CHAPTER III

MAXIMAL STRONGLY FACTORIZABLE SUBSEMIGROUPS OF SYMMETRIC INVERSE SEMIGROUPS

In this chapter, we characterize maximal strongly factorizable subsemigroups of the symmetric inverse semigroup on a finite set.

Since a regular semigroup in which any two idempotents commute is an inverse semigroup, it follows that a regular subsemigroup of an inverse semigroup is an inverse subsemigroup. Then, a strongly factorizable subsemigroup of an inverse semigroup S is a strongly factorizable inverse subsemigroup of S. Hence, for a subsemigroup T of an inverse semigroup S, T is a maximal strongly factorizable subsemigroup of S if and only if T is a maximal strongly factorizable inverse subsemigroup of S.

If X is a set, then for a nonempty subset A of X, let 1_A denote the identity map on A and let $1_{\emptyset} = 0$, the empty transformation.

Let X be a set. If S is a transformation semigroup on X, then

$$E(S) = \{\alpha \in S \mid \nabla\alpha \subseteq \Delta\alpha \text{ and } x\alpha = x \text{ for all } x \in \nabla\alpha\}.$$

Hence

$$E(I_X) = \{\alpha \in I_X \mid \nabla\alpha = \Delta\alpha \text{ and } x\alpha = x \text{ for all } x \in \nabla\alpha\}$$

$$= \{1_A \mid A \subseteq X\}$$

because for $\alpha \in I_X$, α is a one-to-one map.

A finite group with zero is a strongly factorizable inverse semigroup. Then we have

3.1 Lemma. For a set X, if Y is a finite subset of X, then $G_Y \cup \{0\}$ is a strongly factorizable subsemigroup of the symmetric inverse semigroup on X.

If S is a strongly factorizable inverse semigroup, then for $e, f \in E(S)$, $ef\ (= fe) = e$ or $ef = f$, and hence every nonempty subset of E(S) is a subsemigroup of S, so it has a maximum element by Theorem 1.9.

Let X be a set. For $A \subseteq X$, $B \subseteq X$, we have that $1_A, 1_B \in E(I_X)$ and $1_A 1_B = 1_{A \cap B} \in E(I_X)$. Then for $A, B \subseteq X$, $1_A 1_B = 1_B$ if and only if $B \subseteq A$.

3.2 Lemma. Let X be a set and $\mathcal{C}$ a nonempty set of subsets of X. Let $S = \{1_A \mid A \in \mathcal{C}\}$. Then S is a strongly factorizable subsemigroup of the symmetric inverse semigroup on X, I_X, if and only if $A \cap B \in \mathcal{C}$ for all $A, B \in \mathcal{C}$, and every nonempty subset $\mathcal{S}$ of $\mathcal{C}$ with the property that $A \cap B \in \mathcal{S}$ for all $A, B \in \mathcal{S}$ has a maximum element under the partialorder of set inclusion.

Proof : Assume that S is a strongly factorizable subsemigroup of I_X. Since S is a semigroup, for all $A, B \in \mathcal{C}$, $1_A 1_B = 1_{A \cap B} \in S$ which implies $A \cap B \in \mathcal{C}$. Next, let $\mathcal{S}$ be a nonempty subset of $\mathcal{C}$ such that $A \cap B \in \mathcal{S}$ for all A, B in $\mathcal{S}$. Then $\{1_A \mid A \in \mathcal{S}\}$ is a subsemigroup of S, so it is strongly factorizable. Hence $\{ 1_A \mid A \in \mathcal{S}\}$ has a maximum element under the natural partial order, say 1_M, $M \in \mathcal{S}$. Thus $1_A 1_M = 1_A$ for all $A \in \mathcal{S}$ which implies $A \subseteq M$ for all $A \in \mathcal{S}$.

Conversely, assume that $A \cap B \in \mathcal{C}$ for all $A, B \in \mathcal{C}$, and every nonempty sebset $\mathcal{S}$ of $\mathcal{C}$ such that $A \cap B \in \mathcal{S}$ for all $A, B \in \mathcal{S}$ has a maximum element under the partial order of set inclusion. Because for all $A, B \in \mathcal{C}$, $A \cap B \in \mathcal{C}$, it follows that S is a subsemigroup of I_X. Let T be a subsemigroup of S. Let

$$\mathcal{S} = \{ A \in \mathcal{C} \mid 1_A \in T \}.$$

Since T is a subsemigroup of S, we have that $\mathcal{S} \neq \emptyset$ and $A \cap B \in \mathcal{S}$ for all $A, B \in \mathcal{S}$. By assumption, there exists $M \in \mathcal{S}$ such that $A \subseteq M$ for all $A \in \mathcal{S}$. Then $1_M \in T$ and $1_M 1_A = 1_A$ for all $A \in \mathcal{S}$, and thus $\{1_M\}E(T) = \{1_M\}T = T$. Hence T is factorizable. □

3.3 Lemma. Let X be a set and T a strongly factorizable subsemigroup of the symmetric inverse semigroup on X, I_X. Then for all $\alpha \in T$, $\Delta\alpha = \nabla\alpha$.

Proof : Let $\alpha \in T$. Since T is strongly factorizable, there exists $\beta \in E(T)$ such that $\alpha \mathcal{H} \beta$ in T. Because T is a transformation semigroup and $\alpha \mathcal{H} \beta$ in T, it follows that $\Delta\alpha = \Delta\beta$ and $\nabla\alpha = \nabla\beta$ [Chapter II, page 29]. But $\beta \in E(I_X)$, we have that $\Delta\beta = \nabla\beta$. Hence $\Delta\alpha = \nabla\alpha$. □

It has been showed in [4] that if a semigroup S has an identity 1 and S is factorizable as GE(S), then 1 is the identity of G.

If G is a group such that $|G| > 1$, then it clearly follows that the semigroup $G \cup 1$ [Introduction, page 5] is not factorizable.

Let X be a set and $\alpha \in I_X$. Suppose that $\Delta\alpha = \nabla\alpha$. Then α is a permutation on $\Delta\alpha$, that is, $\alpha \in G_{\Delta\alpha}$. If $|\Delta\alpha| < \infty$, then the cyclic subsemigroup generated by α, $\langle\alpha\rangle$, is a subsemigroup of $G_{\Delta\alpha}$, and hence $\langle\alpha\rangle$ is a subgroup of $G_{\Delta\alpha} \subseteq I_X$ with identity $1_{\Delta\alpha}$.

3.4 __Lemma__. Let X be a set and $\alpha \in I_X \smallsetminus E(I_X)$ such that $\Delta\alpha = \nabla\alpha$ which is finite. Let Y be a subset of X such that $\Delta\alpha \subseteq Y$. Then $\langle\alpha\rangle \cup \{1_Y\}$ is a factorizable subsemigroup of I_X if and only if $Y = \Delta\alpha$.

__Proof__ : Let $S = \langle\alpha\rangle \cup \{1_Y\}$. Since $\Delta\alpha \subseteq Y$, S is a subsemigroup of I_X having 1_Y as its identity. Because $\Delta\alpha = \nabla\alpha$ and $|\Delta\alpha| < \infty$, $\langle\alpha\rangle$ is a subgroup of I_X with identity $1_{\Delta\alpha}$.

If $Y = \Delta\alpha$, then $S = \langle\alpha\rangle$ which is a group, so it is factorizable.

Assume that $\Delta\alpha \subsetneq Y$. Then $S \cong \langle\alpha\rangle \cup 1$. Because $\alpha \notin E(I_X)$ and $\Delta\alpha = \nabla\alpha$ which is finite, it follows that $\langle\alpha\rangle$ is a subgroup of I_X and $|\langle\alpha\rangle| > 1$. Then the semigroup $\langle\alpha\rangle \cup 1$ is not factorizable. Hence S is not factorizable. □

3.5 __Lemma__. Let Y be a subset of a finite set X. Then $\langle G_X \cup \{1_Y\}\rangle$ is a strongly factorizable subsemigroup of I_X if and only if $Y = \emptyset$ or $Y = X$.

__Proof__ : Assume $\langle G_X \cup \{1_Y\}\rangle$ is a strongly factorizable subsemigroup of I_X. If $|X| \leq 1$, then $Y = \emptyset$ or $Y = X$. Assume $|X| > 1$. Claim that $Y = \emptyset$ or $Y = X$. Suppose not, then $Y \neq \emptyset$ and $Y \subsetneq X$. Let $Y = \{a_1, a_2, \ldots, a_m\}$ and $X = \{a_1, a_2, \ldots, a_n\}$ where m and n are positive integers such that $m < n$ and $a_i \neq a_j$ if $i \neq j$ in $\{1,2,\ldots, n\}$. Define the map $\alpha : X \to X$ as follows, $a_1\alpha = a_2$, $a_2\alpha = a_3$, ..., $a_{n-1}\alpha = a_n$, $a_n\alpha = a_1$. Then $\alpha \in G_X$. Let $\beta = 1_Y\alpha$. Then $\beta \in \langle G_X \cup \{1_Y\}\rangle$, hence by Lemma 3.3, $\Delta\beta = \nabla\beta$. But $\Delta\beta = \Delta 1_Y\alpha = (Y \cap X)\, 1_Y^{-1} = Y$ and $\nabla\beta = \nabla 1_Y\alpha = (Y \cap X)\alpha = \{a_2, a_3, \ldots, a_m, a_{m+1}\}$, hence $\Delta\beta \neq \nabla\beta$ which is a contradiction. This proves that $Y = \emptyset$ or $Y = X$.

Conversely, assume $Y = \emptyset$ or $Y = X$. If $Y = \emptyset$, then $\langle G_X \cup \{1_Y\}\rangle = G_X \cup \{0\}$ which is a strongly factorizable semigroup by Lemma 3.1. If $Y = X$, then $\langle G_X \cup \{1_Y\}\rangle = G_X$ which is a strongly factorizable semigroup because G_X is a finite group. □

3.6 <u>Lemma</u>. Let Y be a subset of a set X. Then $G_Y \cup \{1_X\}$ is a strongly factorizable subsemigroup of I_X if and only if Y is finite and either $|Y| \leq 1$ or $Y = X$.

<u>Proof</u> : Assume that $G_Y \cup \{1_X\}$ is a strongly factorizable semigroup. Since G_Y is a subsemigroup of $G_Y \cup \{1_X\}$, G_Y is a strongly factorizable semigroup. By Theorem 2.3, Y is a finite set. Claim that $|Y| \leq 1$ or $Y = X$. Suppose that $|Y| > 1$ and $Y \subsetneq X$. Since $Y \subsetneq X$, G_Y and $\{1_X\}$ are the only maximal subgroups of the semigroup $G_Y \cup \{1_X\}$. But $\{1_X\}E(G_Y \cup \{1_X\}) = \{1_Y, 1_X\} \neq G_Y \cup \{1_X\}$ since $|Y| > 1$, and $G_YE(G_Y \cup \{1_X\}) = G_Y \neq G_Y \cup \{1_X\}$ since $Y \subsetneq X$. Then the semigroup $G_Y \cup \{1_X\}$ is not factorizable, a contradiction.

Conversely, assume that Y is finite and either $|Y| \leqslant 1$ or $Y = X$. If $|Y| \leqslant 1$, then $G_Y \cup \{1_X\} = \{1_Y, 1_X\}$ which is a strongly factorizable semigroup. If $Y = X$, then $G_Y \cup \{1_X\} = G_Y$ which is a strongly factorizable semigroup since G_Y is a finite group. □

Let S be a semilattice. If S is strongly factorizable, then by Corollary 1.12, S is a chain under the natural partial order. If S is a finite chain, then any subsemigroup T of S has a maximum element e and $\{e\}E(T) = eT = T$.

Hence a finite semilattice is strongly factorizable if and only if it is a chain.

3.7 Lemma. Let X be a finite set with $|X| = n$. For each $i \in \{0, 1, \ldots, n\}$, let Y_i be a subset of X such that $\emptyset = Y_o \subseteq Y_1 \subseteq Y_2 \subseteq \ldots \subseteq Y_n = X$ and $|Y_i| = i$. Let Z be a subset of X. Then $\langle \{ 1_{Y_i} \mid i = 0,1,\ldots, n \} \cup \{1_Z\} \rangle$ is a strongly factorizable subsemigroup of I_X if and only if $Z = Y_i$ for some $i \in \{0, 1, \ldots, n\}$.

Proof : Let $T = \langle \{ 1_{Y_i} \mid i = 0, 1, \ldots, n \} \cup \{1_Z\} \rangle$. Then $T = \{1_{Y_i} \mid i = 0, 1, \ldots, n \} \cup \{ 1_{Z \cap Y_i} \mid i = 0, 1, \ldots, n \}$ which is a finite semilattice.

Assume that T is strongly factorizable. Then T is a chain and thus $\{Y_i \mid i = 0, 1, \ldots, n \} \cup \{ Z \cap Y_i \mid i = 0, 1, \ldots, n \}$ is a chain of sets under the partial order of set inclusion. Let $|Z| = k$. Then $Z \cap Y_n = Z \subseteq Y_k$ or $Y_k \subseteq Z$. Since $|Y_k| = k$, it follows that $Z = Y_k$.

Conversely, assume that $Z = Y_k$, then $T = \{1_{Y_i} \mid i = 0, 1, \ldots, n\}$ which is a finite chain, so T is strongly factorizable. ◻

For any set X, let T be a strongly factorizable subsemigroup of I_X. Then for $\alpha \in T$, $1_{\Delta\alpha} = 1_{\nabla\alpha} \in T$. To prove this, let $\alpha \in T$. Then by Lemma 3.3, $\Delta\alpha = \nabla\alpha$. Since T is an inverse subsemigroup of I_X, $\alpha^{-1} \in T$. Hence $\alpha\alpha^{-1} = 1_{\Delta\alpha} \in T$. Thus $1_{\Delta\alpha} = 1_{\nabla\alpha} \in T$.

Let X be a finite set with $|X| > 1$. Let n be a nonnegative integer and $Z_o, Z_1, \ldots, Z_n$, Y subsets of X such that
$\emptyset = Z_o \subseteq Z_1 \subseteq \ldots \subseteq Z_n \subseteq Y$, $|Z_{i+1} \smallsetminus Z_i| = 1$ for all $i \in \{0, 1, \ldots, n-1\}$ and either $Z_n = Y = X$ or $|Y \smallsetminus Z_n| > 1$. Then for each $i \in \{0, 1, \ldots, n\}$, $|Z_i| = i$. Let

$$T(Z_o, Z_1, \ldots, Z_n; Y) = \{1_{Z_i} \mid i = 0, 1, \ldots, n\} \cup \{\alpha \in G_Y \mid a\alpha = a \text{ for all } a \in Z_n\}.$$

3.8 Theorem. $T(Z_o, Z_1, \ldots, Z_n; Y)$ is a maximal strongly factorizable subsemigroup of the symmetric inverse semigroup on X.

Proof : It is easy to see that $T(Z_o, Z_1, \ldots, Z_n; Y)$ is a subsemigroup of I_X with identity 1_Y and

$$E(T(Z_o, Z_1, \ldots, Z_n; Y)) = \{1_{Z_o}, 1_{Z_1}, \ldots, 1_{Z_n}, 1_Y\}.$$

Since $Z_o \subseteq Z_1 \subseteq \ldots \subseteq Z_n \subseteq Y$, $E(T(Z_o, Z_1, \ldots, Z_n; Y))$ is a chain. If $\alpha \in G_Y$ such that $a\alpha = a$ for all $a \in Z_n$, so $\alpha^{-1} \in G_Y$ and $a\alpha^{-1} = (a\alpha)\alpha^{-1} = a$ for all $a \in Z_n$. Hence

$$T(Z_o, Z_1, \ldots, Z_n; Y) = \{1_{Z_o}\} \cup \ldots \cup \{1_{Z_n}\} \cup \{\alpha \in G_Y \mid a\alpha = a \text{ for all } a \in Z_n\}.$$

which is a union of subgroups of $T(Z_o, Z_1, \ldots Z_n; Y)$. Then $T(Z_o, Z_1, \ldots, Z_n; Y)$ is a finite inverse semigroup. By Corollary 1.13, $T(Z_o, Z_1, \ldots, Z_n; Y)$ is strongly factorizable.

Now, to show that $T(Z_o, Z_1, \ldots, Z_n; Y)$ is a maximal strongly factorizable subsemigroup of I_X, let K be a strongly factorizable subsemigroup of I_X containing $T(Z_o, Z_1, \ldots, Z_n; Y)$. Claim that $K = I_X$ or $K = T(Z_o, Z_1, \ldots, Z_n; Y)$. Since $|X| > 1$, by Theorem 2.4, I_X is not strongly factorizable, it follows that $K \neq I_X$.

Assume that for each $k \in \{1, 2, \ldots, n\}$, $Z_k = \{a_1, a_2, \ldots, a_k\}$, $a_i \neq a_j$ if $i \neq j$.

<u>Case 1 : $Z_n = Y = X$</u>. Then $T(Z_o, Z_1, \ldots, Z_n; Y) = \{1_{Z_i} \mid i = 0, 1, \ldots, n\}$. Let $\alpha \in K \setminus \{0\}$. Thus $1_{\Delta\alpha} = 1_{\nabla\alpha} \in K$, so $\langle\{1_{Z_i} \mid i = 0, 1, \ldots, n\} \cup \{1_{\Delta\alpha}\}\rangle$ is a subsemigroup of K. Then $\langle\{1_{Z_i} \mid i = 0, 1, \ldots, n\} \cup \{1_{\Delta\alpha}\}\rangle$ is strongly factorizable. Hence by Lemma 3.7, $\Delta\alpha = Z_k$ for some $k \in \{1, 2, \ldots, n\}$. Thus by Lemma 3.3, $\Delta\alpha = \nabla\alpha = Z_k$. To show that $\alpha = 1_{Z_k}$, suppose not. Then there exists $i \in \{1, 2, \ldots, k\}$ such that $a_i\alpha \neq a_i$. Let i_o be the smallest positive integer such that $a_{i_o}\alpha \neq a_{i_o}$. Let $\beta = 1_{Z_{i_o}}\alpha$. Then $\beta \in K$, so by Lemma 3.3, $\Delta\beta = \nabla\beta$. But $\Delta\beta = \Delta 1_{Z_{i_o}}\alpha = (Z_{i_o} \cap Z_k)1_{Z_{i_o}}^{-1} = Z_{i_o}$ and $\nabla\beta = \nabla 1_{Z_{i_o}}\alpha = (Z_{i_o} \cap Z_k)\alpha = \{a_1, a_2, \ldots, a_{i_o-1}, a_{i_o}\alpha\}$, thus $\Delta\beta \neq \nabla\beta$ which is a contradiction. This shows that

$\alpha = 1_{Z_k}$, so $\alpha \in T(Z_o, Z_1, \ldots, Z_n;Y)$. Hence $K \smallsetminus \{0\} \subseteq T(Z_o, Z_1,\ldots,Z_n;Y)$, so $K \subseteq T(Z_o, Z_1, \ldots, Z_n;Y)$; and thus $K = T(Z_o, Z_1, \ldots, Z_n;Y)$.

Case 2 : $|Y \smallsetminus Z_n| \geq 1$ and $Z_n = \emptyset$. Then $T(Z_o, Z_1,\ldots,Z_n;Y) = G_Y \cup \{0\}$ and $|Y| > 1$. Since K is a factorizable inverse semigroup, K has an identity. Let $A \subseteq X$ be such that 1_A is the identity of K. Since $K \neq \{0\}$, $A \neq \emptyset$. Let K be factorizable as $K = GE(K)$. Then 1_A is the identity of G and G is the unit group of K. Hence $G = K \cap G_A$ and thus $K \subseteq G_A E(K)$. Since $1_Y \in T(Z_o, Z_1, \ldots, Z_n;Y) \subseteq K$, $1_Y 1_A = 1_Y$ and thus $Y \subseteq A$. Hence $G_Y \cup \{1_A\}$ is a subsemigroup of K, so it is strongly factorizable. By Lemma 3.6, $|Y| \leq 1$ or $Y = A$. But $|Y| > 1$, then $Y = A$. Thus $G_Y \cup \{0\} \subseteq K \subseteq G_Y E(K)$. Let $\alpha \in K \smallsetminus \{0\}$. Then $\alpha = \beta\gamma$ for some $\beta \in G_Y$ and $\gamma \in E(K)$. Therefore $\beta^{-1}\alpha = 1_Y\gamma = \gamma = 1_{\Delta\gamma}$, so $\Delta\gamma = \Delta\beta^{-1}\alpha \subseteq \Delta\beta^{-1} = Y$. Since $\langle G_Y \cup \{1_{\Delta\gamma}\}\rangle$ is a subsemigroup of K, so it is strongly factorizable. By Lemma 3.5, $\Delta\gamma = \emptyset$ or $\Delta\gamma = Y$. But $\alpha \neq 0$, so $\gamma \neq 0$ and thus $\Delta\gamma \neq \emptyset$. Hence $\Delta\gamma = Y$. Thus $\alpha = \beta\gamma = \beta 1_{\Delta\gamma} = \beta 1_Y = \beta \in G_Y$. This shows that $K = G_Y \cup \{0\}$.

Case 3 : $|Y \smallsetminus Z_n| \geq 1$ and $Z_n \neq \emptyset$. Let $|Y| = m$ and $Y = \{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_m\}$. Then $m - n \geq 2$. Let $\alpha \in K \smallsetminus \{0\}$. Then $\emptyset \neq \Delta\alpha = \nabla\alpha$ and $1_{\Delta\alpha} \in K$.

Subcase 1 : $|\Delta\alpha| \leq n$. Let $|\Delta\alpha| = k$. Since $\{1_{\Delta\alpha}, 1_{Z_k}, 1_{\Delta\alpha \cap Z_k}\}$ is a subsemigroup of K, it is strongly factorizable. By Lemma 3.2, $\{\Delta\alpha, Z_k, \Delta\alpha \cap Z_k\}$ has a maximum element under the partial order of set inclusion. Then $\Delta\alpha \subseteq Z_k$ or $Z_k \subseteq \Delta\alpha$. But since $|\Delta\alpha| = |Z_k| = k$, it follows that $\Delta\alpha = Z_k$. Suppose $\alpha \notin E(I_X)$. Because $\langle\alpha\rangle \cup \{1_Y\}$ is a subsemigroup of K, $\langle\alpha\rangle \cup \{1_Y\}$ is factorizable. By Lemma 3.4, $Y = \Delta\alpha$. Hence $Y = Z_k$, a contradiction. This shows that $\alpha \in E(I_X)$, so $\alpha = 1_{Z_k} \in T(Z_o, Z_1, \ldots, Z_n;Y)$.

<u>Subcase 2 : $n < |\Delta\alpha| < m$</u>. Since $\{1_{\Delta\alpha}, 1_{Z_n}, 1_{\Delta\alpha \cap Z_n}\}$ and $\{1_{\Delta\alpha}, 1_Y, 1_{\Delta\alpha \cap Y}\}$ are subsemigroups of K. By Lemma 3.2, ($\Delta\alpha \subseteq Z_n$ or $Z_n \subseteq \Delta\alpha$) and ($\Delta\alpha \subseteq Y$ or $Y \subseteq \Delta\alpha$). But since $|Z_n| = n < |\Delta\alpha| < m = |Y|$, it follows that $Z_n \subsetneq \Delta\alpha \subsetneq Y$. Because $Z_n = \{a_1, a_2, \ldots, a_n\}$ and $Y = \{a_1, \ldots, a_n, a_{n+1}, \ldots, a_m\}$, without loss of generality, we may assume that

$$\Delta\alpha = \{a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+\ell}\}$$

for some positive integer $\ell < m - n$. Define the map $\beta : Y \to Y$ by

$$x\beta = \begin{cases} a_{n+\ell} & \text{if } x = a_m, \\ a_m & \text{if } x = a_{n+\ell}, \\ x & \text{otherwise.} \end{cases}$$

Then $\beta \in T(Z_o, Z_1, \ldots, Z_n;Y)$. Let $\gamma = 1_{\Delta\alpha}\beta$. Then $\gamma \in K$, so by Lemma 3.4, $\Delta\gamma = \nabla\gamma$. But $\Delta\gamma = \Delta 1_{\Delta\alpha}\beta = (\Delta\alpha \cap Y)1_{\Delta\alpha}^{-1} = \Delta\alpha$ and $\nabla\gamma = \nabla 1_{\Delta\alpha}\beta = (\Delta\alpha \cap Y)\beta = \{a_1, \ldots, a_n, a_{n+1}, \ldots, a_{n+\ell-1}, a_m\}$, hence $\Delta\gamma \neq \nabla\gamma$ which is a contradiction. Thus this subcase cannot occur.

<u>Subcase 3 : $n < |\Delta\alpha| > m$</u>. Since $\{1_{\Delta\alpha}, 1_Y, 1_{\Delta\alpha \cap Y}\}$ is a subsemigroup of K, it is strongly factorizable. By Lemma 3.2, $\Delta\alpha \subseteq Y$ or $Y \subseteq \Delta\alpha$. Since $|\Delta\alpha| > |Y|$, $Y \subsetneq \Delta\alpha$. Let $\beta : Y \to Y$ be defined by

$$x\beta = \begin{cases} a_m & \text{if } x = a_{m-1}, \\ a_{m-1} & \text{if } x = a_m, \\ x & \text{otherwise.} \end{cases}$$

Then $\beta \in T(Z_o, Z_1, \ldots, Z_n;Y)$ and $\beta\beta = 1_Y$. Let $T = \{\beta, 1_Y, 1_{\Delta\alpha}\}$. Then T is a subsemigroup of K, so T is factorizable. All maximal subgroups of T are $\{\beta, 1_Y\}$ and $\{1_{\Delta\alpha}\}$, and $E(T) = \{1_Y, 1_{\Delta\alpha}\}$. But $\{\beta, 1_Y\}E(T) = \{\beta, 1_Y\} \neq T$ and $\{1_{\Delta\alpha}\}E(T) = \{1_Y, 1_{\Delta\alpha}\} \neq T$. Hence T is not factorizable which is a contradiction. Then this subcase cannot occur.

Subcase 4 : $n < |\Delta\alpha| = m$. Since $\{1_{\Delta\alpha}, 1_Y, 1_{\Delta\alpha \cap Y}\}$ is a subsemigroup of K, it is strongly factorizable. By Lemma 3.2, $\Delta\alpha \subseteq Y$ or $Y \subseteq \Delta\alpha$. But $|\Delta\alpha| = |Y| = m$, then $\Delta\alpha = Y$, so $\Delta\alpha = \nabla\alpha = Y$. Claim that $a\alpha = a$ for all $a \in Z_n$. Suppose there exists $i \in \{1, 2, \ldots, n\}$ such that $a_i\alpha \neq a_i$. Let i_o be the smallest positive integer such that $a_{i_o}\alpha \neq a_{i_o}$. Let $\beta = 1_{Z_{i_o}}\alpha$. Then $\beta \in K$, so by Lemma 3.3, $\Delta\beta = \nabla\beta$. But $\Delta\beta = \Delta 1_{Z_{i_o}}\alpha = (Z_{i_o} \cap Y)1_{Z_{i_o}}^{-1} = Z_{i_o}$ and $\nabla\beta = \nabla 1_{Z_{i_o}}\alpha = (Z_{i_o} \cap Y)\alpha = \{a_1, a_2, \ldots, a_{i_o-1}, a_{i_o}\alpha\}$, then $\Delta\beta \neq \nabla\beta$ which is a contradiction. This shows that $a\alpha = a$ for all $a \in Z_n$. Hence $\alpha \in T(Z_o, Z_1, \ldots, Z_n;Y)$.

Hence, the theorem is completely proved. □

Let X be a finite set.

If $X = \emptyset$, then $I_X = \{0\}$, hence there are no maximal strongly factorizable subsemigroups of I_X.

If $|X| = 1$, then $I_X = \{0, 1_X\}$, hence all of the maximal strongly factorizable subsemigroups of I_X are $\{0\}$ and $\{1_X\}$.

3.9 <u>Theorem</u>. Let X be a finite set with $|X| > 1$ and T a maximal strongly factorizable subsemigroup of I_X. Then there are a nonnegative integer n and some sets $Z_o, Z_1, \ldots, Z_n, Y \subseteq X$ such that

$\emptyset = Z_o \subsetneq Z_1 \subsetneq \ldots \subseteq Z_n \subsetneq Y$, $|Z_{i+1} \smallsetminus Z_i| = 1$ for all $i \in \{0, 1, \ldots, n-1\}$, either $Z_n = Y = X$ or $|Y \smallsetminus Z_n| > 1$ and $T = T(Z_o, Z_1, \ldots, Z_n;Y)$.

Proof : Since T is a strongly factorizable semigroup, by Theorem 1.1, T^o is also strongly factorizable. Then $T \subseteq T^o \subseteq I_X$. Since $|X| > 1$, by Theorem 2.4, I_X is not strongly factorizable. But since T is a maximal strongly factorizable subsemigroup of I_X, then $T = T^o$, it follows that $0 \in T$. Because $\{0, 1_X\}$ is a strongly factorizable subsemigroup of I_X and $\{0, 1_X\} \neq I_X$, it follows that $T \neq \{0\}$.

Because T is a factorizable inverse semigroup, T has an identity. Let $Y \subseteq X$ be such that 1_Y is the identity of T. Since $T \neq \{0\}$, $Y \neq \emptyset$. Let T be factorizable as $T = GE(T)$. Then G is the unit group of T with identity 1_Y. Thus $G = G_Y \cap T$, and so $T \subseteq G_Y E(T)$. Because $0, 1_Y \in T$, we have $\{0, 1_Y\} \subsetneq E(T) \subseteq T$.

Case 1 : $\{0, 1_Y\} = E(T) = T$. Let $a \in Y$ and $K = \{0, 1_{\{a\}}, 1_Y, 1_X\}$. Then K is a finite chain, so K is a strongly factorizable subsemigroup of I_X. Since $|X| > 1$ and $Y \subseteq X$, $\{0, 1_Y\} = T \subsetneq K \subsetneq I_X$, this contradicts the maximality of T. Hence Case 1 cannot occur.

Case 2 : $\{0, 1_Y\} = E(T) \subsetneq T$. Let $\alpha \in T \smallsetminus \{0\}$. Since $T \subsetneq G_Y E(T)$, $\alpha = \beta\gamma$ for some $\beta \in G_Y$ and $\gamma \in E(T)$. But $\alpha \neq 0$ and $E(T) = \{0, 1_Y\}$, it follows that $\gamma = 1_Y$. Then $\alpha = \beta\gamma = \beta 1_Y = \beta \in G_Y$. This shows that $T \smallsetminus \{0\} \subseteq G_Y$, so $T \subsetneq G_Y \cup \{0\}$. Since from Lemma 3.1, $G_Y \cup \{0\}$ is a strongly factorizable subsemigroup of I_X and $G_Y \cup \{0\} \neq I_X$, it follows that $T = G_Y \cup \{0\}$. If $|Y| = 1$, then $G_Y \cup \{0\} = \{0, 1_Y\}$. Hence $\{0, 1_Y\} = E(T) \subsetneq T = G_Y \cup \{0\} = \{0, 1_Y\}$, which is a contradiction. Therefore $|Y| > 1$ since

$Y \neq \emptyset$. Then $T = G_Y \cup \{0\} = T(\emptyset; Y)$.

Case 3 : $\{0, 1_Y\} \subsetneq E(T) = T$. Then there are distinct subsets $\emptyset = Z_o, Z_1, \ldots, Z_n$ of X such that $T = \{1_{Z_i} \mid i = 0, 1, \ldots, n\}$. Because T is strongly factorizable, T is a chain, so $\{Z_i\}_{i=0,1,\ldots,n}$ is a chain under the partial order of set inclusion. Then we may assume $\emptyset = Z_o \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n$. Then 1_{Z_n} is the identity of T which implies $1_{Z_n} = 1_Y$, so $Z_n = Y$. To show that $Y = X$, suppose not. Then $Y \subsetneq X$. Thus we have $\emptyset = Z_o \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n = Y \subsetneq X$. Therefore $T \cup \{1_X\}$ is a finite chain and thus $T \cup \{1_X\}$ is a strongly factorizable subsemigroup of I_X. Since $|X| > 1$, $T \cup \{1_X\} \neq I_X$. But $T \subsetneq T \cup \{1_X\}$, this contradicts the maximality of T. This shows that $Y = X$, so $Z_n = Y = X$. Claim that for all $k \in \{0, 1, \ldots, n-1\}$, $|Z_{k+1} \smallsetminus Z_k| = 1$. Suppose that there exists $k \in \{0, 1, \ldots, n-1\}$ such that $|Z_{k+1} \smallsetminus Z_k| > 1$. Let A be a nonempty proper subset of $Z_{k+1} \smallsetminus Z_k$. Then we have $\emptyset = Z_o \subsetneq \ldots \subsetneq Z_k \subsetneq Z_k \cup A \subsetneq Z_{k+1} \subsetneq \ldots \subsetneq Z_n$, so $T \cup \{1_{Z_k \cup A}\}$ is a finite chain, so it is a strongly factorizable subsemigroup of I_X. Since $|X| > 1$, $T \cup \{1_{Z_k \cup A}\} \neq I_X$. But $T \subsetneq T \cup \{1_{Z_k \cup A}\}$, this contradicts the maximality of T. This shows that for each $k \in \{0, 1, \ldots, n-1\}$, $|Z_{k+1} \smallsetminus Z_k| = 1$. Hence $T = T(Z_o, Z_1, \ldots, Z_n; Y)$.

Case 4 : $\{0, 1_Y\} \subsetneq E(T) \subsetneq T$. In this case claim that $|X| > 2$. Suppose not, then $|X| = 2$. Assume $X = \{a, b\}$. Then

$$I_X = \{0, 1_X, \{a\}_a, \{b\}_b, \{a\}_b, \{b\}_a, (a,b)\},$$

$E(I_X) = \{0, 1_X, \{a\}_a, \{b\}_b\}$ and the multiplication of I_X is given as follows :

$\circ$	0	1_X	$\{a\}_a$	$\{b\}_b$	$\{a\}_b$	$\{b\}_a$	(a, b)
0	0	0	0	0	0	0	0
1_X	0	1_X	$\{a\}_a$	$\{b\}_b$	$\{a\}_b$	$\{b\}_a$	(a, b)
$\{a\}_a$	0	$\{a\}_a$	$\{a\}_a$	0	$\{a\}_b$	0	$\{a\}_b$
$\{b\}_b$	0	$\{b\}_b$	0	$\{b\}_b$	0	$\{b\}_a$	$\{b\}_a$
$\{a\}_b$	0	$\{a\}_b$	0	$\{a\}_b$	0	$\{a\}_a$	$\{a\}_a$
$\{b\}_a$	0	$\{b\}_a$	$\{b\}_a$	0	$\{b\}_b$	0	$\{b\}_b$
(a,b)	0	(a,b)	$\{b\}_a$	$\{a\}_b$	$\{b\}_b$	$\{a\}_a$	1_X

Since T is a strongly factorizable subsemigroup of I_X, by Lemma 3.3, for all $\alpha \in T$, $\Delta\alpha = \nabla\alpha$, it follows that $\{a\}_b$ and $\{b\}_a$ are not elements of T. Because $E(T) \subsetneq T$, there exists an α in $T \smallsetminus E(T)$, then $\alpha = (a,b)$. But $(a,b)(a,b) = 1_X$ which implies $1_X \in T$. It follows that $1_Y = 1_X$ since 1_Y is the identity of T. Because $\{0, 1_Y\} \subsetneq E(T)$, there is a β in $E(T) \smallsetminus \{0, 1_Y\}$, then $\beta = \{a\}_a$ or $\{b\}_b$. If $\beta = \{a\}_a$, then $\beta(a,b) = \{a\}_b \in T$, it is a contradiction. If $\beta = \{b\}_b$, then $\beta(a,b) = \{b\}_a \in T$, it is a contradiction. This shows that $|X| > 2$.

Let $\emptyset = Z_0, Z_1, \ldots, Z_n$ be distinct subsets of X such that $Z_i \neq Y$ for all $i \in \{0, 1, \ldots, n\}$ and $E(T) = \{1_{Z_i} \mid i = 0, 1, \ldots, n\} \cup \{1_Y\}$. Since 1_Y is the identity of T, for all $i \in \{0, 1, \ldots, n\}$ $1_Y 1_{Z_i} = 1_{Z_i}$, so $Z_i \subsetneq Y$ for all $i \in \{0, 1, \ldots, n\}$. Because T is strongly factorizable, E(T) is a subsemigroup of T, hence E(T) is also strongly factorizable, so E(T) is a finite chain and thus

$\{Z_i\}_{i=0,1,\ldots,n}$ is a chain under set inclusion. We may assume $\emptyset = Z_o \subsetneq Z_1 \subsetneq \ldots \subsetneq Z_n \subsetneq Y$. Then $|Y \smallsetminus Z_n| > 1$.

Let $\alpha \in T \smallsetminus E(T)$. Since $1_Y\alpha = \alpha$, $\Delta\alpha \subseteq Y$. By Lemma 3.3, $\Delta\alpha = \nabla\alpha \subseteq Y$. Since $\langle\alpha\rangle \cup \{1_Y\}$ is a subsemigroup of T, $\langle\alpha\rangle \cup \{1_Y\}$ is factorizable. Then by Lemma 3.4, $\Delta\alpha = \nabla\alpha = Y$. Claim that $a\alpha = a$ for all $a \in Z_n$. Let $\beta = 1_{Z_n}\alpha$. Then $\beta \in T$ and $\Delta\beta = \Delta 1_{Z_n}\alpha = (Z_n \cap Y)1_{Z_n}^{-1} = Z_n$, so by Lemma 3.3, $\Delta\beta = \nabla\beta = Z_n$. If $\beta \neq 1_{Z_n}$, then $\beta \notin E(I_X)$. Since $\langle\beta\rangle \cup \{1_Y\}$ is a subsemigroup of T, $\langle\beta\rangle \cup \{1_Y\}$ is factorizable, thus by Lemma 3.4, $\Delta\beta = \nabla\beta = Y$ which is a contradiction since $\Delta\beta = Z_n \subsetneq Y$. This shows that $\beta = 1_{Z_n}$. Because $\beta = 1_{Z_n}\alpha$, $a\alpha = a$ for all $a \in Z_n$. Hence for all $\alpha \in T \smallsetminus E(T)$, $\Delta\alpha = \nabla\alpha = Y$ and $a\alpha = a$ for all $a \in Z_n$. This shows that

$$\begin{aligned} T &= E(T) \cup (T \smallsetminus E(T)) \\ &= \{1_{Z_i} \mid i=0,1,\ldots,n\} \cup \{\alpha \in T \mid \Delta\alpha = \nabla\alpha = Y \text{ and } a\alpha = a \text{ for all } a \in Z_n\}. \end{aligned}$$

Claim that for each $k \in \{0,1,\ldots, n-1\}$, $|Z_{k+1} \smallsetminus Z_k| = 1$. Suppose that there exists $k \in \{0,1,\ldots,n-1\}$ such that $|Z_{k+1} \smallsetminus Z_k| > 1$. Let A be a nonempty proper subset of $Z_{k+1} \smallsetminus Z_k$. Then we have $\emptyset = Z_o \subsetneq \ldots \subsetneq Z_k \subsetneq Z_k \cup A \subsetneq Z_{k+1} \subsetneq \ldots \subsetneq Z_n \subsetneq Y$. Let $\gamma = 1_{Z_k \cup A}$. Then it follows that $T \subsetneq T \cup \{\gamma\}$. It is clear that $T \cup \{\gamma\}$ is a finite inverse semigroup and $E(T \cup \{\gamma\})$ is a chain. Since T is a finite strongly factorizable inverse semigroup, by Corollary 1.13, $T = \bigcup_{e \in E(T)} H_e$ such that $H_e = \{e\}$ if $e \in E(T) \smallsetminus \{1_Y\}$. Since $\{\gamma\}$ is a maximal subgroup of $T \cup \{\gamma\}$, $T \cup \{\gamma\} = \bigcup_{e \in E(T \cup \{\gamma\})} H_e$. Hence by Corollary 1.13, $T \cup \{\gamma\}$ is a strongly factorizable subsemigroup of I_X. But $T \subsetneq T \cup \{\gamma\}$, this contradicts the maximality of T. This shows that for all $k \in \{0,1,\ldots, n-1\}$, $|Z_{k+1} \smallsetminus Z_k| = 1$.

Claim that $|Y \smallsetminus Z_n| > 1$. Suppose that $|Y \smallsetminus Z_n| = 1$. Let $\alpha \in T \smallsetminus E(T)$. Then $\Delta\alpha = \nabla\alpha = Y$ and $a\alpha = a$ for all $a \in Z_n$ which implies $\alpha = 1_Y$ since $|Y \smallsetminus Z_n| = 1$, so $\alpha \in E(T)$, a contradiction. This shows that $|Y \smallsetminus Z_n| > 1$. Then

$$T = \{1_{Z_i} \mid i = 0,1,\ldots,n\} \cup \{\alpha \in T \mid \Delta\alpha = \nabla\alpha = Y \text{ and } a\alpha = a \text{ for all } a \in Z_n\}$$

$$\subseteq \{1_{Z_i} \mid i = 0,1,\ldots,n\} \cup \{\alpha \in G_Y \mid a\alpha = a \text{ for all } a \in Z_n\}$$

$$= T(Z_o,Z_1,\ldots,Z_n;Y).$$

By Theorem 3.8, $T(Z_o,Z_1,\ldots,Z_n;Y)$ is a strongly factorizable subsemigroup of I_X, hence $T = T(Z_o,Z_1,\ldots,Z_n; Y)$.

Hence, the theorem is completely proved. □