CHAPTER IV

GROUPS WHICH ARE UNIONS OF THREE PROPER SUBGROUPS

The materials of this chapter are drawn from reference [1].

1 Introduction.

In this chapter we consider groups which can be written as (set - theoretical) unions of three subgroups.

To specific, suppose the group G is given by the non-trivial union:

$$G = A \cup B \cup C$$

where A, B and C are subgroups of G.

If A is a subgroup of B, we are effectively dealing with two subgroups of G, and this is not possible by 2.1 of Chapter III. Similar for the other cases. Hence we may assume the configuration of the following Figure 1 where A', B' and C' must be nonempty sets.

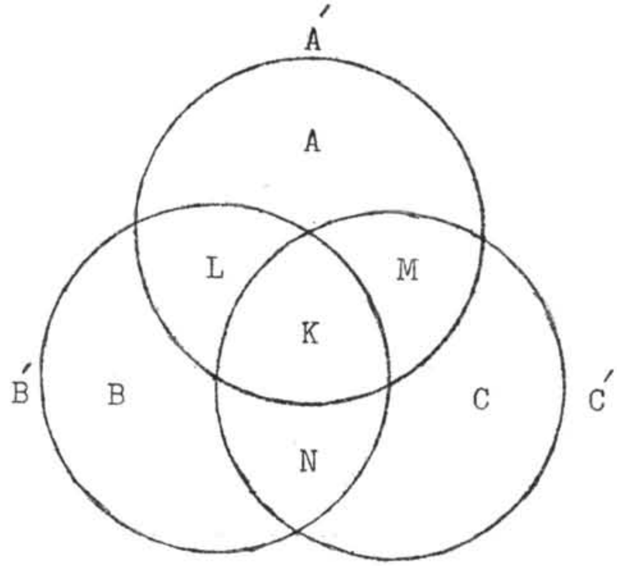

Figure 1.

where $A' = A \setminus (B \cup C)$, $B' = B \setminus (C \cup A)$, $C' = C \setminus (A \cup B)$

$L = (A \cap B) \setminus C$, $M = (C \cap A) \setminus B$, $N = (B \cap C) \setminus A$,

$$K = A \cap B \cap C.$$

For convenience, we will call such a group G a 3-group.

Appling 3.1, 3.3, 3.5, 4.2, 4.3 and 4.5 of Chapter III, we have the followings:

(a) $L = M = N = \emptyset$

(b) A', B', and C' contain their inverses.

(c) If a' is in A' and b' in B', then $a'b'$ is in C'.

(d) If a' is in A', then a'^2 is in $B \cup C$.

(e) If every element of a group G has 2^{nd} root in G, then G can not be an irredundant union of three subgroups.

(f) Let G be a finite group of order N and let 3 be the smallest prime dividing N. Then G is not an irredundant union of three subgroups.

2 Homomorphisms of 3 - groups.

To prove the main theorem, we will introduce three lemmas as follows:

2.1 Lemma. If a' and a'_1 are in A', then $a'a'_1$ is in K.

Proof. The element $a'a'_1$ belobgs either to A' or to K. Suppose that it belongs to A'. Let b' belong to B'. Consider the element $b'a'a'_1$ as follows:

Apply (c) twice; we have $(b'a')a'_1$ is in B'. By assumption and (c) again, $b'(a'a'_1)$ is in C'. Hence we arrive at two contradicting statements and $a'a'_1$ must be in K.

2.2 <u>Lemma</u>. K is a normal subgroup of G.

<u>Proof</u>. It is clear that K is a subgroup of G.

Let a' be in A' and k in K. Then we have ka' belongs either to A' or to K. If ka' is in K, then $k^{-1}(ka') = a'$ is in K. Thus ka' must be in A'. By (b) and 2.1, we have $a'^{-1}ka'$ is in K. Therefore $Ka' = a'K$.

Similary we can show that $Kb' = b'K$ and $Kc' = c'K$ for any b' in B' and c' in C'.

Hence K is a normal subgroup of G.

2.3 <u>Lemma</u>. $a'K = A$ for any a' in A'.

<u>Proof</u>. Let k belong to K. Then $a'k$ is in $A = A' \cup K$. If $a'k$ is in K, then $(a'k)k^{-1} = a'$ is in K. Thus $a'k$ is in A'. Therefore $a'K$ is a subset of A. Suppose that there is an element a'_1 in A' but not in $a'K$. Then by (b) and 2.1 we have $a'^{-1}a'_1$ is in K. Hence $a'_1 = a'k$ for some k in K, which is a contradiction.

Hence the lemma is proved.

2.4 <u>Difinition</u>. If G is the group such that $G = \{1, a_1, a_2, a_3\}$ with the relations $a_1^2 = a_2^2 = a_3^2 = 1$ and $a_1a_2 = a_2a_1 = a_3$, $a_2a_3 = a_3a_2 = a_1$, $a_3a_1 = a_1a_3 = a_2$, then G is said to be <u>the Klein 4 - group</u> and denoted by <u>V</u>.

2.5 Theorem. A group G is an irredundant union of three subgroups if and only if it can be mapped homomorphically onto the Klein 4 - group.

Proof. The "if" part follows clearly from (c), 2.1, 2.2 and 2.3.

To prove the converse, let φ be a homomorphism of G onto $V = \{1, a_1, a_2, a_3\}$. Set

$$A' = \varphi^{-1}(a_1)$$
$$B' = \varphi^{-1}(a_2)$$
$$C' = \varphi^{-1}(a_3)$$

and

$$K = \operatorname{Ker}\varphi .$$

Let

$$A = A' \cup K$$
$$B = B' \cup K$$

and

$$C = C' \cup K.$$

It is obvious to see that A , B and C are subgroups of G and $G = A \cup B \cup C$. Since A', B' and C' are disjoint nonempty sets, G is the irredundant union of subgroups A, B and C.

Hence the theorem is proved.

2.6 Remark. It follows from 2.5 that

$$A'B' = B'A' = C' , \quad B'C' = C'B' = A' , \quad C'A' = A'C' = B'$$

and

$$A'^2 = B'^2 = C'^2 = K.$$

3 Decompositions of 3 - groups.

It follows from 2.5 that G is finite, it must be order 4m. Clearly there are many groups of order 4m which are not unions of three subgroups . For example, the cyclic group C_{4m} of order 4m. More generally, no locally cyclic group can be unions of three subgroups, this follows from 1 of Chapter III.

On the other hand there are many examples of groups of order 4m which are unions of three subgroups. We shall consider a few examples. A <u>decomposition</u> of a group G is a set of subgroups whose union is G and is irredundant.

<u>Example 1</u>. The Klein 4 - group admits a decomposition $\{C_2, C_2, C_2\}$, where as usual C_n denotes the cyclic group of order n.

<u>Example 2</u>. There are five groups of order 8. Disregarding C_8, we are left with $C_4 \times C_2$, $C_2 \times C_2 \times C_2$, D_4 (the dihedral group of order 4) and Q (the quaternian group)

$C_4 \times C_2$ admits a decomposition $\{C_4, C_4, V\}$
$C_2 \times C_2 \times C_2$ admits a decomposition $\{V, V, V\}$
D_4 admits a decomposition $\{C_4, V, V\}$
and Q admits a decomposition $\{C_4, C_4, C_4\}$.

<u>Example 3</u>. For each positive integer m, the dihedral group D_{2m} of order 2m is the group generated by a and b with relations

$$a^{2m} = b^2 = (ab)^2 = 1.$$

D_{2m} admits a decomposition $\{C_{2m}, D_m, D_m\}$.

<u>Example 4</u>. For each even m, the dicyclic group of order 4m generated by a and b with relations

$$a^{2m} = 1, \; a^m = (ab)^2 = b^2$$

admits a decomposition consisting of a C_{2m} and two dicyclic groups, each of order 2m.

Thus far we have characterized the groups which are 3 - groups. The following questions are to be sattled.

Question 1. Can a 3 - groups have "different" decompositions?

Question 2. Can two " different" 3 - groups have the same decomposition ?

As to be expected, both questions are answered affirmatively by the following two examples.

Example 5. $C_2 \times D_4$ has the two distinct decompositions $\{C_4 \times C_2, D_4, D_4\}$ and $\{C_4 \times C_2, C_2 \times C_2 \times C_2, C_2 \times C_2 \times C_2\}$

Example 6. The group of order 16 generated by a, b and c with the relations

$$a^2 = b^2 = c^2 = 1, \; abc = bca = cab$$

has a decomposition $\{C_4 \times C_2, D_4, D_4\}$, the latter is also a decomposition of $C_2 \times D_4$ (cf. Example 5).

If a group is non-abelian, then the subgroups of a decomposition can be abelian ($\mathcal{Q}$) or non-abelian ($\mathcal{N}$). This give four possible types of decompositions; namely $\{\mathcal{Q}, \mathcal{Q}, \mathcal{Q}\}$, $\{\mathcal{Q}, \mathcal{Q}, \mathcal{N}\}$, $\{\mathcal{Q}, \mathcal{N}, \mathcal{N}\}$ and $\{\mathcal{N}, \mathcal{N}, \mathcal{N}\}$. The decomposition $\{\mathcal{Q}, \mathcal{Q}, \mathcal{N}\}$, however, cannot occur.

3.1 Lemma. If the group G has the decomposition $\{A, B, C\}$ with A and B are both abelian, then C is also abelian.

Proof. Let c', c_1' be in C' and k in K. Then there exist a' in A', b' in B' and k_1 in K such that $c' = a'b'$ and $c_1' = c'k_1'$. Since A and B are both abelian, we have

$$c'_1 = (a'b')k_1$$
$$c'c'_1 = (a'b')(a'b'k_1) = (a'b'k_1)(a'b') = c'_1c'$$
and $$c'k = (a'b')k = k(a'b') = kc'.$$

Hence C is abelian.

The remaining three type of decompositions can all occure as the following examples show.

Example 7. The group Q has the decomposition $\{C_4, C_4, C_4\}$ which is of the type $\{\mathfrak{A}, \mathfrak{A}, \mathfrak{A}\}$.

The group D_{2m} has the decomposition $\{C_{2m}, D_m, D_m\}$ which is of the type $\{\mathfrak{A}, \mathfrak{N}, \mathfrak{N}\}$.

The group S_3 x V has the decomposition $\{D_6, D_6, D_6\}$ which is of the type $\{\mathfrak{N}, \mathfrak{N}, \mathfrak{N}\}$.

We summerize these results in the following theorem:

3.2 Theorem. Each decomposition of a 3 - group is one of the type $\{\mathfrak{A}, \mathfrak{A}, \mathfrak{A}\}$, $\{\mathfrak{A}, \mathfrak{N}, \mathfrak{N}\}$ and $\{\mathfrak{N}, \mathfrak{N}, \mathfrak{N}\}$.

If a 3 - group is abelian, then its center is G. For non-abelian 3 - groups we have the following two results:

3.3 Theorem. A non-abelian 3 - group G has an abelian decomposition (i.e., a decomposition of the type $\{\mathfrak{A}, \mathfrak{A}, \mathfrak{A}\}$) if and only if the center of G is K.

Proof. Let G have a decomposition $\{A, B, C\}$ and Z be the center of G.

Firstly, suppose that A, B and C are abelian. Then Z contains K. Suppose that $Z \neq K$; without loss of generality we may let a'_z be in $A' \cap Z$. For each c' in C' there exists a b' in B' such that $c' = a'_z b'$. Since A, B and C are all abelian, we have

$$c'b = (a'_z b')b = b(a'_z b') = bc'$$

for any b in B. Then elements of B and C' commute. It follows that the elements of B and C commute .

Similarly we can show that elements of A and B and elements of A and C commute.

Hence G is abelian, which contradicts the assumption. Therefore Z = K.

Conversely, let Z = K. For any a', a'_1 in A', k in K, there exist a'_2 in A' and k_1 in K such that $a = a'_2 k$ and $a' = a'_1 k_1$. Thus we have

$$a'a'_1 = (a'_1 k_1)a'_1 = a'_1(k_1 a'_1) = a'_1 a'$$

and $$a'k = (a'_2 k)k = k(a'_2 k) = ka'.$$

Hence A is abelian.

Similar arguements show that B and C are abelian.

3.4 <u>Theorem</u>. If G admits a decomposition $\{A, B, C\}$ of the type $\{\mathcal{a}, \mathcal{n}, \mathcal{n}\}$, then the center Z of G is contained in A.

<u>Proof</u>. Suppose that there exists a b'_z in $B' \cap Z$, let b', b'_1 be in B' and k in K. Then there exist k_1, k_2 in K such that $b' = b'_z k_1$ and $b'_1 = b'_z k_2$. So we have

$$b'b'_1 = (b'_z k_1)(b'_z k_2) = (b'_z k_2)(b'_z k_1) = b'_1 b'$$

and $$b'k = (b'_z k_1)k = k(b'_z k_1) = kb'.$$

Hence B is abelian, which contradicts the assumption. Thus $B' \cap Z = \emptyset$. Similary we can show that $C' \cap Z = \emptyset$.

Example 8. For D_6 which has the decomposition $\{C_6, S_3, S_3\}$, we have the center Z is such that $A' \cap Z \neq \emptyset$, that is the center Z contains elements of A'.

If a group G has a decomposition $\{A, B, C\}$ of the type $\{n, n, n\}$, it is clear that Z can not contain elements from A' and B' and not C', since $Z \cap A$ and $Z \cap B$ are groups and a group can not be expressed as an irredundant union of two subgroups (2.1 of Chapter III). But there is the possibility that Z contains elements of A', B' and C' in this case Z is itself necessarily a 3 - group with decomposition $\{Z \cap A, Z \cap B, Z \cap C\}$ (Z may of course be a 3 - group in other cases). The existence of such a decomposition requires that K be non-abelian. Because if K is abelian, there exists an a'_z in $Z \cap A'$ and for any a', a'_1 in A' and k in K, there exist k_1, k_2 in K such that $a' = a'_z k_1$ and $a'_1 = a'_z k_2$. Then we have

$$a'a'_1 = (a'_z k_1)(a'_z k_2) = (a'_z k_2)(a'_z k_1) = a'_1 a'$$

and $$a'k = (a'_z k_1)k = k(a'_z k_1) = ka'.$$

Hence A is abelian, which is a contradiction.

Example 9. For $S_3 \times V$ has a decomposition $\{D_6, D_6, D_6\}$ with none of $Z \cap A'$, $Z \cap B'$, $Z \cap C'$ is empty.

4 Groups of Inner Automorphisms of 3 - groups and Their Degeneracies.

Let a 3 - group G have the decomposition $\{A, B, C\}$. Let $I'(A)$ be the set of inner automorphisms of G defined by elements of A; i.e.,

$I'(A) = \{ i(a) \ / \ a \in A$ and $(i(a))(x) = a^{-1}xa$ for any x in G $\}$.

Then $I'(A)$ is a subgroup of I(G), the group of inner automorphisms of G. Moreover

$$I(G) = I'(A) \cup I'(B) \cup I'(C).$$

4.1 Theorem. The group of inner automorphisms of a 3 - group G with decomposition $\{A, B, C\}$ is a 3 - group or degenerated relative to $\{A, B, C\}$, in the sense that it is one of $I'(A)$, $I'(B)$ or $I'(C)$.

Proof. We have $I(G) = I'(A) \cup I'(B) \cup I'(C)$. If $I(G)$ is the irredundant union of the three subgroups $I'(A)$, $I'(B)$ and $I'(C)$, then $I(G)$ is a 3 - group. On the other hand, if $I(G)$ is not the irredundant union of these subgroups, then by 2.1 of Chapter III, we have $I(G)$ is one of $I'(A)$, $I'(B)$ or $I'(C)$.

We note that the degeneracy (as defined in 4.1) does not necessarily exclude $I(G)$ from being a 3 - group, as is shown by the next example. However, we are considering the structure of $I(G)$ relative to the decomposition $\{A, B, C\}$ of G so that the name "degenerate" is appropriate.

Example 10. Q has a decomposition $\{C_4, C_4, C_4\}$ and

$I(G) = V$ (nondegenerated).

$C_2 \times D_4$ has a decomposition $\{C_4 \times C_2, D_4, D_4\}$ and

$I(G) = I(D_4) = V$ (degenerate).

4.2 Theorem. A non-abelian 3 - groups has an abelian decomposition if and only if the group of inner automorphisms is the Klein 4 - group.

Proof. Let a non-abelian 3 - group G has an abelian decomposition. By 3.3, the center Z of G is K. Let φ be such that

$$\varphi : G \longrightarrow I(G),$$

defined by

$$g \longmapsto i(g^{-1}).$$

Then φ is an onto homomorphism. Thus $G/_{\text{Ker}\,\varphi}$ is isomorphic to I(G). To show that Ker φ = Z, let g be in Ker φ . Then we have $\varphi(g) = i(g^{-1}) = 1$ is in $\varphi(G) = I(G)$ so that $(i(g^{-1}))(g_1) = g_1$ for any g_1 in G and therefore $gg_1g^{-1} = g_1$, which implies that $g_1g = gg_1$. Hence we have Kerφ is a subset of Z.

Again, let k be in Z = K. Since $kg_1k^{-1} = g_1$ for any g_1 in G, $(i(k^{-1}))(g_1) = g_1$. So we have $\varphi(k) = 1$ and k is in Ker φ .

Hence Ker φ = Z so that

$$I(G) = G/_Z = G/_K \;;$$

which is the Klein 4-group by 2.5.

Conversely, let I(G) be the Klein 4-group. From the "if" part, we have $\varphi : G \longrightarrow I(G)$ is an onto homomorphism. If follows from 2.5 that G is a 3-group. By the proof of 2.5, we have constructed a decomposition $\{A, B, C\}$ with $K = A \cap B \cap C = \varphi^{-1}(1)$. To show that K = Z, let k be in K $K = \varphi^{-1}(1)$. Then we have $\varphi(k) = i(k^{-1}) = 1$ so that $(i(k^{-1}))(g) = kgk^{-1} = g$ and therefore gk = kg. Hence K is a subset of Z.

Again, let z be in Z. Since $zgz^{-1} = g$ for any g in G $(i(z^{-1}))(g) = g$. So we have $i(z^{-1}) = \varphi(z) = 1$ and z is in $\varphi^{-1}(1) = K$. Hence K = Z and it follows from 3.3 that G has an abelian decomposition.

A relationship between the 3-group structure and degeneracy of I(G) is given by

<u>4.3 Theorem</u>. Let $\{A, B, C\}$ be a decomposition of the 3-group G. The group inner automorphism of G is degenerated relative to $\{A, B, C\}$ if and only if the center

of G contains elements other than from $K = A \cap B \cap C$.

Proof. To prove the "if" part, we may, without loss of generality, assume $I'(B)$ is a subset of $I'(C)$. Then for each b' in B', there exists a k in K or a c' in C' such that $i(b') = i(k)$ or $i(b') = i(c')$. If $i(b') = i(k)$, then for any g in G we have $b'^{-1}gb' = k^{-1}gk$. Since b' is in B' and k in K, there exists a b'_1 in B' such that $b' = b'_1k$. Then we have

$$b'^{-1}gb' = (b'_1k)^{-1}g(b'_1k) = k^{-1}(b'^{-1}_1gb'_1)k.$$

Therefore we have $b'^{-1}_1gb'_1 = g$ so that $gb'_1 = b'_1g$. Then b'_1 is in $Z \cap B'$.

If $i(b') = i(c')$, then for any g in G we have $b'^{-1}gb' = c'^{-1}gc'$. Since b' is in B' and c' in C', there exists an a' in A' such that $b' = a'c'$. It then follows as before that a' is in $Z \cap A'$. Hence the "if" part is proved.

Conversely, let a'_z be in $Z \cap A'$. For any b' in B', there exists a c' in C' such that $b' = c'a'_z$ and for any g in G, we have

$$b'^{-1}gb' = (c'a'_z)^{-1}g(c'a'_z) = c'^{-1}gc';$$

which implies that $i(b') = i(c')$. Then we have $I'(B')$ is a subset of $I'(C')$. Hence $I(G)$ is degenerated.