CHAPTER IV

SUMMABILITY THEORY

1. Summability

Another satisfactory solution to the problem of representation of functions by their Fourier series is to consider, instead of convergence, some methods of "summability" of Fourier series at individual points. The two best known types of summability are Cesàro and Abel summability. The former (often also referred to as the method of summability by the first arithmetic means or, simply, as (C,1) summability) is defined in the following way :

1.1 Definition. Suppose we are given a nummerical series $u_o + u_1 + \dots$ with partial sums $S_o, S_1, S_2, \dots$. We then form the (C,1) means (or finite arithmetic means)

$$\sigma_n = \frac{S_o + S_1 + \dots + S_n}{n + 1} = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) u_k$$

and say that the series is (C,1) summable to s if $\lim_{n \to \infty} \sigma_n = s$.

1.2 Definition. The Abel means of the series $u_o + u_1 + u_2 + \dots$ are defined for each r, $0 \leq r < 1$, by setting

$$A(r) = u_o + u_1 r + u_2 r^2 + \dots = \sum_{k=0}^{\infty} u_k r^k$$

and we say that the series is Abel summable to s if

$$\lim_{r \to 1^-} A(r) = s .$$

1.3 Theorem. If a series $\sum_{n=0}^{\infty} u_n$ is convergent with sum s, then it is also (C,1) summable to Cesàro sum s.

Proof. Let S_n denote the n-th partial sums of the series, define σ_n as in Definition 1.1, and introduce $t_n = S_n - s$, $T_n = \sigma_n - s$. Then we have

$$T_n = \frac{t_0 + t_1 + \dots + t_n}{n + 1}$$

and we must prove that $T_n \longrightarrow 0$ as $n \longrightarrow \infty$. Choose $A > 0$ so that each $|t_n| \leq A$.

Given $\varepsilon > 0$, choose N so that $n \geq N$ implies that $|t_n| < \frac{\varepsilon}{2}$. Taking $n \geq N$, we obtain

$$|T_n| \leq \frac{|t_0| + \dots + |t_N|}{n + 1} + \frac{|t_{N+1}| + \dots + |t_n|}{n + 1}$$

$$< \frac{(N + 1) A}{n + 1} + \frac{\varepsilon}{2} .$$

If n is sufficiently large, then $\frac{(N + 1) A}{(n + 1)} < \frac{\varepsilon}{2}$, we find that $|T_n| < \varepsilon$ for sufficiently large n. This completes the proof.

1.4 Theorem. If $\sum_{k=0}^{\infty} u_k$ converges to s then $\sum_{k=0}^{\infty} u_k$ is Abel summable to s.

Proof. We must show that $\lim_{x \to 1^-} f(x) = s$ where

$$f(x) = \sum_{k=0}^{\infty} u_k x^k \quad (0 \leq x \leq 1).$$

Since $\sum_{k=0}^{\infty} u_k$ converges, then $\sum_{k=0}^{\infty} u_k x^k$ converge uniformly for $0 \leq x \leq 1$. Hence, f is continuous on $[0,1]$. In particular, f is continuous at 1, $\lim_{x \to 1^-} f(x) = f(1)$. and $f(1) = \sum_{k=0}^{\infty} u_k = s$

We have $\lim_{x \to 1^-} f(x) = s$ which completes the proof.

1.5 Remark and Example. Actually (C,1) and Abel summabilities are strictly more general than convergence as the following shows. The series

$$1 - 1 + 1 - 1 \ldots. = \sum_{k=0}^{\infty} (-1)^k$$

is divergent. This series has sequence of partial sums.

$$S_n = 1 \quad (n = 1,3,\ldots),$$
$$S_n = 0 \quad (n = 2,4,\ldots),$$

The sequence of (C,1) means of the partial sum is giving by the following:

$$\sigma_n = \frac{n+1}{2n} \quad (n = 1,3,5,\ldots)$$
$$\sigma_n = \frac{1}{2} \quad (n = 2,4,6,\ldots),$$

which converges to $\frac{1}{2}$ as $n \to \infty$. It follows that the $\sum_{k=0}^{\infty} (-1)^k$ is (C , 1) summable to $\frac{1}{2}$.

The Abel means of the series are defined by

$$A(r) = 1 - r + r^2 - r^3 + \ldots, \quad \text{for } 0 \leq r < 1,$$

which converges to $\frac{1}{1+r}$ and therefore,

$$\lim_{r \to 1^-} \frac{1}{1+r} = \frac{1}{2}$$

Hence $\sum_{k=0}^{\infty} (-1)^k$ is Abel summable to $\frac{1}{2}$.

1.6 <u>Theorem</u>. If $\sum_{n=0}^{\infty} u_n$ is (C,1) summable to s then $\sum_{n=0}^{\infty} u_n$ is Abel summable to s .

<u>Proof</u>. Let $S_n = u_o + u_1 + \ldots + u_n$ $(n = 0,1,2,\ldots)$. By hypothesis, we have $\lim_{n\to\infty} \sigma_n = s$ where

$$\sigma_n = \frac{S_o + S_1 + \ldots + S_n}{n+1} \qquad (n = 0,1,2,\ldots) .$$

Since $(n+1)\sigma_n - n\,\sigma_{n-1} = (S_o + \ldots + S_n) - (S_o + \ldots + S_{n-1}) = S_n$,

we have

$$\lim_{n\to\infty} \frac{S_n}{n} = \lim_{n\to\infty} \left(\frac{(n+1)\sigma_n}{n} - \sigma_{n-1} \right) = s - s = 0.$$

Hence

$$\lim_{n\to\infty} \frac{u_n}{n} = \lim_{n\to\infty} \left(\frac{S_n - S_{n-1}}{n} \right) = \lim_{n\to\infty} \frac{S_n}{n} - \lim_{n\to\infty} \left(\frac{n-1}{n}\right)\left(\frac{S_{n-1}}{n-1}\right)$$

$$= 0 - 0 = 0 .$$

Since $\left\{ \frac{u_n}{n} \right\}_{n=1}^{\infty}$ is a convergent sequence, then $\left\{ \frac{u_n}{n} \right\}_{n=1}^{\infty}$ is a bounded sequence. This shows that for $0 \leq x < 1$, the $\sum_{n=0}^{\infty} u_n x^n$ is dominated by $\sum_{n=0}^{\infty} Mnx^n$ for some $M > 0$ which converges absolutely for $0 \leq x < 1$. Hence $\sum_{n=0}^{\infty} u_n x^n$ converges absolutely for $0 \leq x < 1$. Let

$$f(x) = u_o + u_1 x + u_2 x^2 + \ldots \qquad (0 \leq x < 1).$$

Then, since $\frac{1}{1-x} = 1 + x + x^2 + \ldots$ .... (1)

is also absolutely convergent for $0 \leq x < 1$, we have

$$\frac{f(x)}{1-x} = \sum_{n=0} c_n x^n \qquad (0 \leq x < 1),$$

where $c_n = u_o.1 + u_2.1 + \ldots + u_n.1$. This is $c_n = S_n$, so that

$$\frac{f(x)}{1-x} = \sum_{n=0}^{\infty} S_n x^n \qquad (0 \leq x < 1) \ldots\ldots\ldots(2)$$

We multiply (2) by (1) to obtain

$$\frac{f(x)}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)\,\sigma_n x^n \qquad (0 \leq x < 1).$$

Hence $$f(x) = (1-x)^2 \sum_{n=0}^{\infty} (n+1)\,\sigma_n x^n \quad (0 \leq x < 1)\ldots(3)$$

But $$\sum_{n=0}^{\infty} (n+1)\, x^n = 1 + 2x + 3x^2 + \ldots = \frac{1}{(1-x)^2} \qquad (-1 < x < 1),$$

so that

$$(1-x)^2 \sum_{n=0}^{\infty} (n+1)\, x^n = 1 \quad (-1 < x < 1),$$

$$s = (1-x)^2 \sum_{n=0}^{\infty} (n+1)\, s x^n \qquad (-1 < x < 1)\ldots\ldots\ldots(4)$$

From (3) and (4) we have

$$f(x) - s = (1-x)^2 \sum_{n=0}^{\infty} (n+1)(\sigma_n - s)\, x^n \quad (0 \leq x < 1) \qquad \ldots\ldots (5)$$

Given $\varepsilon > 0$ choose $N \in \mathbb{Z}\,(>0)$ such that $|\sigma_n - s| < \frac{\varepsilon}{2}$ for all $n \geq N$. Then, from (5)

$$|f(x) - s| \leq (1-x)^2 \sum_{n=1}^{N} (n+1)\,|\sigma_n - s|\,|x^n| + \frac{\varepsilon}{2}(1-x)^2 \sum_{n=N+1}^{\infty} (n+1)\, x^n$$

$$\leq (1-x)^2 \sum_{n=1}^{N} (n+1)\,|\sigma_n - s| + \frac{\varepsilon}{2}(1-x)^2 \sum_{n=0}^{\infty} (n+1) x^n$$

$$\leq (1-x)^2 A + \frac{\varepsilon}{2}$$

where $A = \sum_{n=1}^{N} (n+1)|\sigma_n - s|$. Since $(1-x)^2 < \frac{\varepsilon}{2A}$ if $1 - \delta < x < 1$, where $\delta = \sqrt{\frac{\varepsilon}{2A}}$, we have

$$|f(x) - s| < \frac{\varepsilon}{2A} \cdot A + \frac{\varepsilon}{2} = \varepsilon \quad \text{if } 1 - \delta < x < 1 .$$

This proves $\lim_{x \to 1^-} f(x) = s$, the proof is complete.

Thus, many results involving Abel summability follow from corresponding theorems that deal with Cesàro summability. This happens not only because such series may be Abel summable under weaker conditions on f than are necessary to guarantee their (C,1) summability, but also because Abel summability has special properties, related to the theory of harmonic and analytic functions, that are not enjoyed by Cesàro summability.

Recall the convention that we use the symbol f to denote a complex function over $\mathbb{T}$ and its associated 1-periodic function defined over $\mathbb{R}$.

2. A Theorem of Fejér

2.1 Theorem. If f is 1-periodic and integrable on $[0,1)$, then the (C,1) means and the Abel means of the Fourier series of f converge to

$$\frac{1}{2}\,(f(x_0^+) + f(x_0^-))$$

at every point x_0 where the limits $f(x_0^+)$ and $f(x_0^-)$ exist. In particular, they converge at every point of continuity of f.

Proof. Let us examine the (C,1) means of the Fourier series of a function f. We first obtain an expression of the partial sums of the Fourier series $\sum_{k=-\infty}^{\infty} \hat{f}(k)\, e^{2\pi ikx}$ of f where

$$\hat{f}(k) = \int_0^1 f(t)\, e^{-2\pi ikt}\, dt \quad (k = 0, \pm 1, \pm 2, \ldots),$$

$$\begin{aligned} S_n(x) &= \sum_{k=-n}^{n} \hat{f}(k)\, e^{2\pi ikx} \\ &= \sum_{k=-n}^{n} \left(\int_0^1 f(t)\, e^{-2\pi ikt}\, dt \right) e^{2\pi ikx} \\ &= \int_0^1 \left(\sum_{k=-n}^{n} e^{2\pi ik(x-t)} \right) f(t)\, dt. \end{aligned}$$

If we multiply $D_n(\theta) = \sum_{k=-n}^{n} e^{2\pi ik\theta}$ by $2 \sin \pi\theta = i\,(e^{-i\pi\theta} - e^{i\pi\theta})$, all but the first and last term of the resulting sum cancel and we obtain

$$2D_n(\theta) \sin \pi\theta = i\,(e^{-(2n+1)\pi i\theta} - e^{(2n+1)\pi i\theta}) = 2\sin (2n+1)\, \pi\theta,$$

so that

$$(1) \qquad D_n(\theta) = \frac{\sin (2n+1)\, \pi\theta}{\sin \pi\theta},$$

which is known as the Dirichlet Kernel. Hence

$$\begin{aligned} (2) \qquad S_n(x) &= \int_0^1 f(t)\, D_n(x-t)\, dt \\ &= \int_0^1 f(t)\, \frac{\sin (2n+1)\, \pi\, (x-t)}{\sin \pi\, (x-t)}\, dt \end{aligned}$$

The (C,1) means can then be expressed as :

$$\sigma_n(x) = \frac{S_o(x) + S_1(x) + \dots + S_n(x)}{n + 1}$$

$$= \frac{1}{n+1} \int_0^1 f(t) \left[\sum_{k=0}^{n} D_k(x - t) \right] dt.$$

By multiplying the numerator and denominator of

$$K_n(\theta) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(\theta) = \frac{1}{n+1} \sum_{k=0}^{n} \frac{\sin (2k + 1) \pi\theta}{\sin \pi \theta}$$

by $\sin \pi \theta$ and replacing the products of sines in the numerator by differences of cosines, we obtain

$$(3) \quad K_n(\theta) = \frac{1}{n+1} \sum_{k=0}^{n} \frac{\cos 2k\pi\theta - \cos 2(k + 1) \pi\theta}{2 \sin^2 \pi\theta}$$

$$= \frac{1}{n+1} \frac{1 - \cos 2(n + 1) \pi\theta}{2 \sin^2 \pi\theta}$$

$$= \frac{1}{n+1} \left[\frac{\sin (n + 1) \pi\theta}{\sin \pi\theta} \right]^2$$

called the <u>Fejér kernel</u>. Consequently;

$$(4) \quad \sigma_n(x) = \int_0^1 f(t) K_n(x - t) dt$$

$$= \frac{1}{n+1} \int_0^1 f(t) \left[\frac{\sin (n + 1) \pi (x - t)}{\sin \pi(x - t)} \right]^2 dt .$$

The proof of the theorem follows from the three basic properties of Fejér kernel :

(A) $\int_0^1 K_n(\theta) d\theta = 1;$

(B) $K_n(\theta) \geq 0 ;$

(C) for each $\delta > 0$, $\max_{\delta \leq \theta \leq 1-\delta} K_n(\theta) \longrightarrow 0$ as $n \longrightarrow \infty$.

Property (B) is obvious. Property (A) is a consequence of the corresponding property for the Dirichlet kernel (which is immediate since

$$\int_0^1 D_n(t)\, dt = \sum_{k=-n}^{n} \int_0^1 e^{2\pi i k\theta}\, d\theta = 1 \quad)$$

and the representation $K_n(\theta) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(\theta)$. Finally,

(C) follows from the inequality (see (3))

$$\max_{\delta \leq \theta \leq 1-\delta} K_n(\theta) \leq \frac{1}{(n+1)\sin^2 \pi\delta} .$$

Now, to complete the proof, we suppose x_0 is a point at which the limits $f(x_0^+)$ and $f(x_0^-)$ exist and let $a = \frac{1}{2}\left\{f(x_0^+) + f(x_0^-)\right\}$. Then, using the periodicity of the functions involved, the change of variables $t = x - s$, and property (A),

$$\begin{aligned}
\sigma_n(x_0) - a &= \int_{-1/2}^{1/2} f(s)\, K_n(x_0 - s)\, ds - a.1 \\
&= \int_{-1/2}^{1/2} f(x_0 - t)\, K_n(t)\, dt - a \int_{-1/2}^{1/2} K_n(t)\, dt \\
&= 2\int_0^{\delta} \left[\frac{f(x_0 - t) + f(x_0 + t)}{2} - a\right] K_n(t)\, dt \\
&\quad + \int_{\delta \leq |t| \leq \frac{1}{2}} \left[f(x_0 - t) - a\right] K_n(t)\, dt.
\end{aligned}$$

Hence, if $\delta > 0$ is so chosen that $|f(x_0 - t) + f(x_0 + t) - 2a| \leq \varepsilon$ whenever $|t| \leq \delta$, we have, by (B) and (A),

$$|\sigma_n(x_0) - a| \leq \varepsilon \int_0^{\delta} K_n(t)\, dt + \left[\max_{\delta \leq |t| \leq \frac{1}{2}} K_n(t)\right] \int_{\delta \leq |t| \leq \frac{1}{2}} |f(x_0 - t) - a|\, dt$$

$$\leq \varepsilon \int_{-1/2}^{1/2} K_n(t)\, dt + \left[\max_{\delta \leq |t| \leq \frac{1}{2}} K_n(t)\right] \int_{-1/2}^{1/2} |f(x_0 - t) - a|\, dt$$

$$= \varepsilon . 1 + \left[\max_{\delta \leq |t| \leq \frac{1}{2}} K_n(t)\right] \int_{-1/2}^{1/2} |f(x_0 - t) - a|\, dt \ ;$$

but, by (C), the last term tends to 0 as $n \longrightarrow \infty$. Since $\varepsilon > 0$ is arbitrary we can conclude the $\lim_{n \to \infty} |\sigma_n(x_0) - a| = 0$. By Theorem 1.6, (C,1) summability implies Abel summability, and the theorem is proved.

3. A Theorem of Lebesgue

3.1 The Lebesgue set of a function f.

Before we prove the theorem of Lebesgue, we first introduce the Lebesgue set of a function. Recall that $F'(x) = f(x)$ for almost all x, where $F(x) = \int_0^x f(t)\, dt$. We can indicate this fact by writing :

$$\lim_{h \to 0} \frac{1}{h} \int_0^h \left\{ f(x + t) - f(x) \right\} dt = 0$$

holds for almost all x. In this form, a stronger result is possible :

(1) $$\lim_{h \to 0} \frac{1}{h} \int_0^h |f(x + t) - f(x)|\, dt = 0.$$

holds for almost all x. To show this; fixed a rational number r let E_r be the set of all x such that

$$(2) \quad \lim_{h\to 0} \frac{1}{h}\int_0^h |f(x+t) - r| \, dt = |f(x) - r|$$

fails to hold. Applying Lebesgue's theorem on the differentiation of the integral to $g(t) = |f(t) - r|$ we conclude that E_r has measure 0. Let $E = \cup E_r$, the union being taken over all rational number r. Then E also has measure 0. We claim that if x does not belong to E then (1) holds. Let $\varepsilon > 0$ be given; choose a rational number r_o such that $|f(x) - r_o| < \frac{\varepsilon}{2}$. Then

$$\frac{1}{h}\int_0^h |f(x+t) - f(x)| \, dt \leq \frac{1}{h}\int_0^h |f(x+t) - r_o| \, dt + \frac{1}{h}\int_0^h |f(x) - r_o| \, dt.$$

Since $x \in \mathbb{R} \setminus \cup E_r = \cap (\mathbb{R} \setminus E_r)$, the relation (2) holds for all rationals r so that

$$\frac{1}{h}\int_0^h |f(x+t) - r_o| \, dt < \frac{\varepsilon}{2}$$

if h is close to 0. On the other hand

$$\frac{1}{h}\int_0^h |f(x) - r_o| \, dt < \frac{1}{h}\int_0^h \frac{\varepsilon}{2} \, dt = \frac{\varepsilon}{2}.$$

Thus

$$\frac{1}{h}\int_0^h |f(x+t) - f(x)| \, dt < \varepsilon$$

if h is small. The claim is now proved.

The set of all x such that (1) holds is called the <u>Lebesgue set of f</u>.

3.2 Theorem. (Lebesgue's Theorem) If f is 1-periodic and integrable on $[0,1)$, then the (C,1) means and the Abel means of the Fourier series of f converge to f(x) for almost all x in $[0,1)$.

Proof. We shall show that the (C,1) means of the Fourier series of f converge to f(x) whenever x is a member of the Lebesgue set.

We shall need the following two estimates of the Fejér kernel:

(a) $K_n(t) \leq n + 1$.

(b) $K_n(t) \leq \dfrac{A}{(n + 1)\, t^2}$, for $\dfrac{1}{n + 1} \leq |t| \leq \dfrac{1}{2}$, where A is a

constant depends only on the n.

(a) follows from the obvious estimate on the Dirichlet kernel $|D_k(t)| \leq 2k + 1$; in fact,

$$K_n(t) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(t) \leq \frac{1}{n+1} \sum_{k=0}^{n} (2k+1)$$

$$= \frac{(n+1)^2}{(n+1)} = n + 1.$$

(b) is a consequence of formula (3) in the Proof of Theorem 2.1 :

$$K_n(t) = \frac{1}{n+1} \left[\frac{\sin (n+1) \pi t}{\sin \pi t}\right]^2 \leq \frac{1}{n+1} \left[\frac{1}{\sin(\pi/(n+1))}\right]^2, \text{ for } \frac{1}{n+1} \leq |t| \leq \frac{1}{2},$$

$$= \frac{4}{4(n+1)} \left[\frac{1}{\sin(\pi/(n+1))}\right]^2 \leq \frac{A}{(n+1)t^2} \quad \text{where}$$

$$A = \frac{1}{4} \left[\frac{1}{\sin(\pi/(n+1))}\right]^2 .$$

Now suppose x belongs to the Lebesque set of f. As in the proof of Theorem 2.1, we have, using property (A) in the Proof of Theorem 2.1,

$$\sigma_n(x) - f(x) = \int_{-1/2}^{1/2} \{f(x - t) - f(x)\} K_n(t)\, dt.$$

Thus, using the estimates (a) and (b)

$$|\sigma_n(x) - f(x)| \leq \int_{-1/2}^{1/2} |f(x - t) - f(x)| K_n(t)\, dt$$

$$\leq (n + 1) \int_{|t| \leq 1/(n+1)} |f(x - t) - f(x)|\, dt$$

$$+ \frac{A}{n + 1} \int_{1/(n+1) \leq |t| \leq \frac{1}{2}} \frac{|f(x-t)-f(x)|}{t^2}\, dt.$$

Given $\varepsilon > 0$, let $\delta > 0$ be such that $\frac{1}{h} \int_{|t| \leq h} |f(x-t)-f(x)|\, dt$ $< \varepsilon$ if $h \leq \delta$. Then the first term in the above sum is less than ε whenever $(n + 1)^{-1} \leq \delta$. In order to estimate the second term we break the integral into the sum of two integrals over $\left[-\frac{1}{2}, -(n + 1)^{-1}\right]$ and over $\left[(n + 1)^{-1}, \frac{1}{2}\right]$. We shall show that the first integral tends to 0 as $n \longrightarrow \infty$; a similar argument will then show that the same is true for the second.

Let $G(t) = \int_0^t |f(x - s) - f(x)|\, ds$. Then, integrating by parts, we have

$$\frac{A}{n + 1} \int_{1/(n+1)}^{1/2} \frac{|f(x - t) - f(x)|\, dt}{t^2} \leq \frac{4A}{n+1} G(1/2)$$

$$+ \frac{2A}{n+1} \int_{1/(n+1)}^{\delta} \frac{G(t)}{t^3}\, dt$$

$$+ \frac{2A}{n+1} \int_{\delta}^{1/2} \frac{G(t)}{t^3}\, dt.$$

The first and third terms tends to 0 as $n \longrightarrow \infty$. Since $(1/t)\, G(t) < \varepsilon$ for $|t| \leqslant \delta$ the second term is dominated by

$$\frac{2A\varepsilon}{n+1} \int_{1/(n+1)}^{\delta} \frac{dt}{t^2} < 2A\,\varepsilon .$$

Thus, $|\sigma_n(x) - f(x)|$ can be made as small as we wish by choosing n large enough. This proves the theorem.

4. Able Summability And Harmonic Function

When f is real-valued integrable 1-periodic function, the Fourier series of f

$$(1) \quad \sum_{k=-\infty}^{\infty} \hat{f}\,(k)\, e^{2\pi ikx}$$

where

$$\hat{f}\,(k) = \int_0^1 f(t)\, e^{-2\pi ikt}\, dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

is the real part of the power series

$$(2) \qquad \hat{f}\,(0) + \sum_{k=1}^{\infty} 2\hat{f}\,(k)\, z^k$$

restricted to the unit circle $z = e^{2\pi ix}$. In fact,

$$\hat{f}\,(0) + \sum_{k=1}^{\infty} 2\,\hat{f}\,(k)\, e^{2\pi ikx} = \hat{f}\,(0) + \sum_{k=1}^{\infty} 2\,\hat{f}\,(k)\,(\cos 2\pi kx + i\sin 2\pi kx)$$

$$= \hat{f}\,(0) + \sum_{k=1}^{\infty} 2\left[\int_0^1 f(t)\, e^{-2\pi ikt}\, dt\right](\cos 2\pi kx + i \sin 2\pi kx)$$

$$= \hat{f}\,(0) + \sum_{k=1}^{\infty} 2\left[\int_0^1 f(t)(\cos 2\pi kt - i \sin 2\pi kt)\, dt\right](\cos 2\pi kx + i\sin 2\pi kx)$$

$$= \hat{f}\,(0) + \sum_{k=1}^{\infty}\left[\left\{2\left(\int_0^1 f(t)\cos 2\pi kt\, dt\right)\cos 2\pi kx + 2\left(\int_0^1 f(t)\sin 2\pi kt\, dt\right)\sin 2\pi kx\right\} + i\left\{2\left(\int_0^1 f(t)\cos 2\pi kt\, dt\right)\sin 2\pi kx - 2\left(\int_0^1 f(t)\sin 2\pi kt\, dt\right)\cos 2\pi kx\right\}\right]$$

The real part of $\hat{f}(0) + \sum_{k=1}^{\infty} 2\hat{f}(k)\, e^{2\pi ikx}$, where f is real-valued, is

$$\hat{f}(0) + \sum_{k=1}^{\infty}\left[2\left(\int_0^1 f(t)\cos 2\pi kt\, dt\right)\cos 2\pi kx + 2\left(\int_0^1 f(t)\sin 2\pi kt\, dt\right)\sin 2\pi kx\right].$$

Let $a_k = 2\int_0^1 f(t)\cos 2\pi kt\,dt$, $b_k = 2\int_0^1 f(t)\sin 2\pi kt\,dt$ for $k = 1,2,\ldots,$ and replace $\cos 2\pi kx = \dfrac{e^{2\pi ikx} + e^{-2\pi ikx}}{2}$,

$\sin 2\pi kx = \dfrac{e^{2\pi ikx} - e^{-2\pi ikx}}{2i}$. Then the series

$$\hat{f}(0) + \sum_{k=1}^{\infty}\left[a_k \cos 2\pi kx + b_k \sin 2\pi kx\right]$$

equals to the series

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)\, e^{2\pi ikx} \quad \text{where } \hat{f}(k) = \int_0^1 f(t)\, e^{-2\pi ikt}\, dt,$$

$k = 0, \pm 1, \pm 2, \ldots$

We note that the series (2) defines an analytic function in the interier of the unit circle since the coefficients $\hat{f}(k)$ are uniformly bounded; in fact,

$$|\hat{f}(k)| \leq \int_0^1 |f(t)|\,dt = \|f\|_1 .$$

Thus, the real part of (2) is a harmonic function when $r = |z| < 1$. But this real part is nothing more than the Abel mean of the Fourier series of f :

$$A(r,x) = A_f(r,x) = \hat{f}(0) + \sum_{k=1}^{\infty} r^k \hat{f}(k)(e^{2\pi ikx} + e^{-2\pi ikx})$$

$$= \sum_{k=-\infty}^{\infty} r^{|k|}\, \hat{f}(k)\, e^{2\pi ikx}.$$

The imaginary part of (2) in case $z = e^{2\pi ix}$ is

$$(3) \qquad -i \sum_{k=-\infty}^{\infty} (\operatorname{sgn} k)\, \hat{f}(k)\, e^{2\pi ikx},$$

where, for any nonzero complex number z, sgn $z = \frac{z}{|z|}$ and sgn 0 = 0. This series is called the series <u>conjugate</u> to the Fourier series (1). Though it is not, in general, a Fourier series, this conjugate is closely connected to a function, the <u>conjugate function</u> $\tilde{f}$ <u>of</u> f (see Theorem 5.2.2 for Definition).

As in the case of the (C,1) means, the Abel means A(r,x) have an integral representation; that is a representation similar to (4) in Theorem 2.1. In fact for $0 \leq r < 1$,

$$A(r,x) = \sum_{k=-\infty}^{\infty} r^{|k|}\, \hat{f}(k)\, e^{2\pi ikx}$$

$$= \sum_{k=-\infty}^{\infty} r^{|k|} \left(\int_0^1 f(t)\, e^{-2\pi ikt}\, dt \right) e^{2\pi ikx}$$

$$= \int_0^1 \left(\sum_{k=-\infty}^{\infty} r^{|k|}\, e^{2\pi ik(x-t)} \right) f(t)\, dt,$$

the change in the order of integration and summation can be carried out since the series

$$P(r,\theta) = \sum_{k=-\infty}^{\infty} r^{|k|}\, e^{2\pi ik\theta}$$

converges uniformaly for $0 \leq r < 1$. But, setting $z = re^{2\pi i\theta}$, $P(r,\theta)$ is simply the real part of

$$1 + \sum_{k=1}^{\infty} 2\, r^k\, e^{2\pi ik\theta} = 1 + 2 \sum_{k=1}^{\infty} z^k = \frac{1+z}{1-z}$$

Consequently,

$$P(r,\theta) = \frac{1 - r^2}{1 - 2r\cos 2\pi\theta + r^2}$$

and we obtain the desired integral representation for the Abel means :

$$(4)\quad A(r,x) = \int_0^1 P(r,\ x-t)\ f(t)\ dt$$

$$= \int_0^1 \frac{1-r^2}{1-2r\cos 2\pi\,(x-t)+r^2}\ f(t)\ dt.$$

$P(r,\theta)$ is called the Poisson kernel and the integral (4) is called the Poisson integral of f. This kernel satisfied the three properties, completely analogue to those of the Fejér kernel :

(A') $\int_0^1 P(r,\theta)\,d\theta = 1;$

(B') $P(r,\theta) \geqslant 0$;

(C') for each $\delta > 0$, $\max\limits_{\delta \leqslant \theta \leqslant 1-\delta} P(r,\theta) \longrightarrow 0$ as $r \longrightarrow 1$.

$$\text{Since } \int_0^1 P(r,\theta)\,d\theta = \int_0^1 \sum_{k=-\infty}^{\infty} r^{|k|}\, e^{2\pi i k\theta}\ d\theta$$

$= \sum\limits_{k=-\infty}^{\infty} \int_0^1 r^{|k|}\, e^{2\pi i k\theta}\, d\theta = 1$, and (A') follows.

(B') follows since $1-2\,r\cos 2\pi\theta + r^2 = (1-r\cos 2\pi\theta)^2 + (r\sin 2\pi\theta)^2 > 0$ and $r < 1$. (C') holds because

$$\max_{\delta \leqslant \theta \leqslant 1-\delta} P(r,\theta) \leqslant \frac{1-r^2}{1+r^2-2r\cos 2\pi\delta} \longrightarrow 0 \quad \text{as } r \longrightarrow 1.$$

From this we see that to the proof of Theorem 2.1 in the case of the Cesàro means there corresponds a practically identical proof of this result for the Abel means.

Let us observe that the imaginary part of $\frac{1+z}{1-z}$ has the form

$$Q(r,\theta) = \frac{2r \sin 2\pi\theta}{1 - 2r \cos 2\pi\theta + r^2}$$

and one readily obtains the Abel means of the conjugate Fourier series (3) by the integral

$$(5) \quad \tilde{A}(r,x) = \int_0^1 Q(r, x - t)\, f(t)\, dt = \int_0^1 \frac{2r \sin 2\pi(x - t)}{1 - 2r \cos 2\pi(x - t) + r^2}\, f(t)\, dt.$$

This integral is called the conjugate Poisson integral of f and $Q(r,\theta)$ is known as the conjugate Poisson kernel.

In this discussion we assumed that f was real - valued. It is clear, however, that the Poisson integral formula (4) for the Abel means of the Fourier series of f holds in case f is complex-valued as well. To see this one needs only apply it to real and imaginary parts of f.