CHAPTER II

TRANSFORMATION SEMIGROUPS

The purpose of this chapter is to characterize strongly factorizable transformation semigroups.

Throughout this chapter, the following notation are adopted : For a set X, let

T_X = the partial transformation semigroup on X,

$\mathcal{T}_X$ = the full transformation semigroup on X,

I_X = the symmetric inverse semigroup on X or the 1-1 partial transformation semigroup on X,

U_X = the semigroup of all almost identical partial transformations of X,

V_X = the semigroup of all almost identical transformations of X,

W_X = the semigroup of all almost identical 1-1 partial transformations of X,

E_X = the semigroup of all onto transformations of X,

M_X = the semigroup of all one-to-one transformations of X,

G_X = the permutation group on X,

C_X = the semigroup of all constant partial transformations of X,

F_X = the semigroup of all constant transformations of X.

Let X be a set. For a nonempty subset A of X and for $x \in X$, let A_x denote the partial transformation of X with $\Delta A_x = A$ and $\nabla A_x = \{x\}$. Then

$$C_X = \{A_x \mid \emptyset \neq A \subseteq X,\ x \in X\} \cup \{0\},$$

$$F_X = \{X_x \mid x \in X\} \text{ if } X \neq \emptyset$$

and

$$F_X = \{0\} \quad \text{if} \quad X = \emptyset.$$

Let X be a set. Then for $a,b \in X$, $X_a X_b = X_b$. Then the semigroup of all constant transformations of X is a right zero semigroup. Hence the semigroup F_X is strongly factorizable [Chapter I, page 12].

Therefore we have

2.1 <u>Theorem</u>. For any set X, the semigroup of all constant transformations of X is strongly factorizable.

The next theorem gives necessary and sufficient conditions for the permutation group on a set to be a strongly factorizable semigroup. The following lemma is required :

2.2 <u>Lemma</u>. For any set X, the permutation group on X, G_X, is periodic if and only if X is finite.

Proof : Assume that X is an infinite set. Then X contains a denumerable subset, say A. Then $|\mathbb{Z}| = |A|$ where $\mathbb{Z}$ is the set of all integers. Then there exists a one-to-one map ψ from $\mathbb{Z}$ onto A. Thus $A = \{ n\psi \mid n \in \mathbb{Z}\}$ and $m\psi \neq n\psi$ if $m \neq n$ in $\mathbb{Z}$. Define the map $\alpha : X \to X$ by

$$x\alpha = \begin{cases} (n+1)\psi & \text{if } x = n\psi \text{ for } n \in \mathbb{Z}, \\ x & \text{if } x \notin A. \end{cases}$$

Clearly, α is a permutation on X. By the definition of α, we have that for any positive integer k,

$$x\alpha^k = \begin{cases} (n+k)\psi & \text{if } x = n\psi \text{ for } n \in \mathbb{Z}, \\ x & \text{if } x \notin A. \end{cases}$$

Then $\alpha^k \neq \alpha^\ell$ for $k, \ell \in \{1,2,3,\ldots\}$ such that $k \neq \ell$. Hence G_X is not periodic.

Conversely, if X is a finite set, then G_X is a finite group, so it is periodic. □

2.3 Theorem. For any set X, the permutation group on X, G_X, is a strongly factorizable semigroup if and only if X is a finite set.

Proof : If the permutation group G_X is a strongly factorizable semigroup, then G_X is periodic [Theorem 1.7] , so by Lemma 2.2, X is finite.

Conversely, if X is a finite set, then G_X is a finite group, so it is a strongly factorizable semigroup [Chapter I, page 11].

For a set X, for $\alpha \in T_X$, α is an idempotent of T_X if and only if $\nabla\alpha \subseteq \Delta\alpha$ and $x\alpha = x$ for all $x \in \nabla\alpha$.

Let S be a transformation semigroup on a set X, that is, S is a subsemigroup of T_X. Then

$$E(S) = \{\alpha \in S \mid \nabla\alpha \subseteq \Delta\alpha \text{ and } x\alpha = x \text{ for all } x \in \nabla\alpha\}.$$

Let α, $\beta \in S$ such that $\alpha \mathcal{H} \beta$ in S. Then $\alpha \mathcal{L} \beta$ and $\alpha \mathcal{R} \beta$, so $\alpha = \gamma\beta$, $\beta = \gamma'\alpha$, $\alpha = \beta\lambda$ and $\beta = \alpha\lambda'$ for some $\gamma, \gamma', \lambda, \lambda' \in S$. Hence $\nabla\alpha = \nabla\gamma\beta \subseteq \nabla\beta$, $\nabla\beta = \nabla\gamma'\alpha \subseteq \nabla\alpha$, $\Delta\alpha = \Delta\beta\lambda \subseteq \Delta\beta$ and $\Delta\beta = \Delta\alpha\lambda' \subseteq \Delta\alpha$. It follows that $\Delta\alpha = \Delta\beta$ and $\nabla\alpha = \nabla\beta$. If $\alpha, \beta \in S$ such that $\alpha \mathcal{H} \beta$ and $\beta \in E(S)$, then $\nabla\alpha = \nabla\beta \subseteq \Delta\beta = \Delta\alpha$ and thus $\nabla\alpha \subseteq \Delta\alpha$. Therefore, for $\alpha \in S$, if α belongs to a subgroup of S, then $\nabla\alpha \subseteq \Delta\alpha$. Hence, if S is a union of groups, then $\nabla\alpha \subseteq \Delta\alpha$ for all $\alpha \in S$.

2.4 <u>Theorem</u>. Let X be a set and let S be T_X, I_X, U_X, W_X or C_X. Then the semigroup S is strongly factorizable if and only if $|X| \leqslant 1$.

<u>Proof</u> : Suppose that $|X| \geqslant 2$. Let a and b be two distinct elements of X. Then $\{a\}_b \in S$. Since $\Delta\{a\}_b = \{a\} \neq \{b\} = \nabla\{a\}_b$, it follows that S is not a union of subgroups of S. By Theorem 1.7, S is not strongly factorizable.

Conversely, if $|X| \leqslant 1$, then $S = \{0\}$ or $S = \{0, 1_X\}$, so it is clear that S is strongly factorizable. □

Let X be a set. The following theorem characterizes the semigroups $\mathcal{T}_X$ and V_X which are strongly factorizable in term of cardinality of X.

2.5 Theorem. Let X be a set and let S be $\mathcal{T}_X$ or V_X. Then the semigroup S is strongly factorizable if and only if $|X| \leq 2$.

Proof : Suppose $|X| \geq 3$. Let a,b and c be three distinct elements in X. Define the maps $\alpha, \beta : X \to X$ by

$$x\alpha = \begin{cases} c & \text{if } x \in \{a,b\}, \\ x & \text{otherwise}, \end{cases}$$

$$x\beta = \begin{cases} b & \text{if } x = a, \\ c & \text{if } x = b, \\ x & \text{otherwise}. \end{cases}$$

Then $\alpha, \beta \in S$, $\alpha \in E(S)$, $\alpha\beta = \alpha = \beta\alpha$ and $\beta\beta = \alpha$. Hence $A = \{\alpha,\beta\}$ is a subsemigroup of S with $E(A) = \{\alpha\}$ and $\{\alpha\}$ is the only subgroup of A. But $\{\alpha\}E(A) = \{\alpha\} \neq A$, hence A is not factorizable. Thus S is not strongly factorizable. This proves that if the semigroup S is strongly factorizable, then $|X| \leq 2$.

Conversely, assume that $|X| \leq 2$. If $|X| = 1$, then $|S| = 1$, so S is strongly factorizable.

Assume that $|X| = 2$, let $X = \{a,b\}$. Then $V_X = \mathcal{T}_X$, so $S = \mathcal{T}_X = V_X$. Let 1 be the identity map on X. Then $S = \{1, X_a, X_b, (a,b)\}$ where (a,b) is the permutation on $X = \{a,b\}$ with $a(a,b) = b$ and $b(a,b) = a$. Thus $E(S) = \{1, X_a, X_b\}$ and the multiplication on S is given by the following table :

$\circ$	1	X_a	X_b	(a,b)
1	1	X_a	X_b	(a,b)
X_a	X_a	X_a	X_b	X_b
X_b	X_b	X_a	X_b	X_a
(a,b)	(a,b)	X_a	X_b	1

Since $\{1,(a,b)\}$,$\{X_a\}$ and $\{X_b\}$ are maximal subgroups of S, we have that S is a union of subgroups of S, $H_1 = \{1, (a,b)\}$, $H_{X_a} = \{X_a\}$ and $H_{X_b} = \{X_b\}$. From the table of multiplication on S, E(S) is a subsemigroup of S. It is easy to see that every nonempty subset of E(S) is a subsemigroup of E(S) having a left idenlity. Moreover, $1X_a = X_a1 = X_a$, $1X_b = X_b1 = X_b$, $X_aX_b = X_b$, $X_bX_a = X_a$, $H_1X_a = H_{X_a}$, $H_1X_b = H_{X_b}$, $H_{X_a}X_b = H_{X_b}$, $H_{X_b}X_a = H_{X_a}$, $|H_{X_a}| = 1 = |H_{X_b}|$. Since S is finite, S is periodic. Hence by Theorem 1.11, S is strongly factorizable. □

It is known that for any set X, $E_X = G_X$ if and only if X is finite, and $M_X = G_X$ if and only if X is finite. For any set X, if $S = E_X$ or M_X , then S is regular if and only if X is finite. To prove this, first suppose $S = E_X$ is regular. To show X is finite, suppose not. Let $a \in X$. Then $|X \setminus \{a\}| = |X|$, so there exists a one-to-one and onto map $\alpha : X \setminus \{a\} \longrightarrow X$. Let $b \in X \setminus \{a\}$ such that $b\alpha = a$. Define the map $\beta : X \longrightarrow X$ by

$$x\beta = \begin{cases} a & \text{if } x = a, \\ x\alpha & \text{if } x \in X \setminus \{a\}. \end{cases}$$

Then $\beta \in E_X$ and $a\beta = a = b\alpha = b\beta$. Since E_X is regular, there exists $\gamma \in E_X$ such that $\beta = \beta\gamma\beta$ and $\gamma = \gamma\beta\gamma$. Because γ is onto X, there exist $x,y \in X$ such that $x\gamma = a$ and $y\gamma = b$. But $a = x\gamma = x\gamma\beta\gamma = a\beta\gamma = a\gamma$ and $b = y\gamma = y\gamma\beta\gamma = b\beta\gamma = a\gamma$, hence $a = a\gamma = b$. It is a contradiction since $a \neq b$. Then X is finite.

Now, suppose $S = M_X$ is regular. To show X is finite, suppose X is infinite. Let $a \in X$. Then $|X \setminus \{a\}| = |X|$, so there exists a one-to-one map α from X onto $X \setminus \{a\}$. Then $\alpha \in M_X$. Since M_X is regular, there exists $\beta \in M_X$ such that $\alpha = \alpha\beta\alpha$ and $\beta = \beta\alpha\beta$. Thus $a\beta = a\beta\alpha\beta$ which implies $a = (a\beta)\alpha$ since β is one-to-one. Hence $a \in \nabla\alpha$, which is a contradiction.

This proves that if S is regular, then X is finite.

Conversely, assume that X is finite. Then $S = G_X$ which is a group, so S is regular.

From the above fact, the following theorem is obtained.

2.6 <u>Theorem</u>. Let X be a set and let S be E_X or M_X. Then the semigroup S is strongly factorizable if and only if X is finite.

<u>Proof</u> : Assume that the semigroup S is strongly factorizable. Hence S is factorizable, so S is regular [4, Proposition 2.2]. Therefore X is finite.

Conversely, assume that X is finite. Then $S = G_X$ which is a finite group. Hence S is strongly factorizable since every finite group is a strongly factorizable semigroup. □