CHAPTER I

STRONGLY FACTORIZABLE SEMIGROUPS

The purpose of this chapter is to study the algebraic structure of strongly factorizable semigroups and to characterize strongly factorizable semigroups.

Recall that a semigroup S is said to be factorizable if there exists a subgroup G of S such that $S = GE(S)$ where $E(S)$ is the set of all idempotents of S, and a semigroup in which each subsemigroup is factorizable is called a strongly factorizable semigroup.

Every factorizable inverse semigroup S has an identity and if S is factorizable as $S = GE(S)$, then G is the unit group of S [1, Lemma 2.1].

Let a factorizable semigroup S be factorizable as $S = GE(S)$. Then the identity of G is a left identity of S [4, Lemma 2.1] and G is a maximal subgroup of S (that is, an $\mathcal{H}$ - class of S) [4, Theorem 2.4].

Every factorizable semigroup is regular [4, Proposition 2.2].

Observe from the definition of a strongly factorizable semigroup that every subsemigroup of a strongly factorizable semigroup is strongly factorizable, and every strongly factorizable semigroup is factorizable. But a factorizable semigroup need not be strongly factorizable. The group of all integers under usual addition, $(\mathbb{Z},+)$, is a counterexample since $\mathbb{N} = \{1,2,3,\ldots\}$ is a subsemigroup of $(\mathbb{Z},+)$ which is not factorizable.

Every group is a factorizable semigroup. It is known that every subsemigroup of a finite group is a subgroup. Then every finite group is a strongly factorizable semigroup. A question arises. Is there an infinite group which is a strongly factorizable semigroup ? The following example is the answer:

<u>Example</u>. Let $\mathbb{Z}$ be the set of all integers and $\mathbb{N}$ the set of all positive integers. Let p be a fixed prime and $G = \{ \frac{m}{p^n} \mid m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$. Then G is an infinite commutative group under usual addition and $\mathbb{Z}$ is a subgroup of G. Let $\mathbb{Z}(p^\infty) = G/\mathbb{Z}$. Because $\frac{1}{p^m} + \mathbb{Z} \neq \frac{1}{p^n} + \mathbb{Z}$ for $m,n \in \mathbb{N}$ such that $m \neq n$, it follows that $\mathbb{Z}(p^\infty)$ is an infinite group having $\mathbb{Z}$ as its identity. Claim that the group $\mathbb{Z}(p^\infty)$ is a strongly factorizable semigroup. To prove this, let T be a subsemigroup of $\mathbb{Z}(p^\infty)$. Let $\frac{m}{p^n} + \mathbb{Z} \in T$. If $n = 0$, then $\frac{m}{p^n} + \mathbb{Z} = \mathbb{Z} \in T$. If $n > 0$, then $(\frac{m}{p^n} + \mathbb{Z}) + (\frac{m}{p^n} + \mathbb{Z}) + \ldots + (\frac{m}{p^n} + \mathbb{Z})$ ($p^n - 1$ times) is an element of T and it is the inverse of the element $\frac{m}{p^n} + \mathbb{Z}$. This shows that T is a subgroup of the group $\mathbb{Z}(p^\infty)$.

Hence, the group $\mathbb{Z}(p^\infty)$ is an infinite group which is a strongly factorizable semigroup. □

Observe that the group in the above example is periodic. In fact, every subsemigroup of a periodic group G is a subgroup of G. To prove this, let T be a subsemigroup of G. Let $a \in T$. Since G is periodic, there exists a positive integer k such that $a^k = e$ where e is the identity of G. Then $a^{2k} = e$. Because $2k - 1$ is a positive integer, $a^{2k-1} \varepsilon$ T. Also, $aa^{2k-1} = a^{2k} = e \varepsilon T$. It then follows that T is a subgroup of G. This proves that every periodic group is a strongly factorizable semigroup.

An example of a strongly factorizable semigroup which is not a group is the semigroup $\mathbb{N}$ of all positive integers under the operation $*$ defined by

$$m*n = \text{maximum } \{m,n\}$$

for all $m,n \varepsilon \mathbb{N}$. Observe that every element of $\mathbb{N}$ is an idempotent of the semigroup $(\mathbb{N}, *)$. To prove the semigroup $(\mathbb{N}, *)$ is strongly factorizable, let T be a subsemigroup of $(\mathbb{N}, *)$. Let $n_o \varepsilon T$ be such that n_o is the smallest positive integer belonging to T. Then $T = \{n_o\}*E(T)$. This shows that T is factorizable. Therefore, the semigroup $(\mathbb{N}, *)$ is a strongly factorizable semigroup.

A semigroup S is called a <u>right</u> [left] <u>zero</u> <u>semigroup</u> if ab = b [ab = a] for all a,b ε S. Then every right [left] zero semigroup is a band. Observe that every nonempty subset of a right [left] zero semigroup is a subsemigroup.

If T is a subsemigroup of a right zero semigroup S, then for a ε T, T = {a}E(T). Thus every right zero semigroup is strongly factorizable.

If S is a left zero semigroup, then for a ε S, {a}E(S) = {a}. Hence, a left zero semigroup S is strongly factorizable if and only if $|S| = 1$.

The first theorem shows that if S is a strongly factorizable semigroup, then the semigroup S^{o} is strongly factorizable.

1.1 <u>Theorem</u>. A semigroup S is a strongly factorizable semigroup if and only if the semigroup S^{o} is strongly factorizable.

<u>Proof</u> : If S has a zero, then $S^{o} = S$, so we are done. Suppose that S has no zero. Let T be a subsemigroup of S^{o}. Then $T = \{0\}$ or $T \setminus \{0\}$ is a subsemigroup of S. If $T = \{0\}$, T is clearly factorizable. If $T \setminus \{0\}$ is a subsemigroup of S, then $T \setminus \{0\} = GE(T \setminus \{0\})$ for some subgroup G of $T \setminus \{0\}$, so $T = GE(T)$ and G is a subgroup of T. This proves that the semigroup S^{o} is strongly factorizable.

Because S is a subsemigroup of S^{o}, the converse follows. □

In general, if a semigroup S is a strongly factorizable semigroup, the semigroup S^{1} need not be a strongly factorizable semigroup.

<u>Example</u>. Let $G = \{e,a\}$ and $H = \{\bar{e},\bar{a}\}$ be two disjoint groups of order 2 with identities e and $\bar{e}$, respectively. Let $S = G \cup H$, and define the operation $*$ on S by

$$x*y = xy$$

$$\bar{x}*\bar{y} = \overline{xy}$$

$$x*\bar{y} = \overline{xy}$$

$$\bar{x}*y = xy$$

for all $x,y \in \{e,a\}$. Then $(S, *)$ is a semigroup and $E(S) = \{e,\bar{e}\}$. All subsemigroups of the semigroup $(S, *)$ are S, G, H, $T_1 = \{e\}$, $T_2 = \{\bar{e}\}$ and $T_3 = \{e,\bar{e}\}$. Since $S = G*E(S)$ and $T_3 = \{e\}*E(T_3)$, the semigroup

$(S, *)$ is strongly factorizable. Let $A = \{1, e, a\}$. Then A is a sub-semigroup of the semigroup S^1 and $E(A) = \{1, e\}$. All maximal subgroups of A are $\{1\}$ and $\{e, a\}$. Because $\{1\}*E(A) = \{1, e\} \neq A$ and $\{e, a\}*E(A) = \{e, a\} \neq A$, A is not factorizable. Hence the semigroup S^1 is not strongly factorizable. □

Let $\psi : G \to S$ be a homomorphism from a group G into a semigroup S. It is clear that $G\psi$ is a subgroup of S. Using this fact, the following theorem is obtained :

1.2 Theorem. A homomorphic image of a strongly factorizable semigroup is strongly factorizable.

Proof : Let $\psi : S \to T$ be a homomorphism from a strongly factorizable semigroup S onto a semigroup T. Let A be a subsemigroup of T. Then $A\psi^{-1}$ is a subsemigroup of S. Since S is strongly factorizable, $A\psi^{-1}$ is a factorizable subsemigroup of S, hence there exists a subgroup G of $A\psi^{-1}$ such that $A\psi^{-1} = GE(A\psi^{-1})$. Then $G\psi$ is a subgroup of A. To show $A = (G\psi)E(A)$, let $a \in A$. Then $a\psi^{-1} \subseteq A\psi^{-1}$. Let $x \in a\psi^{-1}$. Since $A\psi^{-1} = GE(A\psi^{-1})$, $x = ge$ for some $g \in G$, $e \in E(A\psi^{-1})$. Then $e\psi \in E(A)$ and $a = x\psi = (ge)\psi = (g\psi)(e\psi)$, hence $a \in (G\psi)E(A)$. This shows that $A = (G\psi)E(A)$, as required. □

If ρ is a congruence on a semigroup S, then the map $\rho^{\natural} : S \to S/\rho$ defined by $a\rho^{\natural} = a\rho$ is an onto homomorphism. Therefore by Theorem 1.2, the following corollary is directly obtained :

1.3 Corollary. If ρ is a congruence of a strongly factorizable semigroup S, then the semigroup S/ρ is strongly factorizable.

Let $\{S_\alpha\}_{\alpha \in A}$ be a nonempty family of semigroups. The semigroup S defined on the Cartesian product of the sets S_α with coordinatewise multiplication, that is, $(x_\alpha)(y_\alpha) = (x_\alpha y_\alpha)$, is the direct product of the semigroups S_α, $\alpha \in A$. If S is the direct product of the semigroups S_α, $\alpha \in A$, then for $\beta \in A$, the map $\Pi_\beta : S \to S_\beta$ defined by $(x_\alpha)\Pi_\beta = x_\beta$ is an onto homomorphism. By Theorem 1.2, we then have

1.4 Corollary. Let $\{S_\alpha\}_{\alpha \in A}$ be a nonempty family of semigroups. If the direct product of the semigroups S_α, $\alpha \in A$, is a strongly factorizable semigroup, then each semigroup S_α is strongly factorizable.

The converse of Corollary 1.4 is not true even the index set A is finite. An example is given as follows :

Example. Let $\mathbb{N}$ be the set of all positive integers. Then under the operation $*$ defined by $m*n = \text{maximum}\,\{m,n\}$, $(\mathbb{N}, *)$ is a strongly factorizable semigroup. Let $G = \{e, a, a^2, a^3\}$ be a cyclic group generated by a of order 4. Hence G is a strongly factorizable semigroup. Let $\mathbb{N} \times G$ be the direct product of the semigroup $(\mathbb{N}, *)$ and G and let $A = \{(1, e), (2, e), (3, e), (3, a^2)\}$. Then A is a subsemigroup of the semigroup $\mathbb{N} \times G$. All maximal subgroups of A are $\{(1, e)\}$, $\{(2, e)\}$, $\{(3, e),(3, a^2)\}$ and $E(A) = \{(1, e), (2, e), (3, e)\}$. But $\{(1, e)\}E(A) = \{(1, e), (2, e), (3, e)\} \neq A$, $\{(2, e)\}E(A) = \{(2, e), (3, e)\} \neq A$ and $\{(3, e), (3, a^2)\}E(A) = \{(3, e), (3, a^2)\} \neq A$. Hence A is not factorizable. This shows that the semigroup $\mathbb{N} \times G$ is not strongly factorizable. □

Let S be a semigroup and I an ideal of S. Then the relation ρ_I defined by

$$a \, \rho_I \, b \text{ if and only if } a, b \in I \text{ or } a = b \qquad (a, b \in S)$$

is a congruence on S which is called the <u>Rees congruence</u> induced by I, and the semigroup S/ρ_I is denoted by S/I and it is called the <u>Rees quotient semigroup</u> relative to I.

1.5 <u>Theorem</u>. Let I be an ideal of a semigroup S. If S is strongly factorizable, then I and S/I are strongly factorizable.

<u>Proof</u> : Since I is a subsemigroup of S, I is strongly factorizable. The semigroup S/I is strongly factorizable by Corollary 1.3. □

A semigroup S with zero 0 is called a <u>Kronecker semigroup</u> if

$$ab = \begin{cases} a & \text{if } a = b, \\ 0 & \text{if } a \neq b \end{cases}$$

for all elements a, b in S. Observe that a Kronecker semigroup is a band with zero. A Kronecker semigroup of order ≤ 2 is clearly strongly factorizable. If S is a Kronecker semigroup with zero 0, then for $a \in S$, $\{a\}E(S) = \{0, a\}$. This shows that a Kronecker semigroup S is strongly factorizable if and only if $|S| \leq 2$.

The converse of Theorem 1.5 is not true. Let S = {0, a, b} be a Kronecker semigroup of order 3 with zero 0. Then S is not strongly factorizable. Let I = {0, a}. Then I is an ideal of S. The semigroups I and S/I are Kronecker semigroups of order 2, therefore they are both strongly factorizable.

Let S be a semigroup and a an element of S. Then the cyclic subsemigroup of S generated by a is

$$\langle a\rangle = \{a^n \mid n = 1, 2, 3, \ldots\}.$$

Suppose there are distinct positive integers p, q, say $p < q$, such that $a^p = a^q$, then

$$\langle a\rangle = \{a, a^2, \ldots, a^q\}.$$

Thus a has finite order. Let s be the smallest positive integer such that $a^s = a^r$ for some positive integer $r < s$. Let

$$K_a = \{a^r, a^{r+1}, \ldots, a^{s-1}\}.$$

Then K_a is a subgroup of S [2, Theorem 1.9]. The identity element of the group K_a is the only one idempotent of the semigroup $\langle a\rangle$.

A regular semigroup containing exactly one idempotent is a group [2, Exercise for § 1.9(4)].

The next theorem shows that a strongly factorizable semigroup is periodic and a union of groups. The following lemma is required :

1.6 Lemma. If S is a semigroup in which every subsemigroup is regular, then S is periodic and it is a union of groups.

Proof : Let $a \in S$. Then the cyclic subsemigroup of S generated by a, $\langle a\rangle$, is regular. Because $\langle a\rangle = \{a^n \mid n = 1, 2, 3, \ldots\}$, there exists a positive integer k such that $a = aa^ka$, so $a = a^{k+2}$. Then the element a has a finite order and <a> contains exactly one idempotent. It follows that <a> is a group.

This shows that for each element $a \in S$, a has a finite order and <a> is a subgroup of S. Hence S is periodic and $S = \bigcup_{a \in S} \langle a \rangle$ is a union of groups. □

1.7. Theorem. A strongly factorizable semigroup is a union of periodic groups.

Proof : It has been proved in [4] that every factorizable semigroup is regular. Then every subsemigroup of a strongly factorizable semigroup is regular. Hence, by Lemma 1.6, a strongly factorizable semigroup is periodic and a union of groups. □

A periodic semigroup which is a union of groups need not be strongly factorizable. A left zero semigroup S with $|S| > 1$ is a counterexample.

Suppose a semigroup S is a union of subgroups of S. Then there exist an index set I and some subgroups, G_i, $i \in I$, such that $S = \bigcup_{i \in I} G_i$. For each i, let e_i be the identity of the group G_i. Then for each i, $G_i \subseteq H_{e_i}$ [Introduction, page 6], and hence

$$S = \bigcup_{i \in I} G_i \subseteq \bigcup_{i \in I} H_{e_i} \subseteq \bigcup_{e \in E(S)} H_e \subseteq S.$$

Thus $S = \bigcup_{e \in E(S)} H_e$ which is a disjoint union of groups.

It then follows by Theorem 1.7 that if S is a strongly factorizable semigroup, then $S = \bigcup_{e \in E(S)} H_e$.

Let Y be a semilattice and let a semigroup $S = \bigcup_{\alpha \in Y} G_\alpha$ be a disjoint union of subgroups G_α of S. The semigroup S is said to be a <u>semilattice</u> Y <u>of</u> <u>groups</u> G_α if $G_\alpha G_\beta \subseteq G_{\alpha\beta}$ for all α, β in Y.

A semilattice of groups is an inverse semigroup [3, Corollary 7.53].

Let S be an inverse semigroup and $S = \bigcup_{e \in E(S)} H_e$. Then E(S) is a semilattice. Since S is an inverse semigroup, every $\mathcal{L}$ - class and every $\mathcal{R}$ - class contains exactly one idempotent [2, Theorem 1.17]. But each $\mathcal{L}$ - class and each $\mathcal{R}$- class of S is a union of $\mathcal{H}$ - classes of S. Then $\mathcal{L} = \mathcal{H} = \mathcal{R}$. But $\mathcal{L}$ is right compatible and $\mathcal{R}$ is left compatible which implies $\mathcal{H}$ is a congruence. Let e, f ε E(S). If a ε H_e and b ε H_f, then a $\mathcal{H}$ e and b $\mathcal{H}$ f, and thus ab $\mathcal{H}$ ef since $\mathcal{H}$ is a congruence on S, which implies ab ε H_{ef}. This proves that $H_e H_f \subseteq H_{ef}$. Therefore, S is a semilattice E(S) of groups H_e.

Hence, the following corollary is directly obtained from Theorem 1.7.

1.8 <u>Corollary</u>. A strongly factorizable inverse semigroup is a semilattice of periodic groups.

A semilattice of periodic groups need not be strongly factorizable, as shown by the following example : Define the operation o on the set $\mathbb{N}$ of positive integers by mon = minimum {m, n}. Then ($\mathbb{N}$, o) is a semilattice, so it is a semilattice of periodic groups. Because the semigroup ($\mathbb{N}$, o) is an inverse semigroup without identity, we have that it is not factorizable, and hence it is not strongly factorizable.

Let S be a strongly factorizable semigroup. Let A be a nonempty set of idempotents of S. Then the subsemigroup of S generated by A, $\langle A \rangle$, is factorizable. Thus $\langle A \rangle$ has a left identity, say e. We then have ea = a for all a ε A. Since e ε $\langle A \rangle$, there are $e_1, e_2, \ldots, e_n$ ε A such that $e = e_1 e_2 \ldots e_n$. Hence $e = ee_n = e_n$ ε A.

Therefore, we have

1.9 <u>Theorem</u>. Let S be a strongly factorizable semigroup. If A is a nonempty set of E(S), then there is an element e ε A such that ea = a for all a ε A. Hence, for any idempotents e, f of S, ef = f or fe = e.

A semigroup S is called an <u>orthodox semigroup</u> if S is regular and the set of all idempotents of S forms a subsemigroup of S. Hence, a regular semigroup is orthodox if and only if for e, f ε E(S), ef ε E(S).

Let S be a strongly factorizable semigroup and e, f ε E(S). Then ef = f or fe = e by Theorem 1.9. If ef = f, then ef ε E(S). If ef ≠ f, then fe = e and thus $(ef)^2 = e(fe)f = eef = ef$ ε E(S). This shows that every strongly factorizable semigroup S is an orthodox semigroup.

If S is a strongly factorizable semigroup, then for e, f ε E(S), ef = f or fe = e, but they need not occur in the same time. A finite group with zero is a counterexample.

A question arises. Are the necessary conditions of a strongly factorizable semigroup in Theorem 1.7 and Theorem 1.9 sufficient for a semigroup to be strongly factorizable ? The following example is the answer : Let G be a nontrivial periodic group. Let S = G∪1 [Introduction, page 5]. Then the semigroup S is the union of periodic groups G and {1}. and E(S) = {e, 1} where e is the identity of the group G. The semigroup S satisfies the necessary condition in Theorem 1.9. But S is not factorizable because $\{1\}E(S) = \{1, e\} \neq S$ and $GE(S) = G \neq S$.

The next theorem shows properties of Green's relations on any strongly factorizable semigroup.

1.10 Theorem. Let S be a strongly factorizable semigroup. Then the following hold :

(a) $\mathscr{H} = \mathscr{L}$.

(b) For idempotents e, f of S, ef = f implies (i) $H_e f = H_f$ and (ii) fe = e if $|H_f| > 1$.

Proof : (a) Since $S = \bigcup_{e \in E(S)} H_e$ and $\mathscr{H} \subseteq \mathscr{L}$, it follows that $\mathscr{H} = \mathscr{L}$ if and only if for $e, f \in E(S)$, $e \mathscr{L} f$ implies e = f.

Let $e, f \in E(S)$ such that $e \mathscr{L} f$. Then e = xf and f = ye for some $x, y \in S$. By Theorem 1.9, ef = f or fe = e. If ef = f, then e = xf implies f = ef = xff = xf = e. If fe = e, then f = ye implies e = fe = yee = ye = f.

(b) Assume $a, b \in E(S)$ such that ab = b. Let $x \in H_a$. Then $x \mathscr{H} a$. Hence $x \mathscr{L} a$. Because $\mathscr{L}$ is right compatible, $xb \mathscr{L} ab$, and hence $xb \mathcal{H} b$ since $\mathcal{H} = \mathcal{L}$ and $ab = b$. Thus $xb \in H_b$. This proves that if $a, b \in E(S)$ such that $ab = b$, then $H_a b \subseteq H_b$.

Next, let $e, f \in E(S)$ such that $ef = f$. Then $H_e f \subseteq H_f$. If $e = f$, (i) and (ii) follow clearly. Assume $e \neq f$.

Case $|H_f| = 1$. Then $H_e f \subseteq H_f = \{f\}$ which implies $H_e f = H_f$.

Case $|H_f| > 1$. Let $T = \{e\} \cup H_f \cup H_{fe}$. Since $ef = f$, we have that $eH_f = H_f$, $eH_{fe} = efeH_{fe} = feH_{fe} = H_{fe}$ and $H_{fe}e = H_{fe}fee = H_{fe}fe = H_{fe}$. Because $f(fe) = fe$, we have that $H_f e = H_f fe \subseteq H_{fe}$ and $H_f H_{fe} = (H_f fe)H_{fe} \subseteq H_{fe}H_{fe} = H_{fe}$. Also, from $(fe)f = f$, it follows that $H_{fe}H_f = (H_{fe}f)H_f \subseteq H_f H_f = H_f$. Hence T is a subsemigroup of S with $E(T) = \{e, f, fe\}$. To show that $fe = e$, suppose $fe \neq e$. Since T is factorizable, $T = GE(T)$ for some subgroup G of T. Then the identity of G is a left identity of S. Since $E(S) = \{e, f, fe\}$ and $fe \neq e$, e is the only left identity of T. Then $e \in G$. Because $e \neq f$ and $e \neq fe$, it follows that $G = \{e\}$. Hence $T = \{e\}\{e, f, fe\} = \{e, f, fe\}$, which is a contradiction because $H_f \subseteq T$ and $|H_f| > 1$. Therefore $fe = e$. Thus $H_f e \subseteq H_e$ which implies $H_f = H_f ef \subseteq H_e f$. But $H_e f \subseteq H_f$, so we have $H_e f = H_f$. □

The following theorem gives a characterization of a strongly factorizable semigroup.

1.11 Theorem. Let S be a semigroup. Then S is strongly factorizable if and only if it satisfies the following three conditions :

(1) S is a union of periodic groups.

(2) For any nonempty set A of idempotents of S, there exists an element $e \in A$ such that $ea = a$ for all $a \in A$.

(3) For any $e, f \in E(S)$, $ef = f$ implies (i) $H_e f = H_f$ and (ii) $fe = e$ if $|H_f| > 1$.

<u>Proof</u> : If S is strongly factorizable, then (1), (2) and (3) follow from Theorem 1.7, Theorem 1.9 and Theorem 1.10, respectively.

Conversely, assume that S satisfies the conditions (1), (2) and (3). To show S is strongly factorizable, let T be a subsemigroup of S. By (1), $E(T) \neq \emptyset$. Then, by (2), there is an element $e \in E(T)$ such that $ea = a$ for all $a \in E(T)$. Because $T \cap H_e$ is a subsemigroup of the periodic group H_e, it follows that $T \cap H_e$ is a subgroup of H_e. Thus $T \cap H_e$ is a subgroup of T. Claim that $T = (T \cap H_e)E(T)$. Let $x \in T$. Then $x \in H_f$ for some $f \in E(S)$ since S is a union of groups. Since H_f is a periodic group, there exists a positive integer k such that $x^k = f$. Then $f \in E(T)$. If $|H_f| = 1$, then $x = f$ which implies $x = ex \in (T \cap H_e)E(T)$. If $|H_f| > 1$, then $fe = e$ by (3)(ii), and it then follows from (3)(i) that $H_f e = H_e$. Hence $xe \in H_e$. Thus $x = xf = x(ef) = (xe)f \in (T \cap H_e)E(T)$. This proves that $T = (T \cap H_e)E(T)$, as required. □

Let S be a semilattice. The relation $\leqslant$ defined on S by $a \leqslant b$ if and only if $a = ab$ $(=ba)$ is a partial order on S which is called the <u>natural partial order</u> on S.

Let S be an inverse semigroup. Then E(S) is a semilattice. If 1 is the identity of S, 1 is clearly the maximum element of E(S) under the natural partial order on $E(S)$. Assume e is the maximum element of $E(S)$. Then $ef = fe = f$ for all $f \in E(S)$. Then for $a \in S$, $ea = (eaa^{-1})a = aa^{-1}a = a = aa^{-1}a = a(a^{-1}ae) = ae$, so e is the identity of S.

Let S be an inverse semigroup. Suppose S is strongly factorizable. Then S has an identity, say 1. Then for each $e \in E(S)$ such that $e \neq 1$, $1e = e$ but $e1 \neq 1$. By Theorem 1.10(b), $|H_e| = 1$, so $H_e = \{e\}$.

1.12 Corollary. Let S be an inverse semigroup. Then S is strongly factorizable if and only if it satisfies the following conditions:

(1) Every nonempty set of idempotents of S has a maximum element.

(2) $S = \bigcup_{e \in E(S)} H_e$ such that H_1 is periodic and $H_e = \{e\}$ if $e \in E(S)$ such that $e \neq 1$ where 1 is the identity of S.

Proof : If the semigroup S is strongly factorizable, then (1) and (2) follow from Theorem 1.11 and the above proof.

To prove the converse, by Theorem 1.11, it suffices to show that $H_1e = H_e$ for all $e \in E(S)$. Because S is an inverse semigroup which is a union of groups, it follows that S is a semilattice $E(S)$ of groups H_e. Hence, for each $e \in E(S)$ such that $e \neq 1$, $H_1e = H_1H_e \subseteq H_{1e} = H_e = \{e\} \subseteq H_1e$, thus $H_1e = H_e$. □

1.13 Corollary. A finite inverse semigroup is strongly factorizable if and only if it satisfies the following two conditions :

(1) $E(S)$ is a chain.

(2) $S = \bigcup_{e \in E(S)} H_e$ such that $H_e = \{e\}$ if $e \in E(S)$ such that $e \neq 1$ where 1 is the identity of S.

Proof : It follows directly from Corollary 1.12. □

A semigroup is called a right group if it is right simple and left cancellative. A left group is defined dually. A semigroup S is a right group if and only if $S = \bigcup_{e \varepsilon E(S)} H_e$ and ef = f for all e, f ε E(S) [2, Exercise for § 1.11(2)] . Dually, a semigroup S is a left group if and only if $S = \bigcup_{e \varepsilon E(S)} H_e$ and ef = e for all e, f ε E(S).

Observe that if S is a left group, then E(S) has a left identity if and only if $|E(S)| = 1$. Hence, by Theorem 1.11, a left group S is strongly factorizable if and only if S is a periodic group.

Let S be a right group. Let e, f ε E(S). Then ef = f. Claim that $H_e f = H_f$. Let a ε H_e. Then a $\mathcal{H}$ e, so a $\mathcal{L}$ e. Since $\mathcal{L}$ is right compatible, af $\mathcal{L}$ ef , so af $\mathcal{L}$ f. Since S is right simple, $afS^1 = S = fS^1$. Thus af $\mathcal{R}$ f. Hence af $\mathcal{H}$ f, so af ε H_f. This shows that $H_e f \subseteq H_f$ for all e, f ε E(S). Then for e, f ε E(S), $H_f = H_f ef = (H_f e)f \subseteq H_e f \subseteq H_f$ which implies $H_e f = H_f$. It thus follows from Theorem 1.11 that a right group is strongly factorizable if and only if S is periodic.

1.14 Corollary. (a) A left group is strongly factorizable if and only if it is a periodic group.

(b) A right group is strongly factorizable if and only if it is periodic.