CHAPTER VI

# CURRENT DENSITY OF S-ELECTRONS

## VI.1 Green's Functions of the s- Electron

We start from the Hamiltonian ( 5.6 )

$$
\begin{aligned}
H = & \sum_{l,\sigma'} \epsilon_l c^+_{l\sigma'} c_{l\sigma'} + \sum_{j,\sigma'} (E_j + U\langle n_\sigma \rangle) d^+_{j\sigma'} d_{j\sigma'} \\
& - \frac{1}{2} \sum_{j\sigma'} \Delta_g \, d^+_{j\sigma'} d^+_{j-\sigma'} - \frac{1}{2} \sum_{j\sigma'} \Delta^*_g \, d^+_{j-\sigma'} d_{j\sigma'} \\
& + \sum_{j,l,\sigma'} \left( V_{jl} c^+_{l\sigma'} d_{j\sigma'} + V^*_{lj} d^+_{j\sigma'} c_{l\sigma'} \right) ,
\end{aligned}
$$

where $\epsilon_l$ and $E_j$ are measured with respect to Fermi energy. The Green's function is defined as ( see Appendix A )

$$
\omega \langle\langle c_{k\sigma}, c^+_{k'\sigma'} \rangle\rangle_\omega = \frac{1}{2\pi} \langle \{ c_{k\sigma}, c^+_{k'\sigma'} \} \rangle + \langle\langle [c_{k\sigma}, H]; c^+_{k'\sigma'} \rangle\rangle_\omega
$$

Since

$$
\begin{aligned}
[c_{k\sigma}, H] &= \epsilon_k c_{k\sigma} + \sum_j V_{jk} d_{j\sigma} \\
[c^+_{k\sigma}, H] &= -\epsilon_k c^+_{k\sigma} - \sum_j V^*_{kj} d_{j\sigma} ,
\end{aligned}
$$

we get the Green's functions for s- electrons as followings

$$
\begin{aligned}
(\omega - \epsilon_k) \langle\langle c_{k\sigma}; c^+_{k'\sigma'} \rangle\rangle &= \frac{1}{2\pi} \delta_{kk'} \delta_{\sigma\sigma'} + \sum_j V_{jk} \langle\langle d_{j\sigma}; c^+_{k'\sigma'} \rangle\rangle \\
(\omega + \epsilon_k) \langle\langle c^+_{-k-\sigma}; c_{-k'-\sigma'} \rangle\rangle &= \frac{1}{2\pi} \delta_{kk'} \delta_{\sigma\sigma'} - \sum_j V^*_{-kj} \langle\langle d^+_{j-\sigma}; c_{-k'-\sigma} \rangle\rangle \\
(\omega - \epsilon_k) \langle\langle c_{k\sigma}; c_{-k'-\sigma'} \rangle\rangle &= \sum_j V_{jk} \langle\langle d_{j\sigma}; c_{-k'-\sigma'} \rangle\rangle \\
(\omega + \epsilon_k) \langle\langle c^+_{-k-\sigma}; c^+_{k'\sigma'} \rangle\rangle &= -\sum_j V^*_{-kj} \langle\langle d^+_{j-\sigma}; c^+_{k'\sigma'} \rangle\rangle
\end{aligned}
$$

Or in the matrix notation,

$$\begin{bmatrix} \omega-\epsilon_k & 0 \\ 0 & \omega+\epsilon_k \end{bmatrix} \begin{bmatrix} \langle\langle c_{k\sigma}; c^+_{k'\sigma}\rangle\rangle & \langle\langle c_{k\sigma}; c_{-k'-\sigma}\rangle\rangle \\ \langle\langle c^+_{-k-\sigma}; c^+_{k'\sigma}\rangle\rangle & \langle\langle c^+_{-k-\sigma}; c_{-k'-\sigma}\rangle\rangle \end{bmatrix} =$$

$$\frac{1}{2\pi}\begin{bmatrix} \delta_{kk'} & 0 \\ 0 & \delta_{kk'} \end{bmatrix} + \sum_j \begin{bmatrix} V_{jk} & 0 \\ 0 & -V^*_{kj} \end{bmatrix} \begin{bmatrix} \langle\langle d_{j\sigma}; c^+_{k'\sigma}\rangle\rangle & \langle\langle d_{j\sigma}; c_{-k'-\sigma}\rangle\rangle \\ \langle\langle d^+_{j-\sigma}; c^+_{k'\sigma}\rangle\rangle & \langle\langle d^+_{j-\sigma}; c_{-k'-\sigma}\rangle\rangle \end{bmatrix} \qquad (6.1)$$

Similarly we can get the Green's functions appearing in the last term of Eq. ( 6.1 )

$$\begin{bmatrix} \omega-E_j-U\langle n_\sigma\rangle & \Delta_g \\ \Delta_g^* & \omega+E_j+U\langle n_\sigma\rangle \end{bmatrix} \begin{bmatrix} \langle\langle d_{j\sigma}; c^+_{k'\sigma}\rangle\rangle & \langle\langle d_{j\sigma}; c_{-k'-\sigma}\rangle\rangle \\ \langle\langle d^+_{j-\sigma}; c^+_{k'\sigma}\rangle\rangle & \langle\langle d^+_{j-\sigma}; c_{-k'-\sigma}\rangle\rangle \end{bmatrix} =$$

$$\sum_l \begin{bmatrix} V^*_{lj} & 0 \\ 0 & -V_{j-l} \end{bmatrix} \begin{bmatrix} \langle\langle c_{l\sigma}; c^+_{k'\sigma}\rangle\rangle & \langle\langle c_{l\sigma}; c_{-k'-\sigma}\rangle\rangle \\ \langle\langle c^+_{-l-\sigma}; c^+_{k'\sigma}\rangle\rangle & \langle\langle c^+_{-l-\sigma}; c_{-k'-\sigma}\rangle\rangle \end{bmatrix} \qquad (6.2)$$

Using the notation

$$\hat{G}(l,k) = \begin{bmatrix} \langle\langle c_{l\sigma}; c^+_{k\sigma}\rangle\rangle & \langle\langle c_{l\sigma}; c_{-k'-\sigma}\rangle\rangle \\ \langle\langle c^+_{-l\sigma}; c^+_{k\sigma}\rangle\rangle & \langle\langle c^+_{-l-\sigma}; c_{-k'-\sigma}\rangle\rangle \end{bmatrix}$$

$$\hat{G}_0^{-1}(k) = \begin{bmatrix} \omega-\epsilon_k & 0 \\ 0 & \omega+\epsilon_k \end{bmatrix}$$

$$\hat{V}_{(kj)} = \begin{bmatrix} V_{jk} & 0 \\ 0 & -V^*_{-kj} \end{bmatrix} \quad ; \quad \hat{\delta}_{kk'} = \begin{bmatrix} \delta_{kk'} & 0 \\ 0 & \delta_{kk'} \end{bmatrix}$$

$$\hat{V}^{+}_{(kj)} = \begin{bmatrix} V^*_{kj} & 0 \\ 0 & -V_{j-k} \end{bmatrix}$$

$$\hat{M}_{(jk')} = \begin{bmatrix} \langle\langle d_{j\sigma}; c^+_{k'\sigma} \rangle\rangle & \langle\langle d_{j\sigma}; c_{-k'-\sigma} \rangle\rangle \\ \langle\langle d^+_{j-\sigma}; c^+_{k'\sigma} \rangle\rangle & \langle\langle d^+_{j-\sigma}; c_{-k'-\sigma} \rangle\rangle \end{bmatrix}$$

equation ( 6.1 ) becomes

$$\hat{G}_0^{-1}(k) \; \hat{G}(kk') = \frac{1}{2\pi} \hat{\delta}_{kk'} + \sum_j \hat{V}_{kj} \hat{M}(jk') \qquad ( 6.3 )$$

or

$$\hat{G}(kk') = \frac{1}{2\pi} \hat{G}_0(k) \hat{\delta}_{kk'} + \sum_j \hat{G}_0(k) \hat{V}_{kj} \hat{M}(jk') \qquad ( 6.4 )$$

With

$$\hat{M}_0^{-1}(j) = \begin{bmatrix} \omega - E_j - U\langle n_\sigma \rangle & \Delta_g \\ \Delta_g^* & \omega + E_j + U\langle n_\sigma \rangle \end{bmatrix}$$

Eq. ( 6.2 ) becomes

$$\hat{M}_0^{-1}(j) \hat{M}(jk') = \sum_l \hat{V}^{+}(lj) \hat{G}(lk') \qquad ( 6.5 )$$

From ( 6.4 ), change $k$ to $l$

$$\vec{G}(l,k') = \frac{1}{2\pi}\hat{G}_0(l)\,\hat{\delta}_{lk'} + \sum_{j'} \hat{G}_0(l)\,\hat{V}_{lj'}\,\hat{M}(j'k') \; ;$$

and substituting this into the Eq. ( 6.5 ) we get

$$\begin{aligned}\hat{M}_0^{-1}(j)\,\hat{M}(jk') &= \sum_{l} \hat{V}^{+}_{lj}\left( \frac{1}{2\pi}\hat{G}_0(l)\,\hat{\delta}_{lk'} + \sum_{j'} \hat{G}_0(l)\,\hat{V}_{lj'}\,\hat{M}(j'k')\right) \\ &= \frac{1}{2\pi}\hat{V}^{+}(k'j)\,\hat{G}_0(k') + \sum_{l,j'} \hat{V}^{+}(lj)\,\hat{G}_0(l)\,\hat{V}(lj')\,\hat{M}(j'k') .\end{aligned}$$

Because both $\hat{V}$ and $\vec{G}_0$ are diagonal matrices they commute, thus we have

$$\begin{aligned}\hat{M}_0^{-1}(j)\,\hat{M}(jk') &= \frac{1}{2\pi}\hat{V}^{+}(k'j)\,\hat{G}_0(k') + \sum_{l} \hat{V}^{+}(lj)\,\hat{G}_0(l)\,\hat{V}(lj)\,\hat{M}(jk) \\ &\quad + \sum_{l}\sum_{j'\neq j} \hat{V}^{+}(lj)\,\vec{G}_0(l)\,\hat{V}(lj')\,\hat{M}(j'k') \\ &= \frac{1}{2\pi}\hat{V}^{+}(k'j)\,\hat{G}_0(k') + \sum_{l} |\hat{V}(lj)|^2\,\hat{G}_0(l)\,\hat{M}(jk') \\ &\quad + \sum_{l}\sum_{j'\neq j} \hat{V}^{+}(lj)\,\hat{V}(lj')\,\vec{G}_0(l)\,\hat{M}(j'k') \; ;\end{aligned}$$

or

$$\begin{aligned}\left[\hat{M}_0^{-1}(j) - \sum_{l} |\hat{V}(lj)|^2\,\vec{G}_0(l)\right]\hat{M}(jk') &= \frac{1}{2\pi}\hat{V}^{+}(k'j)\,\vec{G}_0(k') \\ &\quad + \sum_{l}\sum_{j\neq j'} \hat{V}^{+}(lj)\,\hat{V}(lj')\,\vec{G}_0(l)\,\hat{M}(j'k') .\end{aligned}$$

Define

$$\left(\hat{M}_0'(j)\right)^{-1} = \left[\hat{M}_0^{-1}(j) - \sum_{l} |\hat{V}(lj)|^2\,\hat{G}_0(l)\right],$$

then

$$\hat{M}(jk') = \frac{1}{2\pi}\hat{M}_0'(j)\,\hat{V}^{+}(k'j)\,\vec{G}_0(k') + \hat{M}_0'(j)\sum_{l}\sum_{j'\neq j} \hat{V}^{+}(lj)\,\hat{V}(lj')\,\vec{G}_0(l)\,\hat{M}(j'k')$$

For higher order

$$\hat{M}(jk') = \frac{1}{2\pi}\hat{M}'_0(j)\,\hat{V}^+(k'j)\,\hat{G}_0(k')$$
$$+\frac{1}{2\pi}\hat{M}'_0(j)\sum_{j'\neq j}\sum_{l}|V_{lj'}|^2\,\hat{G}_0(l)\,\hat{M}'_0(j')\,\hat{V}^+(k'j)\,\hat{G}_0(k')$$
$$+\Big\{\hat{M}'_0(j)\sum_{j'\neq j}\sum_{l}|V_{lj'}|^2\,G_0(l)\,\hat{M}'_0(j')$$
$$\times\sum_{j''\neq j'}\sum_{h}|V_{hj''}|^2\,G_0(h)\,M(j''k')\Big\}$$

Substituting this into Eq. ( 6.4 ) yields

$$\hat{G}(kk') = \frac{1}{2\pi}\hat{G}_0(k)\,\delta_{kk'} + \sum_j \hat{G}_0(k)\,\hat{V}(kj)\,\hat{M}(jk')$$
$$= \frac{1}{2\pi}\hat{G}_0(k)\,\delta_{kk'} + \sum_j \hat{G}_0(k)\,\hat{V}(kj)\,\frac{1}{2\pi}\hat{M}'_0(j)\,\hat{V}^+(k'j)\,\hat{G}_0(k')$$
$$+\Big\{\sum_j \hat{G}_0(k)\,\hat{V}(kj)\,\frac{1}{2\pi}\hat{M}'_0(j)\sum_{j'\neq j}\sum_{l}|V_{lj'}|^2$$
$$\times\hat{G}_0(l)\,\hat{M}'_0(j')\,\hat{V}^+(kj')\,\hat{G}_0(k')\Big\}$$
$$+\ \text{high order terms.} \qquad (6.6)$$

Compare the Eq. ( 6.6 ) to the Dyson equation[1]

$$G = G_0 + G_0\Sigma G_0 + G_0\Sigma G_0\Sigma G_0 + \cdots\cdots$$

$$G^{-1} = G_0^{-1} - \Sigma \qquad (6.7)$$

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[1]Abrikosov, A.A., L.R. Gorkov and I.Y. Dzyaloshinskii, Quantum Field Theoretical Methods In Statistical Physics, 2nd Ed., Pergamon Press, Oxford ( 1965 ), Sect.16.1 .

where $\Sigma$ = self energy. We see that for our case the self energy is

$$\Sigma_{(kk')} = \frac{1}{2\pi}\sum_j \hat{V}(kj)\,\hat{M}'_0(j)\,\hat{V}^+(k'j).$$

$$= \frac{1}{2\pi}\sum_j \begin{bmatrix} V_{jk} & 0 \\ 0 & -V^*_{-kj} \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{bmatrix} V^*_{k'j} & 0 \\ 0 & -V_{-j-k'} \end{bmatrix}$$

$$= \frac{1}{2\pi}\sum_j \begin{bmatrix} |V_{kj}|^2 M_{11} & -V_{jk}V_{j-k'}M_{12} \\ -V^*_{-kj}V^*_{k'j}M_{21} & |V_{j-k}|^2 M_{22} \end{bmatrix}$$

Let

$$\hat{G}^{-1}_{kk'} = \begin{bmatrix} \langle\langle c_{k\sigma};c^+_{k'\sigma}\rangle\rangle & \langle\langle c_{k\sigma};c_{-k'-\sigma}\rangle\rangle \\ \langle\langle c^+_{-k-\sigma};c^+_{k'\sigma}\rangle\rangle & \langle\langle c^+_{-k-\sigma};c_{-k'-\sigma}\rangle\rangle \end{bmatrix}^{-1}$$

From ( 6.7 )

$$\hat{G}^{-1}_{kk} = \begin{bmatrix} \omega-\epsilon_k & 0 \\ 0 & \omega+\epsilon_k \end{bmatrix} - \frac{1}{2\pi}\sum_j \begin{bmatrix} |V_{kj}|^2 M_{11} & -V_{jk}V_{j-k}M_{12} \\ -V^*_{-kj}V^*_{kj}M_{21} & |V_{j-k}|^2 M_{22} \end{bmatrix}$$

or

$$\hat{G}^{-1}_{kk} = \begin{bmatrix} \omega-\epsilon_k-\frac{1}{2\pi}\sum_j |V_{kj}|^2 M_{11} & \sum_j V_{jk}V_{j-k}\,M_{12}\left(\frac{1}{2\pi}\right) \\ \sum_j V^*_{-kj}\,V^*_{kj}\,M_{21}\left(\frac{1}{2\pi}\right) & \omega+\epsilon_k-\frac{1}{2\pi}\sum_j |V_{j-k}|^2 M_{22} \end{bmatrix} \qquad (6.8)$$

Since

$$\left(\hat{M}_0'(j)\right)^{-1} = \left[\hat{M}_0^{-1}(j) - \sum_l |\hat{V}(l,j)|^2 \, G_0(l)\right]$$

$$= \begin{bmatrix} \omega - E_j - U\langle n_\sigma\rangle & \Delta_g \\ \Delta_g & \omega + E_j + U\langle n_\sigma\rangle \end{bmatrix}$$

$$- \sum_l \begin{bmatrix} V_{lj} & 0 \\ 0 & -V^*_{-lj} \end{bmatrix} \cdot \begin{bmatrix} V^*_{lj} & 0 \\ 0 & -V_{j-l} \end{bmatrix} \begin{bmatrix} \frac{\omega+\epsilon_l}{\omega^2-\epsilon_l^2} & 0 \\ 0 & \frac{\omega-\epsilon_l}{\omega^2-\epsilon_l^2} \end{bmatrix}$$

$$= \begin{bmatrix} \omega - E_j - U\langle n_\sigma\rangle - \sum_l \frac{|V_{lj}|^2}{\omega-\epsilon_l} & \Delta_g \\ \Delta_g^* & \omega + E_j + U\langle n_\sigma\rangle - \sum_l \frac{|V_{-lj}|^2}{\omega+\epsilon_l} \end{bmatrix}$$

we can find its inverse matrix as following

$$\hat{M}_0'(j) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \quad , \text{ where}$$

$$M_{11} = \left[\omega + E_j + U\langle n_\sigma\rangle - \sum_l \frac{|V_{-lj}|^2}{\omega+\epsilon_l}\right]\left[\mathrm{Det}\, M^{-1}\right]^{-1}$$

$$M_{12} = -\Delta_g \left[\mathrm{Det}\, M^{-1}\right]^{-1}$$

$$M_{21} = -\Delta_g^* \left[\mathrm{Det}\, M^{-1}\right]^{-1}$$

$$M_{22} = \left[\omega - E_j - U\langle n_\sigma\rangle - \sum_l \frac{|V_{lj}|^2}{\omega-\epsilon_l}\right]\left[\mathrm{Det}\, M^{-1}\right]^{-1}$$

and $[\mathrm{Det}\, M^{-1}] = \left(\omega - E_j - U\langle n_\sigma\rangle - \sum_\ell \frac{|V_{\ell j}|^2}{\omega - \epsilon_\ell}\right)\left(\omega + E_j + U\langle n_\sigma\rangle - \sum_\ell \frac{|V_{-\ell j}|^2}{\omega + \epsilon_\ell}\right) - |\Delta_g|^2$

$$(6.9)$$

Similarly we can find the inverse of $\hat{G}^{-1}_{kk}$

$$\hat{G}_{kk} = \begin{bmatrix} \langle\langle c_{k\sigma}; c^+_{k\sigma}\rangle\rangle & \langle\langle c_{k\sigma}; c_{-k-\sigma}\rangle\rangle \\ \langle\langle c^+_{-k-\sigma}; c^+_{k\sigma}\rangle\rangle & \langle\langle c^+_{-k-\sigma}; c_{-k-\sigma}\rangle\rangle \end{bmatrix}$$

Thus from ( 6.8 ) we have

$$\hat{G}_{kk} = \begin{bmatrix} \dfrac{\omega + \epsilon_k - \frac{1}{2\pi}\sum_j |V_{j-k}|^2 M_{22}}{\mathrm{Det}\, G^{-1}} & \dfrac{-\sum_j V_{jk} V_{-kj} M_{12}\left(\frac{1}{2\pi}\right)}{\mathrm{Det}\, G^{-1}} \\ \dfrac{-\sum_j V^*_{-kj} V^*_{kj} M_{21}\left(\frac{1}{2\pi}\right)}{\mathrm{Det}\, G^{-1}} & \dfrac{\omega - \epsilon_k - \frac{1}{2\pi}\sum_j |V_{jk}|^2 M_{11}}{\mathrm{Det}\, G^{-1}} \end{bmatrix}$$

$$(6.10)$$

where $[\mathrm{Det}\, G^{-1}] = \left(\omega - \epsilon_k - \frac{1}{2\pi}\sum_j |V_{kj}|^2 M_{11}\right)\left(\omega + \epsilon_k - \frac{1}{2\pi}\sum_j |V_{j-k}|^2 M_{22}\right)$

$$- \left(\sum_j V_{jk} V_{j-k} M_{12} \sum_j V^*_{-kj} V^*_{kj} M_{21}\right)\left(\frac{1}{2\pi}\right)^2 \qquad (6.11)$$

$\hat{G}_{kk}$ is the set of the Green's functions of s- electron at T = 0°K. For the case T > 0°K we only change $\omega$ in $\hat{G}_{kk}$ to $i\omega$ [2].

$$\hat{\mathcal{G}}_{kk} = \hat{G}_{kk}\,(\omega \longrightarrow i\omega).$$

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[2] Ref. 1 Sect. 14.1

## VI.2 The Current Density

Assume that a superconductor with a plane surface occupies the half space $z < 0$, and is situated in a constant magnetic field, directed parallel to its surface.

The Fourier component of the current density is in the form[3]

$$\breve{j}(k') = -\frac{N e^2}{m} \bar{Q}(k') A(k')$$

$\bar{Q}(k')$ is the response kernel and is in the form

$$\bar{Q}(k') = 1 + \frac{3T}{4} \sum_{\omega} \int_0^{\pi} \sin^3\theta \, d\theta \int_{-\infty}^{\infty} d\epsilon \left( \mathcal{G}_{\omega}(k_+) \mathcal{G}_{\omega}(k_-) + \mathcal{F}_{\omega}(k_+) \mathcal{F}^+_{\omega}(k_-) \right)$$

where

$$\vec{k}_{\pm} = \vec{k} \pm \frac{\vec{k}' \cdot \vec{v}}{2}$$

$$\mathcal{G}_{\omega}(k) = \langle\langle c_{k\sigma} ; c^+_{k\sigma} \rangle\rangle_{i\omega}$$

$$\mathcal{F}^+_{\omega}(k) = \langle\langle c^+_{-k\sigma} ; c^+_{k\sigma} \rangle\rangle_{i\omega}$$

$$\mathcal{F}_{\omega}(k) = \langle\langle c_{k\sigma} ; c_{-k\sigma} \rangle\rangle_{i\omega}$$

By adding and substracting $\bar{Q}(k')$ for normal metal ( $\Delta_g = 0$ ) to the response kernel we get[3]

$$\bar{Q}(k') = \frac{3T}{4} \sum_{\omega} \int_0^{\pi} \sin^3\theta \, d\theta \int_{-\infty}^{\infty} d\epsilon \; \mathcal{F}_{\omega}(k_+) \mathcal{F}^+_{\omega}(k_-) \qquad (6.12)$$

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[3] Ref.1 Chapter 7

Substituting $\mathcal{F}^{\dagger}_{\omega}(k_{-})$ and $\mathcal{F}_{\omega}(k_{+})$ which are in the form ( 6.10 ) into the Eq. ( 6.12 ) yields

$$\bar{Q}(k') = \frac{3T}{4} \sum_{\omega} \int_0^{\pi} \sin^3\theta \, d\theta \int_{-\infty}^{\infty} d\epsilon \frac{\sum_j \sum_j |V_{jk}|^2 \, |V_{-kj}|^2 \, |\Delta_g|^2 \, g^{-2}_{(i\omega)}}{[\mathrm{Det}\, G^{-1}]_+ [\mathrm{Det}\, G^{-1}]_-} \left(\frac{1}{2\pi}\right)^2 \qquad (6.13)$$

where

$$g_{(i\omega)} = [\mathrm{Det}\, M^{-1}]$$

$$= \left[\left( i\omega - E_j - U\langle n_\sigma\rangle - \sum_l \frac{|V_{lj}|^2}{i\omega - \epsilon_l} \right) \times \left( i\omega + E_j + U\langle n_\sigma\rangle - \sum_l \frac{|V_{-lj}|^2}{i\omega + \epsilon_l} \right)\right] - |\Delta_g|^2$$

$$[\mathrm{Det}\, G^{-1}]_{\pm} = \left[\left( i\omega - \epsilon_{k\pm} - \frac{1}{2\pi} \sum_j |V_{kj}|^2 M_{11} \right) \times \left( i\omega + \epsilon_{k\pm} - \frac{1}{2\pi} \sum_j |V_{j-k}|^2 M_{22} \right)\right] - \sum_j |V_{jk}|^2 \sum_j |V_{-kj}|^2 |\Delta_g|^2 g^{-2}_{(i\omega)}$$

$$M_{11} = \left( i\omega + E_j + U\langle n_\sigma\rangle - \sum_l \frac{|V_{-lj}|^2}{i\omega + \epsilon_l} \right) \left( g_{(i\omega)} \right)^{-1}$$

$$M_{22} = \left( i\omega - E_j - U\langle n_\sigma\rangle - \sum_l \frac{|V_{lj}|^2}{i\omega - \epsilon_l} \right) \left( g_{(i\omega)} \right)^{-1}$$

Changing $\sum_l$ to $\int d\epsilon$ we get

$$\sum_l \frac{|V_{lj}|^2}{i\omega - \epsilon_l} = \frac{2mP_0}{(2\pi)^3} \int \frac{|V_{lj}|^2}{i\omega - \epsilon} \, d\epsilon \quad ;$$

where $P_0$ is Fermi momentum of s- electron.

Since $\epsilon$ is only in the region $\omega_D$ above and below the Fermi energy ( see Sect. IV.1 ), thus

$$\sum_l \frac{|V_{lj}|^2}{i\omega - \epsilon_l} = \frac{2mP_0 V^2}{(2\pi)^3} \left[ \ln \frac{\omega - \omega_D}{\omega + \omega_D} \right]$$

and

$$\sum_{\ell} \frac{|V_{-\ell j}|^2}{i\omega+\epsilon_\ell} = \frac{2mP_0V^2}{(2\pi)^3}\left[\ln \frac{\omega+\omega_D}{\omega-\omega_D}\right]$$

Assume that $|V_{\ell j}|^2 = |V_{-\ell j}|^2 = V^2$ , one gets

$$\sum_{\ell} \frac{|V_{-\ell j}|^2}{i\omega+\epsilon_\ell} = -\sum_{\ell} \frac{|V_{\ell j}|^2}{i\omega-\epsilon_\ell} = \text{a real number}$$

Thus

$$\mathcal{G}(i\omega) = -(\omega^2 + B^2 + |\Delta_g|^2)$$

where

$$B = E + U\langle n_r\rangle + A$$

$$A = \sum_{\ell} |V_{\ell j}|^2/(i\omega-\epsilon_\ell)$$

$$E_j = E \quad ; \quad E = \text{constant}$$

If we write $\frac{1}{2\pi}\sum_j |V_{kj}|^2 M_{11}$ as $G_+$ ; and $\frac{1}{2\pi}\sum_j |V_{j-k}|^2 M_{22}$ as $G_-$ ; then the equation for poles of the integrand in the Eq. ( 6.13 ) is

$$\epsilon_{k_\pm}^2 + \epsilon_{k_\pm}(G_+ - G_-) + \omega^2 + \Delta'^2 - i\omega(G_- + G_+) - G_+G_- = 0$$

Let N be the density of lattice sites, one can find that

$$G_+ - G_- = -\frac{NV^2}{\pi}\cdot\frac{B}{(\omega^2+B^2+|\Delta_g|^2)} = \text{a real number}$$

$$G_+ + G_- = -\frac{NV^2}{\pi}\cdot\frac{i\omega}{(\omega^2+B^2+|\Delta_g|^2)} = \text{a pure imaginary number}$$

$$G_+G_- = -\frac{N^2V^4}{4\pi^2}\cdot\frac{\omega^2+B^2}{(\omega^2+B^2+|\Delta_g|^2)} = \text{a real number}$$

Now we find that the four poles of the integrand are located at

$$\epsilon_1 = \frac{NV^2}{2\pi} \cdot \frac{B}{\omega^2+B^2+\Delta_g^2} - \frac{1}{2}\, v k' \beta + i\sqrt{\omega^2+\Delta'^2+C^2}$$

$$\epsilon_2 = \frac{NV^2}{2\pi} \cdot \frac{B}{\omega^2+B^2+\Delta_g^2} - \frac{1}{2}\, v k' \beta - i\sqrt{\omega^2+\Delta'^2+C^2}$$

$$\epsilon_3 = \frac{NV^2}{2\pi} \cdot \frac{B}{\omega^2+B^2+\Delta_g^2} + \frac{1}{2}\, v k' \beta + i\sqrt{\omega^2+\Delta'^2+C^2}$$

$$\epsilon_4 = \frac{NV^2}{2\pi} \cdot \frac{B^2}{\omega^2+B^2+\Delta_g^2} + \frac{1}{2}\, v k' \beta - i\sqrt{\omega^2+\Delta'^2+C^2}$$

where

$$C^2 = -\left( \frac{NV^2\omega^2}{\pi(\omega^2+B^2+\Delta_g^2)} - \frac{N^2V^4}{4\pi^2} \cdot \frac{\omega^2}{(\omega^2+B^2+\Delta_g^2)^2} \right)$$

$$\Delta'^2 = \left(\frac{1}{2\pi}\right)^2 N^2 \, |V_{jk}|^2 \, |V_{-kj}|^2 \, |\Delta_g|^2 \, g^{-2}(i\omega)$$

$$B = E + U\langle n\rangle + A$$

$$A = \frac{2mP_0V^2}{(2\pi)^3} \ln\left(\frac{\omega-\omega_D}{\omega+\omega_D}\right)$$

$$\vec{k}'\cdot\vec{v} = vk'\cos\theta = vk'\beta$$

The $\epsilon$ integration from $-\infty$ to $+\infty$ is performed by integrating around the contour C which enclose the two poles $\epsilon_1$ and $\epsilon_3$ ( Fig. 9 )

$$\int_{-\infty}^{\infty} d\epsilon \, (\cdots) = 2\pi i \sum_{\epsilon_1,\epsilon_3} \text{Residue}$$

The result is

$$\int_{-\infty}^{\infty} d\epsilon \, \frac{|\Delta'|^2}{[\mathrm{Det}\,G^{-1}]_+[\mathrm{Det}\,G^{-1}]_-} = \frac{\pi\,|\Delta'|^2}{2\sqrt{\omega^2+|\Delta'|^2+C^2}\,\left(\frac{v^2k'^2\beta^2}{4}+\omega^2+\Delta'^2+C^2\right)}$$

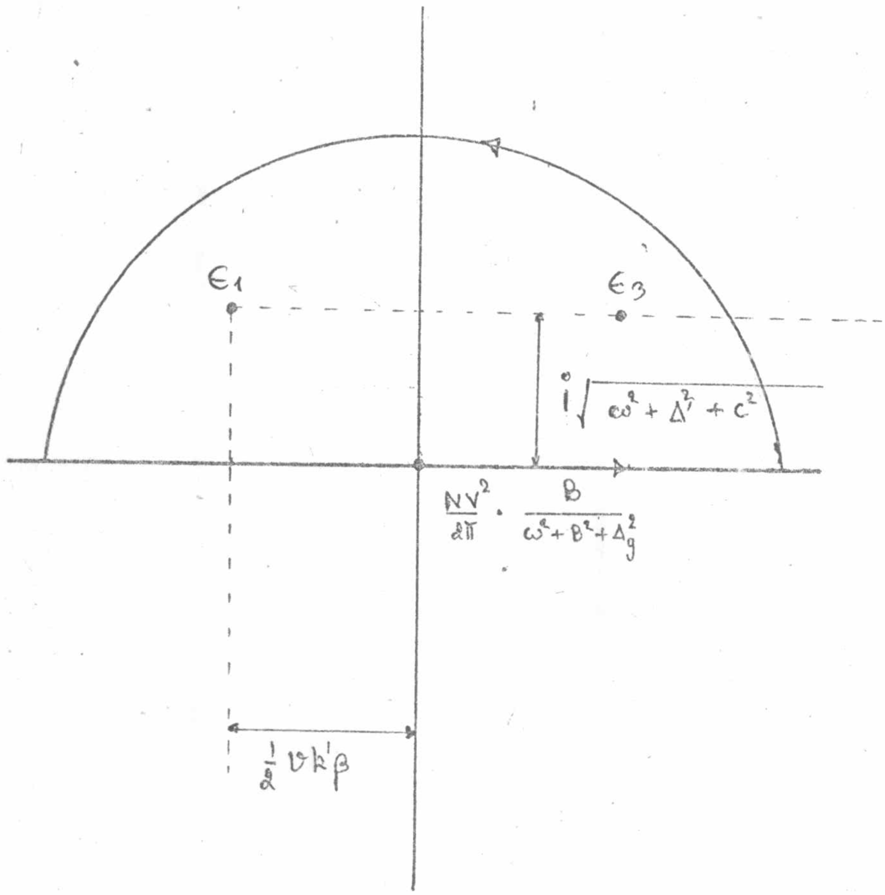


Fig. 9 The Contour for Integration of ( 6.13 )

Thus the response kernel ( 6.13 ) becomes

$$\tilde{Q}(k') = \frac{\pi T}{2} \sum_{\omega} \frac{|\Delta'|^2}{\sqrt{\omega^2 + \Delta'^2 + C^2}\,\left( v^2 k'^2 \beta^2/4 + \omega^2 + \Delta'^2 + C^2 \right)} \qquad (6.14)$$

In the case of London superconductor ( most of transition metal superconductors are London or type-II superconductors ) $v^2 k'^2 \beta^2/4$ is small compare to $\omega^2$ and we can neglect this term,[4] then

$$\tilde{Q}(k') = \frac{\pi T}{2} \sum_{\omega} \frac{|\Delta'|^2}{(\omega^2 + \Delta'^2 + C^2)^{3/2}} \qquad (6.15)$$

[4]Ref. 1 Chapter 7