CHAPTER III

GROUPS WHICH ARE UNIONS OF THEIR PROPER SUBGROUPS

Alomost all the materials of this chapter are drawn from reference [2].

1 Introduction.

It is evident that any group G which is not cyclic is expressible as a union of proper subgroups; for example, such a group G is the union of its cyclic subgroups, which by hypothesis are all proper. Conversely, if a group is a union of proper subgroups of itself, it clearly can not be cyclic.

It is not easy to characterize groups which are finite unions of proper subgroups. That there exist noncyclic groups which are not such finite unions is seen from the example of the additive group $\mathbb{Q}^+$ (the set of rational numbers with the usual addition). Indeed suppose that $\mathbb{Q}^+$ is the union of its proper subgroups H_1, H_2, ..., H_n. We can assume that this union is irredundant. Then there exists an $r = \frac{m}{n}$ in H_1 for some integers m, n $\neq$ o and r does not belong to $\bigcup_{i=2}^{n} H_i$. If $\frac{r}{p}$ is in $\bigcup_{i=2}^{n} H_i$ for some integer $p \neq o$, then r is in $\bigcup_{i=2}^{n} H_i$. Then for any nonzero integer p, $\frac{r}{p}$ is in H_1. Thus we have $\frac{r}{m} = \frac{1}{n}$ is in H_1. Let $\frac{c}{d}$ be in $\mathbb{Q}^+$ where c and d $\neq$ o are integers, and since $\frac{1}{nd}$ is in H_1, so we have $cn(\frac{1}{nd}) = \frac{c}{d}$ is in H_1. Hence $\mathbb{Q}^+ = H_1$, which is a contradiction.

More generally, no locally cyclic group can be a finite union of its proper subgroups, which implies that no subgroup of $\mathbb{Q}^+$ can be an irredundant union of its subgroups. A proof is given as follows:

Suppose that a locally cyclic group G is the irredundant union of its proper subgroups $G_1, G_2, \ldots, G_n$. Then there exist g_1 and g_2 belong to $G_1 \setminus \bigcup_{i=2}^{n} G_i$ and $G_2 \setminus \bigcup_{\substack{i=1 \\ i\neq 2}}^{n} G_i$; respectively. Since G is locally cyclic, $[g_1, g_2] = [g]$ for some g in G. If g belongs to G_1, then $[g]$ is a subgroup of G_1 so that g_1g_2 belongs to G_1; this implies that g_2 belongs to G_1, which is impossible. Similary g can not belong to G_2. Then g must belong to $\bigcup_{i=3}^{n} G_i \setminus (G_1 \cup G_2)$ so that $[g]$ is a subset of $\bigcup_{i=3}^{n} G_i$ and it follows that g_1 and g_2 belong to $\bigcup_{i=3}^{n} G_i$, which is a contradiction, since g_1 and g_2 do not belong to $\bigcup_{i=3}^{n} G_i$.

2 Groups can not be Unions of Two Proper Subgroups.

2.1 Theorem. No group is a union of two of its proper subgroups.

Proof. Suppose that a group G is the union of its proper subgroups A and B. Then the union must be irredundant. Therefore there exist a in $A \setminus B$ and b in $B \setminus A$. So ab is in G, it follows that ab is either in A or B. If ab is in B, then a must be in B and if ab is in A, then b must be in A. In either case, we have a contradiction.

3 Groups Which are Irredundant Unions of Arbitrary Number of Their Subgroups.

3.1 Theorem. Let G be the irredundant union of the subgroups H_α's. Then for each α H_α contains the intersection of all the remaining H's.

Proof. Let a be in H_α and not in the other H's and let b be an element contained in the intersection of all the other H's. If ab is in H_β for some $\beta \neq \alpha$, then $(ab)b^{-1} = a$ is in H_β, which is a contradiction. Therefore ab must belong to H_α so that $a^{-1}(ab) = b$ is in H_α.
Hence the theorem is proved.

3.2 Theorem. Let G be a group and I an index set such that

$G = \bigcup_{\alpha \in I} H_\alpha$ is the irredundant union of its subgroups H_α and let

$I = I_1 \cup I_2$ be such that $I_1 \cap I_2 = \emptyset$.

If $\left(\bigcap_{\alpha \in I_1} H_\alpha \setminus \bigcup_{\beta \in I_2} H_\beta\right) \neq \emptyset$, then $\bigcap_{\alpha \in I_1} H_\alpha \setminus \bigcup_{\beta \in I_2} H_\beta$ contains its inverses.

Proof. Let a belong to $\bigcap_{\alpha \in I_1} H_\alpha \setminus \bigcup_{\beta \in I_2} H_\beta$. Then a belongs to H_α for all α in I_1. Since H_α is a subgroup of G, a^{-1}, the inverse of a belongs to H_α for all α in I_1. If a^{-1} is in H_β for some β in I_2, then a is in H_β, which is a contradiction. Hence a^{-1} is in $\bigcap_{\alpha \in I_1} H_\alpha \setminus \bigcup_{\beta \in I_2} H_\beta$.

From 3.2, the following corollary follows immediately.

3.3 <u>Corollary</u>. Let G be a group and I an index set. If $G = \bigcup_{\alpha \in I} H_\alpha$ is the irredundant union of its subgroups H_α, then for each β in I,

$$H'_\beta = H_\beta \setminus \bigcup_{\substack{\alpha \in I \\ \alpha \neq \beta}} H_\alpha$$

contains its inverses.

3.4 <u>Theorem</u>. Let G be a group and I an index set such that $G = \bigcup_{\alpha \in I} H_\alpha$ is the irredundant union of its subgroups H_α and let $I = I_1 \cup I_2$ be such that $I_1 \cap I_2 = \emptyset$. If $\bigcap_{\alpha \in I_1} H_\alpha \setminus \bigcup_{\beta \in I_2} H_\beta \neq \emptyset$, then $\bigcap_{\beta \in I_2} H_\beta \setminus \bigcup_{\alpha \in I_1} H_\alpha = \emptyset$.

<u>**Proof.**</u> Let $\bigcup_{\alpha \in I_1} H_\alpha \setminus \bigcap_{\beta \in I_2} H_\beta \neq \emptyset$. Then there exists an a in $\bigcap_{\alpha \in I_1} H_\alpha \setminus \bigcup_{\beta \in I_2} H_\beta$. Suppose there exists a b in $\bigcap_{\beta \in I_2} H_\beta \setminus \bigcup_{\alpha \in I_1} H_\alpha$. Since ab is in G, ab is either in H_α for some α in I_1 or in H_β for some β in I_2. If ab is in H_α some α in I_1, then $a^{-1}(ab) = b$ is in H_α and if ab is in H_β for some β in I_2, then a is in H_β. In either case, we have a contradiction.

Hence the theorem is proved.

3.5 <u>Theorem</u>. Let G be a group and I an index set such that $G = \bigcup_{\alpha \in I} H_\alpha$ is the irredundant union of its subgroups H_α.

If a is in $H_p \setminus \bigcup_{\substack{\alpha \in I \\ \alpha \neq p}} H_\alpha$ and b in $H_q \setminus \bigcup_{\substack{\alpha \in I \\ \alpha \neq q}} H_\alpha$ for $p \neq q$, then ab is in $G \setminus H_p \cup H_q$.

Proof. Suppose that ab is in $H_p \cup H_q$. If ab is in H_p, then $a^{-1}(ab) = b$ is in H_p, and if ab is in H_q, then $(ab)b^{-1} = a$ is in H_q. In either case, we have a contradiction. Hence the conclusion follows.

3.6 Theorem. Let I be an index set and $A = \bigcup_{\alpha \in I} A_\alpha$ the irredundant union of subgroups A_α. If a group G can be mapped homomorphically onto A, then for each α in I, there exist G_α, a proper subgroup of G and $G = \bigcup_{\alpha \in I} G_\alpha$ is the irredundant union of subgroups G_α.

Proof. Let $\varphi : G \longrightarrow A$ be an onto homomorphism. For each in I we define G_α as follows;

$$G_\alpha = \varphi^{-1}(A_\alpha)$$

Then G_α is a subgroup of G and

$$G = \bigcup_{\alpha \in I} G_\alpha .$$

Finally, we will show that $G = \bigcup_{\alpha \in I} G_\alpha$ is the irredundant union of the G_α. Let β be in I. Since $A_\beta \setminus \bigcup_{\substack{\alpha \in I \\ \alpha \neq \beta}} A_\alpha \neq \emptyset$,

there exists an a'_β in $A_\beta \setminus \bigcup_{\substack{\alpha \in I \\ \alpha \neq \beta}} A_\alpha$. If $G_\beta \subset \bigcup_{\substack{\alpha \in I \\ \alpha \neq \beta}} G_\alpha$, then

$\varphi^{-1}(a'_\beta) \subset \bigcup_{\substack{\alpha \in I \\ \alpha \neq \beta}} G_\alpha$. Let y be an element in $\varphi^{-1}(a'_\beta)$.

Then y must belong to G_α for some α in I and $\alpha \neq \beta$, so that

$$a'_\beta = \varphi(y) \quad A_\alpha$$

which is impossible. Hence the union is irredundant.

4. Groups Which are Irredundant Unions of Finite Number of Their Subgroups.

4.1 Theorem. Let a group $G = \bigcup_{i=1}^{n} G_i$ be the irredundant union of its subgroups G_i and $K = \bigcap_{i=1}^{n} G_i$ is a normal subgroup of G. Then $o(G/K)$, the order of G/K is not a prime.

Proof. Let φ be the mapping such that

$$\varphi: G \longrightarrow G_{/K},$$

defined by $g \longmapsto gK.$

Then φ is an onto homomorphism . If $o(G_{/K})$ is a prime, then $G_{/K}$ has no proper subgroup. For each j in $\{1, 2, \ldots n\}$, $G_j \setminus \bigcup_{\substack{i=1\\ i\neq j}}^{n} G_i \neq \emptyset$. Then there exists an a'_j in $G_j \setminus \bigcup_{\substack{i=1\\ i\neq j}}^{n} G_j$. Let

$$A = \{ aK \mid a \in G_2 \},$$

which is a subgroup of $G_{/K}$. Since a'_2 is not in K, $a'_2K \neq K$. Then $A \neq \{K\}$. Suppose that a'_1K is in A, then there exists a_2 in G_2 such that $a'_1K = a_2K$, so that $a'_1a_2^{-1}$ is in K. Then $a'_1a_2^{-1}$ belongs to G_2 which implies that a'_1 is in G_2; a contradiction. Thus a'_1K is not in A. So we have A is a proper subgroup of $G_{/K}$. Hence $o(G_{/K})$ is not a prime.

4.2 Lemma. Let a group G be the irredundant union of subgroups $A_1, A_2, \ldots A_n (n > 2)$ and set $M = A_2 \cup A_3 \cup \ldots \cup A_n$. Then if a is not in M, we have a^k in M for some k in $\{1,2,\ldots n-1\}$.

Proof. Since a is not in M, a is in A_1. Let b belong to $A_2 \setminus (\bigcup_{i=3}^{n} A_i \cup A_1)$. If a^jb is in A_1 for some j in $\{0,1,\ldots n-1\}$, then b is in A_1, which is a contradiction. Thus a^jb is in M for all j in $\{0,1,\ldots n-1\}$. If $a^jb = a^mb$ for some j, m in $\{0,1,\ldots n-1\}$ and suppose that $j > m$, then we have $a^{j-m} = 1$ which is in M. In the case when $ab, a^2b, \ldots, a^{n-1}b, b$ are all distinct, it follows that there exist k_1 and k_2, $k_1 \neq k_2$ where $0 \leqslant k_1, k_2 \leqslant n-1$ such that $a^{k_1}b$ and $a^{k_2}b$ are in the same A_i for some $i \geqslant 2$. Suppose that $k_1 > k_2$, so we have $a^{k_1-k_2}$ is in A_i which is a subset of M.

Hence the lemma is proved completely.

4.3 Theorem. Suppose the k^{th} roots can be taken in the group G for every positive integer k less than a certain n. Then G is not an irredundant union of n (or fewer!) of it subgroups.

Proof. Suppose that G is the irredundant union of n proper subgroups and adopt the notation of the lemma. Let a be not in M and b be the $(n-1)!^{th}$ root of a which exists in G by hypothesis. If b is in M, then $b^{(n-1)!} = a$ is in M. So b is not in M. By 4.2 there exists a k in $\{1,2,\ldots n-1\}$ such that b^k is in M. Then $(b^k)^r = a$ is in M where $r = \frac{(n-1)!}{k}$, which is a contradiction.

The "fewer part" is an obvious consequence of the above proof.

4.4 Lemma. Let G be a finite group of order N, r is a prime to N, then each element of G has an r^{th} root.

Proof. Let a be in G. Since r and N are relatively prime, there exist integers h and k such that $hr + kN = 1$. Therefore $a^{hr} \cdot a^{kN} = a$ so that $(a^h)^r = a$. Let $b = a^h$ which is in G. It follows that $b^r = a$. Then a has an r^{th} root in G.

4.5 Corollary. Let G be a finite group of order N, p the smallest prime dividing N. Then G is not a union of p or fewer of its proper subgroups.

Proof. Let k be a positive integer and k less than p. Since p is the smallest prime dividing N, k and N are relatively prime. Then it follows from 4.4 that each element of G has $k^{\underline{th}}$ root. Hence, by 4.3 G is not an irredundant union of p or fewer of its proper subgroups.

The criterion of 4.3 can not be strengthened. Indeed let G be the abelian group generated by two elements a, b with the relations $a^p = b^p = 1$, then G is the union of the p+1 proper subgroups generated by $a, b, ab, a^2b, \ldots, a^{p-1}b$, respectively.

For finite groups, the case in which (as in the example just given) the minimum imposed by 4.5 is actually assumed may be partially characterized by

4.6 <u>Theorem</u>. With notations as in Corollary 4.5, suppose that G is the union of exactly p+1 proper subgroups S_i, then at least one of the S's, say S_j has index p. If moreover, this S_j is normal, then all the S_i have index p and p^2 divides N.

<u>Proof</u>. Suppose the order of G/S_i, $o(G/S_i)$ is not p for all i. Since p is the smallest prime that divides N and $o(G/S_i)$ must divide N, we have $o(G/S_i) > p$ so that $\frac{o(G)}{p+1} \geqslant o(S_i)$ for all i. Thus we have

$$N < \sum o(S_i) \leqslant (p+1)\frac{o(G)}{p+1} = o(G) = N;$$ a contradiction. Hence there exists at least one of S's, say S_j has index p.

Suppose further that S_j is normal. For $i \neq j$ S_iS_j is a subgroup of G. If S_iS_j is a proper subgroup of G, then G is the union of S_iS_j and the p-1 proper subgroups S_k for $k = 1, 2, \ldots, n+1$, $k \neq i$, $k \neq j$.

By Corollary 4.5, it is impossible. Thus $S_iS_j = G$. Since we have S_iS_j/S_j is isomophic to $S_i/S_i \cap S_j$,

$$\frac{o(G)}{o(S_j)}\, o(S_i \cap S_j) = o(S_i),$$

and therefore

$$p\, o(S_i \cap S_j) = o(S_i)$$

for $i \neq j$. Let $o(S_i) = \frac{N}{q_i}$ for $i = 1, 2, \ldots p+1$. Since p is the smallest prime that divides N, $q_i \geqslant p$. Suppose there exists an l such that $q_l > p$, so we have

$$\begin{aligned}
N = o(G) &\leqslant o(S_j) + \sum_{k\neq j} \left[o(S_k) - o(S_k \cap S_j) \right] \\
&\leqslant \frac{N}{p} + \sum_{k\neq j} \left[\frac{N}{q_k} - \frac{N}{pq_k} \right] \\
&= \frac{N}{p} + \sum_{k\neq j} \frac{1}{q_k} \left[N - \frac{N}{p} \right] \\
&< \frac{N}{p} + \sum_{k\neq j} \frac{1}{p} \left[N - \frac{N}{p} \right] \\
&= \frac{N}{p} + p\,\frac{1}{p} \left[N - \frac{N}{p} \right] \\
&= N,
\end{aligned}$$

which is a contradiction. So we have $q_i = p$ for all $i \neq j$ and $o(S_i \cap S_j) = \frac{N}{p^2}$. Hence all the S_i have index p and p^2 divides N.

5 Groups of Inner Automorphisms of Groups Which are Irredundant Unions of Their Subgroups.

5.1 Theorem. Let G be a group and let I(G) be the group of inner automorphisms of G. Let A be an index set.

If $I(G) = \bigcup_{\alpha \in A} I_\alpha$ is the irredundant union of its subgroups I_α, then for each α in A, there exists G_α, a subgroup of G such that $G = \bigcup_{\alpha \in A} G_\alpha$ is the irredundant union of the G_α.

Proof. Let

$I(G) = \{ i(g) \ / \ g \in G \text{ and } i(g)(x) = g^{-1}xg \text{ for all } x \text{ in } G \}$, which is the group of inner automorphisms of G. Let φ be the mapping such that

$$\varphi : G \longrightarrow I(G),$$

defined by $g \longmapsto i(g^{-1})$.

It is clear that φ is an onto homomorphism. Since $I(G) = \bigcup_{\alpha \in A} I_\alpha$ is the irredundant union of its subgroups I_α by 3.6, for each α in A, there exists G_α, a subgroup of G and $G = \bigcup_{\alpha \in A} G_\alpha$ is the irredundant union of the G_α.

Hence the theorem is proved.

5.2 Remark. It is obvious to see that the converse of 5.1 is not true, since the group of inner automomorphisms of an abelian group is the trivial group, i.e., if G is an abelian group, then $I(G) = \{1\}$.