem can be generalized to the case of system of differential equations. This is done in the following theorem.

4.1.2 <u>Theorem</u>. Given n functions $f_i(x, y_1, \ldots, y_n)$ defined and continuous on (n + 1) dimensional domain G containing the point $(x_0, y_{01}, \ldots, y_{0n})$. Suppose that each f_i satisfies a Lipschitz condition, i.e.

$$| f_i (x, y_1,\dots, y_n) - f_i(x, \tilde{y}_1,\dots, \tilde{y}_n) | \leqslant M \max_{1 \leqslant i \leqslant n} | y_i - \tilde{y}_i |$$

for all $x, y_1,\dots, y_n, \tilde{y}_1,\dots, \tilde{y}_n$ such that $(x, y_1,\dots, y_n) \in G$ and $(x, \tilde{y}_1,\dots, \tilde{y}_n) \in G$. Then there is an interval $[x_o - \delta , x_o + \delta]$ in which the system of differential equations.

$$\frac{dy_i}{dx} = f_i(x, y_1,\dots, y_n) \qquad \dots\dots\dots\dots(4)$$

for $i = 1, 2, \dots, n$, has the unique solution

$$y_1 = \varphi_1(x) , \dots, y_n = \varphi_n(x)$$

satisfying the system of initial conditions

$$\varphi_1 (x_o) = y_{o1},\dots, \varphi_n(x_o) = y_{on} . \dots\dots(5)$$

Proof. The system of differential equations(4) and initial conditions (5) are equivalent to the system of integral equations

$$\varphi_i(x) = y_{oi} + \int_{x_o}^{x} f_i(t, \varphi_1(t),\dots, \varphi_n(t))dt \dots(6)$$

for $i = 1, 2,\dots, n$. Let $\varepsilon > 0$ be given. Since f_i are continuous for $i = 1, 2,\dots, n$, hence there exists δ_o such that for any $(x, y_1,\dots, y_n) \in G$ $|(x,y_1,\dots,y_n) - (x_o,y_{o1},\dots,y_{on})| < \delta_o$ implies $| f_i(x, y_1,\dots, y_n) - f_i(x_o,y_{o1},\dots,y_{on}) | < \varepsilon$. But

$$|f_i(x,y_1,\dots,y_n)| - |f_i(x_o,y_{o1},\dots,y_{on})|$$

$$\leqslant | f_i(x, y_1,\dots, y_n) - f_i(x_o, y_{o1},\dots,y_{on})| .$$

Therefore there exists a real number K such that

$|f_i(x, y_1, \ldots, y_n)| \leqslant K$ for $(x, y_1, \ldots, y_n) \in G' =$
$S((x_o, y_{o1}, \ldots, y_{on}), \delta_o)$.

Choose $\delta = \min \left\{ \dfrac{\delta_o}{2(nK + 1)} , \dfrac{1}{M + 1} \right\}$.

We claim that

(1) $(x, y_1, \ldots, y_n) \in G'$ if $|x - x_o| \leqslant \delta$, $|y_i - y_{oi}| \leqslant K\delta$,
for $i = 1, 2, \ldots, n$;

(2) $M\delta < 1$.

Since, if $|x - x_o| \leqslant \delta$, $|y_i - y_{oi}| \leqslant K\delta$, for $i = 1, 2, \ldots, n$,
then

$$
\begin{aligned}
|(x, y_1, \ldots, y_n) - (x_o, y_{o1}, \ldots, y_{on})| &\leqslant |x - x_o| + |y_1 - y_{o1}| + \ldots \\
&\quad + |y_n - y_{on}| \\
&\leqslant \delta + nK\delta \\
&= \delta(1 + nK) \\
&\leqslant \frac{\delta_o}{2} \\
&< \delta_o
\end{aligned}
$$

so that $(x, y_1, \ldots, y_n) \in G'$ and since $\delta < \dfrac{1}{M + 1}$, $M\delta < 1$.

Let C_n^* be the space of n-tuples $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n)$ of continuous function $\varphi_1, \varphi_2, \ldots, \varphi_n$ defined on the closed interval $[x_o - \delta, x_o + \delta]$ such that $|\varphi_i(x) - y_{oi}| \leqslant K\delta$, with the metric

$$d(\varphi, \tilde{\varphi}) = \sup_{x,i} |\varphi_i(x) - \tilde{\varphi}_i(x)| .$$

We claim that the space C_n^* is complete.

Let Y_{oi} be defined by $Y_{oi}(x) = y_{oi}$ for all $x \in [x_o - \delta, x_o + \delta]$. Then $Y_o = (Y_{o1}, Y_{o2}, \ldots, Y_{on}) \in C^n_{[x_o - \delta, x_o + \delta]}$. It follows that

$$
\begin{aligned}
C_n^* &= \{ \varphi / |\varphi_i(x) - y_o| \leqslant K\delta \text{ for all } x \in [x_o - \delta, x_o + \delta] \text{ and } i = 1, 2, \ldots, n \} \\
&= \{ \varphi / \sup_{x,i} | \varphi_i(x) - Y_{oi}(x) | \leqslant K\delta \} \\
&= \{ \varphi / d(\varphi, Y_o) \leqslant K\delta \},
\end{aligned}
$$

i.e. C_n^* is a closed ball and hence C_n^* is a closed subspace of the complete space $C^n_{[x_o - \delta, x_o + \delta]}$. Therefore by theorem 2.2.12, C_n^* is complete.

For any $\varphi \in C_n^*$, define $A\varphi = (A\varphi_1, A\varphi_2, \ldots, A\varphi_n)$ where

$$
A\varphi_i(x) = y_{oi} + \int_{x_o}^{x} f_i(t, \varphi_1(t), \ldots, \varphi_n(t))dt
$$

for $i = 1, 2, \ldots, n$ and $x \in [x_o - \delta, x_o + \delta]$.

We claim that A is a contraction mapping carry C_n^* into itself.

If $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in C_n^*$ and $x \in [x_o - \delta, x_o + \delta]$ then, for $i = 1, 2, \ldots, n$

$$
\begin{aligned}
|A\varphi_i(x) - y_{oi}| &= \left| \int_{x_o}^{x} f_i(t, \varphi_1(t), \ldots, \varphi_n(t))dt \right| \\
&\leqslant \int_{x_o}^{x} | f_i(t, \varphi_1(t), \ldots, \varphi_n(t)) | \, |dt| \\
&\leqslant K \int_{x_o}^{x} |dt| \\
&\leqslant K |x - x_o| \\
&\leqslant K\delta
\end{aligned}
$$

so that $A\varphi \in C_n^*$. Therefore A is a mapping carry C_n^* into itself. Moreover,

$$
\begin{aligned}
|A\varphi_i(x) - A\tilde{\varphi}_i(x)| &= \Big|\int_{x_0}^{x}[f_i(t, \varphi_1(t), \ldots, \varphi_n(t)) - f_i(t, \tilde{\varphi}_1(t), \\
&\qquad \ldots, \tilde{\varphi}_n(t))]dt\Big| \\
&\leqslant \int_{x_0}^{x}|f_i(t, \varphi_1(t), \ldots, \varphi_n(t)) - f_i(t, \tilde{\varphi}_1(t), \\
&\qquad \ldots, \tilde{\varphi}_n(t))||dt| \\
&\leqslant M\int_{x_0}^{x}\max_{1\leqslant i\leqslant n}|\varphi_i(t) - \tilde{\varphi}_i(t)||dt| \\
&\leqslant M\max_{1\leqslant i\leqslant n}\sup_{x}|\varphi_i(x) - \tilde{\varphi}_i(x)|\int_{x_0}^{x}|dt| \\
&\leqslant M|x - x_0|\, d(\varphi, \tilde{\varphi}) \\
&\leqslant M\delta\, d(\varphi, \tilde{\varphi})
\end{aligned}
$$

for all $x \in [x_0 - \delta, x_0 + \delta]$ and $i = 1, 2, \ldots, n$. Therefore

$$\sup_{x,i}|A\varphi_i(x) - A\tilde{\varphi}_i(x)| \leqslant M\delta\, d(\varphi, \tilde{\varphi}),$$

i.e. we have

$$d(A\varphi, A\tilde{\varphi}) \leqslant M\delta\, d(\varphi, \tilde{\varphi}).$$

Since $M\delta < 1$, hence A is a contraction mapping carry C_n^* into itself. It follows from Banach's contraction theorem that there exists the unique $\varphi \in C_n^*$ such that $\varphi = A\varphi$. Therefore the system of integral equations(4) has the unique solution satisfying (5).

2. Existence and unique theorem for solution of integral equations.

By a Fredholm's equation (of the second kind) is meant an integral equation of the form

$$f(x) = \lambda \int_a^b K(x, y)\, f(y)\, dy + \psi(x) \qquad \ldots(1)$$

where K is the kernel of the equation and λ is an arbitrary parameter (i.e. λ is a real number). K and ψ are known function, f is unknown.

4.2.1 <u>Theorem</u>. If $K(x, y)$ is continuous function on the square $[a, b]\times[a, b]$, $\psi(x)$ continuous on $[a, b]$ and λ is a real number such that $|\lambda|$ is sufficiently small. Then the equation (1) has a unique solution on $[a, b]$.

<u>Proof</u>. Since $K(x, y)$ is continuous, there exists a real number M such that $|K(x, y)| \leq M$ for all $(x,y) \in [a, b]\times[a, b]$.

If we let A be a mapping of the complete metric space $C_{[a,b]}$ into itself given by

$$Af(x) = \lambda \int_a^b K(x, y)\, f(y)\, dy + \psi(x)$$

and let $f, \tilde{f} \in C_{[a,b]}$. Then

$$\begin{aligned} |A f(x) - A \tilde{f}(x)| &= \left|\lambda \int_a^b K(x,y)\left[f(y) - \tilde{f}(y)\right] dy\right| \\ &\leq |\lambda| \int_a^b |K(x,y)|\, |f(y) - \tilde{f}(y)|\, dy \\ &\leq |\lambda| M \sup_{a\leq x\leq b} |f(x) - \tilde{f}(x)|\, (b - a) \\ &= |\lambda| M (b - a)\, d(f, \tilde{f}) \end{aligned}$$

for all $x \in [a, b]$. Take supremum over x on the left, we have

$$d(Af, A\tilde{f}) \leq |\lambda| M (b - a)\, d(f, \tilde{f}).$$

It follows that A is a contraction mapping if

$$|\lambda| < \frac{1}{M(b-a)} . \qquad \ldots\ldots\ldots(2)$$

By Banach's contraction theorem, there exists the unique solution f such that $f = Af$. Therefore the Fredholm's equation

$$f(x) = \lambda \int_a^b K(x, y)\, f(y)\, dy + \varphi(x)$$

has the unique solution for any value of λ satisfying (2).

Next consider the Volterra's equation

$$f(x) = \lambda \int_a^x K(x, y) f(y)\, dy + \varphi(x) \qquad \ldots\ldots\ldots(3)$$

which differs from Fredholm's equation (1) by having the real variable x instead of the fixed real number b as the upper limit of integration. Volterra's equation can be regarded as a special case of Fredholm's equation by setting $K(x,y) = 0$ if $y > x$. But arbitrary λ can be used for Volterra's equation, not just for sufficiently small $|\lambda|$ as in case of Fredholm's equation.

4.2.2 Theorem. Let $K(x, y)$ be a continuous function on the square $[a, b] \times [a, b]$ and $\varphi(x)$ a continuous function on $[a, b]$. Then the equation (3) has the unique solution for every λ.

Proof. Let A be a mapping of $C_{[a,b]}$ into itself defined by

$$A f(x) = \lambda \int_a^x K(x, y)\, f(y)\, dy + \varphi(x)$$

and let $f, \tilde{f} \in C_{[a,b]}$. Then

$$|A f(x) - A \tilde{f}(x)| \leq |\lambda| \int_a^x |K(x, y)|\, |f(y) - \tilde{f}(y)|\, dy$$

$$\leq |\lambda| M(x - a) \sup_{a \leq x \leq b} |f(x) - \tilde{f}(x)|$$

so that

$$
\begin{aligned}
|A^2 f(x) - A^2\tilde{f}(x)| &\leq |\lambda| \int_a^x |K(x,y)|\,|Af(y) - A\tilde{f}(y)|\,dy \\
&\leq |\lambda| M \int_a^x |Af(y) - A\tilde{f}(y)|\,dy \\
&\leq |\lambda|^2 M^2 \sup_{a \leq x \leq b} |f(x) - \tilde{f}(x)| \int_a^x (y-a)\,dy \\
&= |\lambda|^2 M^2 \frac{(x-a)^2}{2}\, d(f, \tilde{f}) \\
&\leq |\lambda|^2 M^2 \frac{(b-a)^2}{2}\, d(f, \tilde{f}).
\end{aligned}
$$

By induction we can show that

$$|A^m f(x) - A^m \tilde{f}(x)| \leq |\lambda|^m M^m \frac{(b-a)^m}{m!}\, d(f, \tilde{f}).$$

Since $\lim_{m \to \infty} |\lambda|^m M^m \frac{(b-a)^m}{m!} = 0$. Given any λ, we can always choose m large enough to make

$$|\lambda|^m M^m \frac{(b-a)^m}{m!} < 1$$

so that A^m is a contraction map carry $C_{[a,b]}$ into itself for some m. Therefore, by Banach's contraction theorem, (3) has the unique solution.