CHAPTER I

THE LATTICE OF SUBALGEBRAS

1.1. Definitions. A universal algebra or briefly, an algebra, is a pair $\langle A; F \rangle$, where A is a nonvoid set and F is a family of finitary operations on A. F is not neccesarily finite.

A type of an algebra τ is a sequence $\langle n_0, n_1, \ldots, n_\gamma, \ldots \rangle$ of nonnegative integers, $\gamma < O(\tau)$ where $O(\tau)$ is an ordinal, called the order of τ. For every $\gamma < O(\tau)$, we have a symbol f_γ of an n_γ-ary operation on the algebra.

Thus, if $\langle A; F^A \rangle$ and $\langle B; F^B \rangle$ are both algebras of the same type τ, f_γ^A will denote an n_γ-ary operation on A as well as f_γ^B on B.

The class of all algebras of type τ will be denoted by $K(\tau)$; it will be called a similarity class of algebras.

1.2. Definition. Let $\langle A; F \rangle$ be an algebra of type τ and B a subset of A, $\langle B; F \rangle$ is called a subalgebra of $\langle A; F \rangle$ if and only if $b_0, \ldots, b_{n_\gamma - 1} \in B$ implies $f_\gamma(b_0, \ldots, b_{n_\gamma - 1}) \in B$, $\forall \gamma \in O(\tau)$. Let Su(A) denote the set of all subalgebras of $\langle A; F \rangle$. Let X be a subset of A. Let $Sg(X) = \bigcap \{U : X \subseteq U \in Su(A)\}$ denote the subalgebra of $\langle A; F \rangle$ generated by X.

1.3. Definition. Let $\langle A; F^A\rangle$ and $\langle B; F^B\rangle$ be both algebras of the same type τ.

A mapping $\psi : A \to B$ such that

$$f_\gamma^A(a_0,\ldots, a_{n_\gamma-1})\psi = f_\gamma^B(a_0\psi,\ldots, a_{n_\gamma-1}\psi),$$

for all $\gamma \in O(\tau)$, $a_0,\ldots, a_{n_\gamma-1} \in A$, is called a homomorphism of $\langle A; F^A\rangle$ into $\langle B; F^B\rangle$. A homomorphism $\psi : A \to B$ is called an isomorphism if it is 1-1 and onto.

1.4. Definition. A relational system is a pair $\langle A; R\rangle$ where A is a nonvoid set and R is a family of finitary relations on A. A type of a relational system τ is a sequence $\langle m_0, m_1,\ldots, m_\gamma,\ldots\rangle$ of nonnegative integers, $\gamma < O(\tau)$ where $O(\tau)$ is an ordinal, and for every $\gamma < O(\tau)$ we have a symbol r_γ of an m_γ-ary relation on the relational system.

1.5. Definition. Let $\langle A; R^A\rangle$ and $\langle B; R^B\rangle$ be relational systems of the same type τ.

A mapping $\psi : A \to B$ such that

$$r_\gamma^A(a_0,\ldots, a_{m_\gamma-1}) \Rightarrow r_\gamma^B(a_0\psi,\ldots, a_{m_\gamma-1}\psi)$$

for all $\gamma \in O(\tau)$, $a_0,\ldots, a_{m_\gamma-1} \in A$, is called a homomorphism of $\langle A; R^A\rangle$ into $\langle B; R^B\rangle$, and if ψ is also 1-1, onto, and

$$r_\gamma^A(a_0, \ldots, a_{m_\gamma -1}) \Leftrightarrow r_\gamma^B(a_0\psi, \ldots, a_{m_\gamma -1}\psi)$$

for all $\gamma \in O(\tau)$, $a_0, \ldots, a_{m_\gamma -1} \in A$, then it is called an isomorphism of $\langle A; R^A \rangle$ into $\langle B; R^B \rangle$.

1.6. Definition. A partially ordered set is a relational system $\langle P; \leqslant \rangle$ where " $\leqslant$ " is a binary relation on P satisfying the following conditions for all a, b, c in P.

i) $a \leqslant a$

ii) $a \leqslant b$ and $b \leqslant a$ imply $a = b$

iii) $a \leqslant b$ and $b \leqslant c$ imply $a \leqslant c$.

A chain $\langle C; \leqslant \rangle$ is a partially ordered set satisfying the additional condition.

iv) $a \leqslant b$ or $b \leqslant a$ for all a, b in C.

1.7. Definitions. A semigroup $\langle A; \circ \rangle$ is an algebra with one binary operation " $\circ$ " such that

$$a \circ (b \circ c) = (a \circ b) \circ c \quad \text{for all a, b, c in A.}$$

A semigroup $\langle A; \circ \rangle$ is commutative if

$$a \circ b = b \circ a \quad \text{for all a, b in A.}$$

A semilattice $\langle A; \circ \rangle$ is a commutative semigroup in which every element is idempotent, i.e, $a \circ a = a$ for all a in A.

If there exists 0 in a semilattice $\langle A; \circ \rangle$ such that $0 \circ a = 0$ for all a in A, then 0 is called a zero of a semilattice $\langle A; \circ \rangle$.

We can define a semilattice as a partially ordered set $\langle A; \leq \rangle$ in which any two element subset of A has a least upper bound.

The two definitions are naturally equivalent in the following sense.

i) Let $\mathcal{A} = \langle A; \circ \rangle$ be a semilattice. Define a binary relation " $\leq$ " on A by, for all a, b in A, $a \leq b$ if and only if $a \circ b = b$. Then $\mathcal{A}^{\circ} = \langle A; \leq \rangle$ is a partially ordered set, and as a partially ordered set, it is a semilattice; futhermore $\ell.u.b.(\{a,b\}) = a \circ b$.

ii) Let $\mathcal{A} = \langle A; \leq \rangle$ be a partially ordered set which is a semilattice. Set $a \circ b = \ell.u.b(\{a,b\})$.
Then $\mathcal{A}^{+} = \langle A; \circ \rangle$ is a semilattice and $a \leq b$ if and only if $a \circ b = b$.

iii) Let $\mathcal{A} = \langle A; \circ \rangle$ be a semilattice. Then $(\mathcal{A}^{\circ})^{+} = \mathcal{A}$.

iv) Let $\mathcal{A} = \langle A; \leq \rangle$ be a semilattice. Then $(\mathcal{A}^{+})^{\circ} = \mathcal{A}$.

1.8. <u>Definition</u>. An <u>ideal</u> of a semilattice $\langle A; \vee \rangle$ is a nonvoid subset I of A such that, for all a, b in A,

$$a \vee b \in I \text{ if and only if } a, b \in I.$$

We can define an ideal of a semilattice $\langle A; \leq \rangle$ as a nonvoid subset I of A such that

i) $a, b \in I$ implies $\ell.u.b(\{a,b\}) \in I$.

iii) $a \leq b \in I$ implies $a \in I$.

And as in 1.7. these two definitions are equivalent.

1.9. Definition. A lattice is an algebra $\langle A; \vee, \wedge\rangle$ where $\vee$ and $\wedge$ are binary operations on A, called join and meet, respectively, satisfying the following conditions.

i) $\langle A; \vee\rangle$ and $\langle A; \wedge\rangle$ are similattices.

ii) $(a \vee b) \wedge a = a$ and $(a \wedge b) \vee a = a$, for all a, b in A.

We can define a lattice as a partially ordered set $\langle A; \leq\rangle$ in which any two element subset of A has ℓ.u.b. and g.ℓ.b.

The two definitions are equivalent in the following sense.

i) Let $\mathcal{A} = \langle A; \vee, \wedge\rangle$ be a lattice. Define a binary relation " $\leq$ " on A by, for all a, b in A, $a \leq b$ if and only if $a \vee b = b$. Then $A^{\circ} = \langle A; \leq\rangle$ is a partially set, and as a partially set, it is a lattice; furthermore $\ell.u.b.(\{a,b\}) = a \vee b$

and $g.\ell.b.(\{a,b\}) = a \wedge b$.

ii) Let $\mathcal{A} = \langle A; \leq\rangle$ be a lattice. Set $a \vee b = \ell.u.b.(\{a,b\})$ and $a \wedge b = g.\ell.b.(\{a,b\})$.

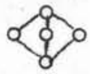
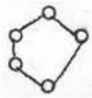
Then $\mathcal{A}^{+} = \langle A; \vee, \wedge\rangle$ is a lattice and $a \leq b$ if and only if $a \vee b = b$.

iii) Let $\mathcal{A} = \langle A; \vee, \wedge\rangle$ be a lattice. Then $(\mathcal{A}^{\circ})^{+} = \mathcal{A}$.

iv) Let $\mathcal{A} = \langle A; \leq\rangle$ be a lattice. Then $(\mathcal{A}^{+})^{\circ} = \mathcal{A}$.

1.10. Definition. If $\mathcal{A}$ is a lattice, then subalgebras of $\mathcal{A}$ are called sublattices of $\mathcal{A}$.

1.11. Definition. A lattice $\langle A; \vee, \wedge \rangle$ is distributive if for all a, b, c in A, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

1.12. Theorem. A lattice is distributive if and only if it does not have  or  as a sublattice.

Proof. (see in [2], page 22;).

1.13. Definition. A lattice $\langle A; \vee, \wedge \rangle$ is modular if for all a,b,c in A, $a \vee c = c$ implies $(a \vee b) \wedge c = a \vee (b \wedge c)$.

Note : Distributive lattices are modular.

The smallest element in a lattice, if it exists, is called zero (0).

The largest element in a lattice, if it exists, is called unit (1).

b is a complement of a in a lattice with zero and unit if $a \vee b = 1$ and $a \wedge b = 0$.

A complemented lattice is a lattice in which every element has a complement.

1.14. Definition. Let $\langle A; \leq \rangle$ be a lattice, if the collection a_i, $i \in I$ of elements of A has a g.ℓ.b (ℓ.u.b) we denote this element by $\wedge \{a_i \mid i \in I\}$ (and $\vee \{a_i \mid i \in I\}$).
If for all $X \subseteq A$, $\wedge X$ and $\vee X$ exist, then the lattice $\langle A; \leq \rangle$ is called complete.

1.15. Definition. Let $\langle A; \leqslant \rangle$ be a complete lattice. Let $a \in A$. The element a is called compact if the following condition is satisfied :

If $a \leqslant \vee\{x_i \mid i \in I\}$ where $x_i \in A$, for each i in I, then there exists $I_1 \subseteq I$ such that I_1 is finite and $a \leqslant \vee\{x_i \mid i \in I_1\}$, i.e. if a is contained in an infinite join, it is already contained in a finite join.

1.16. Definition. A lattice $\langle A; \leqslant \rangle$ is called algebraic if :

(i) it is complete.

(ii) every a in A can be written as

$a = \vee\{x_i \mid i \in I\}$ where x_i is compact in A for all i in I.

1.17. Lemma. Let $\mathcal{F} = \langle F; \vee \rangle$ be a semilattice with zero. Then $\langle I(\mathcal{F}); \subseteq \rangle$ is an algebraic lattice where $I(\mathcal{F})$ is a set of all ideals of $\mathcal{F}$.

Proof. (see in [1], page 22;).

1.18. Lemma. Let $\mathcal{L} = \langle L; \leqslant \rangle$ be an algebraic lattice. Then there exists a semilattice $\mathcal{F} = \langle F; \vee \rangle$ with zero such that $\mathcal{L}$ is isomorphic to $\langle I(\mathcal{F}); \subseteq \rangle$ where $I(\mathcal{F})$ is the set of all ideals of $\mathcal{F}$.

Proof. Let F be the set of all compact elements of L. Since 0 is compact, we have $0 \in F$

To show $\langle F; V\rangle$ is a semilattice with zero, let a, b ε F and $a \vee b \leq \vee\{x_i \mid i \in J\}$ where $x_i \in L$, $i \in J$.

Since $a \leq a \vee b \leq \vee\{x_i \mid i \in J\}$ and

$b \leq a \vee b \leq \vee\{x_i \mid i \in J\}$, which by the compactness of a and b imply that there exists finite subsets J_1, J_2 of J such that

$$a \leq \vee\{x_i \mid i \in J_1\} \text{ and } b \leq \vee\{x_i \mid i \in J_2\}.$$

Therefore $a \vee b \leq \vee\{x_i \mid i \in J_1 \cup J_2\}$ where $J_1 \cup J_2$ is finite, and so $a \vee b$ is in F.

That is, F is closed under V and $\langle F; V\rangle$ has same properties as $\langle L; V\rangle$.

For $a \in L$, set $I_a = \{x \mid x \in F \text{ and } x \leq a\}$.

To show I_a is an ideal of $\mathfrak{F}$, since $0 \in I_a$, we have $I_a \neq \emptyset$.

$x, y \in I_a$ iff $x \leq a$ and $y \leq a$ iff $x \vee y \leq a$ iff $x \vee y \in I_a$.

Therefore I_a is an ideal of $\mathfrak{F}$.

Define $\psi : L \to I(\mathfrak{F})$ by $a\psi = I_a$.

To show ψ is 1-1, If $a \in L$, then there exists H, a set of compact elements such that

$a = \vee\{x \mid x \in H\}$, that is $H \subseteq I_a$, and

$a = \vee\{x \mid x \in H\} \leq \vee\{x \mid x \in I_a\} \leq a$, so

$a = \vee\{x \mid x \in I_a\}$. Let $a, b \in L$ such that $I_a = I_b$, then

$a = \vee\{x \mid x \in I_a\} = \vee\{x \mid x \in I_b\} = b$.

To show ψ is onto, let $I \in I(\mathfrak{F})$ and $a = \vee\{y \mid y \in I\}$. Since for all y in I, y is compact and $y \leq a$, we have $I \subseteq I_a$.

Suppose $x \in I_a$, then x is compact, $x \leq V\{y|y \in I\}$ and there exists I_1, a finite subset of I, such that $x \leq V\{y|y \in I_1\}$. Since $V\{y|y \in I_1\} \in I$ and I is an ideal of $\mathcal{F}$, we have $x \in I$; thus,

$$I_a = I .$$

To show ψ is homomorphism, let $a, b \in L$ such that $a \leq b$, and let $x \in I_a$, we have $x \in F$, $x \leq a \leq b$, and so $x \in I_b$. That is $I_a \subseteq I_b$, i.e. $a\psi \subseteq b\psi$. Let $I_a \subseteq I_b$, $a = V\{x|x \in I_a\} \leq V\{x|x \in I_b\} = b$; $a \leq b$. Therefore $\langle L; \leq\rangle \cong \langle I(\mathcal{F}); \subseteq\rangle$.

1.19. Lemma. For any $\langle A; F\rangle$, $\langle Su(A); \subseteq\rangle$ is an algebraic lattice, denote $\langle Su(A); \subseteq\rangle$ by $Su(A)$. If G is a family of subalgebras, then the meet of G is $\bigcap G$, the join of G is $Sg(\cup G)$.

U in G is compact iff U is finitely generated.

Proof. (see in [2], 103-104;).

1.20. Theorem. (Whaley) If m is a cardinal and $\mathcal{L}$ is a non-trivial algebraic lattice such that for all x in L, x is compact implies x contains at most m compact elements, then there exists an algebra A of 1 binary operation, m unary operations, such that $\mathcal{L} \cong Su(A)$.

Proof. Let F = {non zero compact elements of L}

Let I be any set of order m.

For each $x \in F$, let $f_x : I \xrightarrow{\text{onto}} \{y|y \in F, y \leq x\}$.

Let $A = \langle F; V, F_i(i \in I)\rangle$ where $F_i(x) = f_x(i)$ for all x in F.

Let $\mathcal{F} = \langle F \cup \{0\}; V\rangle$, from Lemma 1.18, $\mathcal{F}$ is a semilattice with zero.

Let $I(\mathcal{F})$ be the set of all ideals of $\mathcal{F}$.

To show $\langle I(\mathcal{F}),\subseteq\rangle \cong Su(A)$, let $\psi : I(\mathcal{F}) \to Su(A)$ be defined by

$\psi(I) = I - \{0\}$ for all I in $I(\mathcal{F})$.

To show $I - \{0\} \in Su(A)$, let $x \in I - \{0\}$.

Since $F_i(x) = f_x(i) = y$ for some $y \leqslant x$, $y \in F$, we have $y \in I - \{0\}$,

so $F_i(x) \in I - \{0\}$.

Let $x, y \in I - \{0\}$, $x \in I$ and $y \in I$, $x \neq 0$, $y \neq 0$.

Since I is an ideal, we have $x \vee y \in I$ and $x \vee y \neq 0$,

then $x \vee y \in I - \{0\}$. Therefore $I - \{0\} \in Su(A)$.

Clearly ψ is 1-1 and preserves order.

To show ψ is onto, let $S \in Su(A)$, consider $S \cup \{0\}$,

let $x, y \in S \cup \{0\}$.

If x or y is zero, then clearly $x \vee y \in S \cup \{0\}$.

If x and y are not zero, then $x, y \in S$ and $x \vee y \in S$,

that is $x \vee y \in S \cup \{0\}$,

let $x, y \in F$ such that $x \vee y \in S \cup \{0\}$.

If $x \vee y = 0$, then clearly $x = y = 0 \in S \cup \{0\}$.

Suppose $x \vee y \neq 0$, Since $f_{xVy}: I \xrightarrow{\text{onto}} \{z \mid z \in F,\ z \leqslant x \vee y\}$ and

$x \leqslant x \vee y$, $y \leqslant x \vee y$, we have $f_{xVy}(i) = F_i(xVy) = x$ and

$f_{xVy}(j) = F_j(xVy) = y$ for some i, j in I.

Therefore $x, y \in S$ and $x, y \in S \cup \{0\}$, and $S \cup \{0\} \in I(\mathcal{F})$.

Hence, $\langle I(\mathcal{F});\subseteq\rangle \cong Su(A)$.

By lemma 1.18. $\mathcal{L} \cong Su(A)$.

Note. If m is uncountable and some compact element of $\mathcal{L}$ contains m compact elements, then all algebras A such that $Su(A) \cong \mathcal{L}$ must have at least m operations.

1.21. Theorem. For any non-trivial algebraic lattice $\mathcal{L}$, the followings are equivalent,

i) each compact element contains only countably many compact elements,

ii) there exists an algebra A of countable similarity type such that $\mathcal{L} \cong Su(A)$,

iii) there exists a commutative groupoid G such that $\mathcal{L} \cong Su(G)$.

Proof. (iii) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i) Suppose not (i), from note, A must have an uncountable number of operations. To show (i) $\Rightarrow$ (iii), let V = {non zero compact elements of L} for each a in V; arrange V into a sequence

$a' = \langle a'_0, a'_1, \ldots \rangle$ where $a'_0 = a$ and a'_i are contained in a. If a contains finite number, n, of elements of V, **let** a' is periodic, with $a'_i = a'_j$ iff $i \equiv j \pmod n$.

For all a, b in V, define

$$a * b = \begin{cases} a'_{i+1} & \text{if } b = a'_i \\ b'_{i+1} & \text{if } a = b'_i \\ a \vee b & \text{otherwise.} \end{cases}$$

Let $G = \langle V; * \rangle$, clearly G is a commutative groupoid and

$Su(G) \cong I(\mathcal{F})$ via $\phi \Rightarrow \{0\}$ and $Su(G) \ni G_i \Rightarrow G_i \cup \{0\} \in I(\mathcal{F})$.

To show $G_1 \cup \{0\} \in I(\mathcal{F})$, let $x, y \in G_1 \cup \{0\}$.

If x or y is zero, then $x \vee y = x$ or y and $x \vee y \in G_1 \cup \{0\}$.

If x and y are not zero, then $x, y \in G_1$ and $x * y \in G_1$.

$$x * y = \begin{cases} x'_{i+1} \text{ if } y = x'_i & \text{........(1)} \\ y'_{i+1} \text{ if } x = y'_i & \text{........(2)} \\ x \vee y \text{ otherwise} & \text{........(3).} \end{cases}$$

In case 1. $x \vee y = x \in G_1 \Rightarrow x \vee y \in G_1 \cup \{0\}$.

In case 2. $x \vee y = y \in G_1 \Rightarrow x \vee y \in G_1 \cup \{0\}$.

In case 3. $x \vee y \in G_1 \Rightarrow x \vee y \in G_1 \cup \{0\}$.

Let $x, y \in V$ such that $x \vee y \in G_1 \cup \{0\}$.

If $x \vee y = 0$, then $x = y = 0 \in G_1 \cup \{0\}$.

If $x \vee y \neq 0$, $x \lesssim x \vee y$, $y \lesssim x \vee y$, then there exist m, n in $\mathbb{N}$

such that $(xVy)^m = x$ and $(xVy)^n = y$. Therefore $x, y \in G_1 \cup \{0\}$.

Hence $G_1 \cup \{0\} \in I(\mathcal{F})$.

Clearly, this mapping preserves order.

Consequently; $Su(G) \cong I(\mathcal{F})$ and by lemma 1.18, $Su(G) \cong \mathcal{L}$.