CHAPTER II

THREE PEARLS OF BANACH SPACE TECHNIQUES

1. Banch Spaces

1.1 Definition. A complex vector space X is said to be a normed linear space if to each $x \in X$ there is a nonnegative real number $\|x\|$, called the norm of x, such that

(a) $\|x + y\| \leq \|x\| + \|y\|$ for all x and $y \in X$,

(b) $\|\alpha x\| = |\alpha| \|x\|$ if $x \in X$ and α is a complex number,

(c) $\|x\| = 0$ implies $x = 0$.

1.2 Theorem. Let X be a normed linear space. Then $d(x, y) = \|x - y\|$ is a metric on X.

Proof. Let x, y, z be any elements in X. $d(x, y) = \|x - y\|$ is always nonnegative real number. By (a) in Definition 1.1, we have

$$\|x - z\| \leq \|x - y\| + \|y - z\|$$

so that $d(x, z) \leq d(x, y) + d(y, z)$.

By taking $\alpha = 0$ in (b) in Definition 1.1, we have

$x = 0$ implies $\|x\| = 0$

and (c) in Definition 1.1, show that

$x = y$ if and only if $d(x, y) = 0$.

Finally, by taking $\alpha = -1$ in (b) in Definition 1.1, we have

$$\|x - y\| = \|y - x\|$$

so that $d(x, y) = d(y, x)$.

This completes the proof.

1.3 Definition. A Banach space is a normed linear space which is complete in the metric defined by its norm.

1.4 Example (a).

For any fixed n, the set $\mathbb{R}^n$ of all n-tuples

$$x = (x_1, x_2, \ldots, x_n),$$

where $x_1, x_2, \ldots, x_n$ are real numbers, is a real Banach space if additive and scalar multiplication are defined by

$$x + y = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$; for any x and $y \in \mathbb{R}^n$,

$$\alpha x = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$$

where $x = (x_1, x_2, \ldots, x_n)$; for any $x \in \mathbb{R}^n$ and any $\alpha \in \mathbb{R}$,

and if $\|x\| = \sqrt{\sum_{i=1}^{n} x_i^2}$.

Proof. Since $\mathbb{R}^n$, under these operations, is clearly a real normed linear space with the norm $\|\cdot\|$, we only need to show completeness.

Let $\{x^m\}$ be any Cauchy sequence in $\mathbb{R}^n$ where x^j is of the form $(x_1^j, x_2^j, \ldots, x_n^j)$, $x_i^j \in \mathbb{R}$ for $i = 1, 2, \ldots, n$, $j \in \mathbb{Z}\ (>0)$. For any $\varepsilon > 0$, there exists $n_o \in \mathbb{Z}\ (>0)$ such that for all $m_1 \geq n_o$, $m_2 \geq n_o$, $\|x^{m_1} - x^{m_2}\| < \varepsilon$. For each $i = 1, 2, \ldots, n$, we have

$$\left| x_i^{m_1} - x_i^{m_2} \right| = \sqrt{(x_i^{m_1} - x_i^{m_2})^2} \leq \sqrt{\sum_{i=1}^{n} (x_i^{m_1} - x_i^{m_2})^2}$$

$$= \| x^{m_1} - x^{m_2} \| < \varepsilon$$

for all $m_1 \geq n_o$, $m_2 \geq n_o$. This shows that, for $i = 1, 2, \ldots, n$, $\{x_i^m\}$ is a Cauchy sequence in $\mathbb{R}$, which is complete. There exists $x_i \in \mathbb{R}$ such that $x_i^m \to x_i$ as $m \to \infty$ for $i = 1, 2, \ldots, n$. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. We claim that $x^m \to x$ as $m \to \infty$. It follows from $x_i^m \to x_i$ as $m \to \infty$, for $i = 1, 2, \ldots, n$, that there exists $n_o^i \in \mathbb{Z}$ (> 0) such that for any $m \geq n_o^i$, $|x_i^m - x_i| < \frac{\varepsilon}{\sqrt{n}}$.

Let $n^* = \max \{ n_o^i \mid i = 1, 2, \ldots, n \}$.

For any $m \geq n^*$, we have

$$\| x^m - x \| = \sqrt{\sum_{i=1}^{n} (x_i^m - x_i)^2} < \sqrt{\sum_{i=1}^{n} \frac{\varepsilon^2}{n}} = \varepsilon .$$

This completes the proof.

1.5 <u>Example (b).</u>

For any fixed n, the set $\mathbb{C}^n$ of all n-tuples

$$x = (x_1, x_2, \ldots, x_n),$$

where $x_1, x_2, \ldots, x_n$ are complex numbers, is a Banach space if addition and scalar multiplication are defined componentwise, as usual, and if

$$\| x \| = \sqrt{\sum_{i=1}^{n} |x_i|^2} .$$

<u>Proof.</u> The proof follows the same pattern as in proof of (a).

2. Bounded Linear Transformation

2.1 Definition. A transformation T from a normed linear space X into a normed linear space Y is called linear if

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

for any $x, y \in X$ and any $\alpha, \beta \in \mathbb{C}$.

2.2 Theorem . Let X and Y be normed linear spaces. The set $\mathcal{L}(X, Y)$ of all linear transformations of X into Y is a complex vector space, under the operations defined in the proof.

Proof. For any $f, g \in \mathcal{L}(X, Y)$, any $x \in X$ and any $\alpha \in \mathbb{C}$, we define

$$(f + g)(x) = f(x) + g(x),$$

$$(\alpha f)(x) = \alpha f(x).$$

For any $f, g \in \mathcal{L}(X, Y)$, any $x, y \in X$ and any $\alpha, \lambda, \beta \in \mathbb{C}$, we have

$$\begin{aligned}(f + g)(\alpha x + \beta y) &= f(\alpha x + \beta y) + g(\alpha x + \beta y)\\ &= \alpha f(x) + \beta f(y) + \alpha g(x) + \beta g(y)\\ &= \alpha(f + g)(x) + \beta(f + g)(y)\end{aligned}$$

so that $f + g \in \mathcal{L}(X, Y)$ and

$$\begin{aligned}(\lambda f)(\alpha x + \beta y) &= \lambda (f(\alpha x + \beta y))\\ &= \lambda (\alpha f(x) + \beta f(y))\\ &= \alpha (\lambda f)(x) + \beta(\lambda f)(y)\end{aligned}$$

so that $\lambda f \in \mathcal{L}(X,Y)$. This completes the proof.

2.3 Definition. For any T in $\mathcal{L}(X,Y)$, T is called a bounded linear transformation if there is a nonnegative real A such that that $\|Tx\| \leq A\,\|x\|$ for all $x \in X$.

The smallest such A is denoted by $\|T\|$, called the norm of T; in particular, $\|Tx\| \leq \|T\|\|x\|$ for all $x \in X$.

2.4 Theorem. The set $B\mathcal{L}(X,Y)$ of all bounded linear transformations of X into Y is a complex vector subspace of the complex vector space $\mathcal{L}(X,Y)$.

Proof. By definition of $B\mathcal{L}(X,Y)$ and $\mathcal{L}(X,Y)$, we see that $B\mathcal{L}(X,Y)$ is a subset of $\mathcal{L}(X,Y)$. Consider for any $f, g \in B\mathcal{L}(X,Y)$, any $x \in X$, we have

$$\begin{aligned} \|(f+g)(x)\| &= \|f(x)+g(x)\| \\ &\leq \|f(x)\| + \|g(x)\| \\ &\leq \|f\|\|x\| + \|g\|\|x\| \\ &= A_1\,\|x\|, \end{aligned}$$

where $A_1 = (\|f\| + \|g\|) \in \mathbb{R}(\geq 0)$, so that $f + g \in B\mathcal{L}(X,Y)$ and for any $\alpha \in \mathbb{C}$,

$$\begin{aligned} \|(\alpha f)(x)\| = \|\alpha(f(x))\| &= |\alpha|\,\|f(x)\| \leq |\alpha|\|f\|\,\|x\| \\ &= A_2\|x\|, \end{aligned}$$

where $A_2 = |\alpha|\|f\| \in \mathbb{R}(\geq 0)$, so that $\alpha f \in B\mathcal{L}(X,Y)$. Then $B\mathcal{L}(X,Y)$ is closed under vector addition and scalar multiplication which are defined in the complex vector space $\mathcal{L}(X,Y)$. Hence $B\mathcal{L}(X,Y)$ is a complex vector subspace of complex vector space $\mathcal{L}(X,Y)$. The proof is complete.

2.5 Theorem. Let T be any element in $B\mathcal{L}(X, Y)$; that is, there is a nonnegative real A such that $\|T(x)\| \leq A\|x\|$ for all $x \in X$. The following formulations of $\|T\|$ are equivalent :

(1) $\|T\| = \text{Inf}\left\{A \in \mathbb{R}(>0) \;\middle|\; \|T(x)\| \leq A\|x\| \text{ for all } x \in X\right\}$.

(2) $\|T\| = \sup\left\{\dfrac{\|T(x)\|}{\|x\|} \;\middle|\; x \in X \setminus \{0\}\right\}$.

(3) $\|T\| = \sup\left\{\|T(x)\| \;\middle|\; x \in X,\ \|x\| = 1\right\}$.

Proof. (1) implies (2).

From equation (1), $\|T\| \geq \dfrac{\|T(x)\|}{\|x\|}$ for $x \in X \setminus \{0\}$ so that $\|T\| \geq \sup\left\{\dfrac{\|T(x)\|}{\|x\|} \;\middle|\; x \in X \setminus \{0\}\right\}$. It remains to show that $\|T\| \leq \sup\left\{\dfrac{\|T(x)\|}{\|x\|} \;\middle|\; x \in X \setminus \{0\}\right\}$. If $\|T\| = 0$ then $\sup\limits_{x \in X \setminus \{0\}} \dfrac{\|T(x)\|}{\|x\|} \geq 0 = \|T\|$. If $\|T\| > 0$ then, for any b such that $0 < b < \|T\|$, since $\|T\|$ is the infimum of $A > 0$ such that $\|T(x)\| \leq A\|x\|$, there is a x in X such that $\|Tx\| > b\|x\|$, this implies $x \neq 0$; otherewise $T(x) = 0$ and $0 > b\|x\|$ which contradicts the fact that $\|x\| \geq 0$. Then we have $\dfrac{\|T(x)\|}{\|x\|} > b$, so that $\sup\limits_{x \in X \setminus \{0\}} \dfrac{\|T(x)\|}{\|x\|} \geq b$ and $\sup\limits_{x \in X \setminus \{0\}} \dfrac{\|T(x)\|}{\|x\|} \geq \|T\|$, since $b \in (0, \|T\|)$. This completes the proof of (1) implies (2).

Next, we want to show that equations (2) and (3) are equivalent. This follows, since for any $y \in X \setminus \{0\}$, $y = \alpha x$ for some $\alpha \in \mathbb{C}$ and for some $x \in X$ such that $\|x\| = 1$, so that the following equalities hold :

$$\sup_{y \in X \setminus \{0\}} \frac{\| T(y) \|}{\| y \|} = \sup_{\substack{x \in X \\ \|x\| = 1}} \frac{\| T(\alpha x) \|}{\| \alpha x \|} = \sup_{\substack{x \in X \\ \|x\| = 1}} \frac{|\alpha| \, \| T(x) \|}{|\alpha| \, \| x \|}$$

$$= \sup_{\substack{x \in X \\ \|x\| = 1}} \| T(x) \| .$$

The theorem will be proved when we show that equation (2) implies equation (1). From equation (2), we have $\| T \| \geq \frac{\| T(x) \|}{\| x \|}$ for all $x \in X \setminus \{0\}$ and for any $A \in \mathbb{R}(>0)$ such that $A \geq \| T \|$, $\| T(x) \| \leq A \| x \|$ for all $x \in X$. Hence $\| T \| = \inf \left\{ A \in \mathbb{R}(>0) \;\middle|\; \| T(x) \| \leq A \| x \| \text{ for all } x \in X \right\}$. Now, the proof is complete.

2.6 Theorem. For any linear transformation T of a normed linear space X into a normed linear space Y, the following three conditions are equivalent :

(1) T is bounded.

(2) T is continuous.

(3) T is continuous at one point $x_0 \in X$.

Proof. (1) implies (2).

If $\| T \| = 0$ then T is the zero transformation which is continuous. Assume $\| T \| \in \mathbb{R}(>0)$. For any $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{\| T \|}$, for any x, x_0 in X such that $\| x - x_0 \| < \delta$ implies $\| T(x) - T(x_0) \| \leq \| T \| \, \| x - x_0 \| < \| T \| \frac{\varepsilon}{\| T \|} = \varepsilon$.

(2) implies (3) is trivially.

(3) implies (1).

Given any $\varepsilon > 0$, there is a $\delta > 0$ such that $\| x - x_0 \| < \delta$

implies $\| T(x) - T(x_0) \| < \varepsilon$. In other words, $\| x \| < \delta$ implies $\| T(x_0 + x) - T(x_0) \| = \| T(x) \| < \varepsilon$ or $\| \frac{x}{\delta} \| \leq 1$ implies $\| T(\frac{x}{\delta}) \| \leq \frac{\varepsilon}{\delta}$. Hence, $\| T \| = \sup_{\| \frac{x}{\delta} \| = 1} \| T(\frac{x}{\delta}) \| \leq \frac{\varepsilon}{\delta}$ implies that T is bounded.

2.7 <u>Theorem</u>. $B\mathcal{L}(X,Y)$ is a normed linear subspace of the linear space $\mathcal{L}(X,Y)$.

<u>Proof</u>. Note that $B\mathcal{L}(X,Y)$ is a complex vector subspace of the linear space $\mathcal{L}(X,Y)$. For any $T \in B\mathcal{L}(X,Y)$, we define

$$\| T \| = \sup_{\substack{x \in X \\ \| x \| = 1}} \| T(x) \| .$$

We claim that $B\mathcal{L}(X,Y)$ with the above norm is a normed linear space. By Theorem 2.5, we see that $\| T \|$ is a nonnegative real number. For any $T_1, T_2, T \in B\mathcal{L}(X,Y)$, any $\alpha \in \mathbb{C}$, we have

$$\| T_1 + T_2 \| = \sup_{\substack{x \in X \\ \| x \| = 1}} \| (T_1 + T_2)(x) \| = \sup_{\substack{x \in X \\ \| x \| = 1}} \| T_1(x) + T_2(x) \|$$

$$\leq \sup_{\substack{x \in X \\ \| x \| = 1}} \| T_1(x) \| + \sup_{\substack{x \in X \\ \| x \| = 1}} \| T_2(x) \| = \| T_1 \| + \| T_2 \| ,$$

and $\| \alpha T \| = \sup_{\substack{x \in X \\ \| x \| = 1}} \| (\alpha T)(x) \| = |\alpha| \sup_{\substack{x \in X \\ \| x \| = 1}} \| T(x) \| = |\alpha| \, \| T \|$.

If $\| T \| = 0$, we have $\| T(x) \| \leq \| T \| \, \| x \| = 0$, for all $x \in X$. This implies that $T(x) = 0$ for all $x \in X$; that is , T is a zero transformation. The proof is complete.

2.8 <u>Theorem</u>. If Y is a Banach space then $B\mathcal{L}(X,Y)$ is a Banach space.

<u>Proof</u>. Let $\{f_n\}$ be a Canchy sequence in $B\mathcal{L}(X,Y)$. Let $\varepsilon > 0$ be given. There is a $n_0 \in \mathbb{Z}\,(>0)$ such that $\| f_m - f_n \| < \frac{\varepsilon}{3}$

for all $m \geq n_o$, $n \geq n_o$. By the definition of $\|\cdot\|$ in $B\mathcal{L}(X,Y)$, we have for any x such that $\|x\| = 1$, $\|f_m(x) - f_n(x)\| < \frac{\varepsilon}{3}$ for $m \geq n_o$, $n \geq n_o$. This shows that $\{f_n(x)\}$ is a Cauchy sequence in Y, which is complete, hence $\{f_n(x)\}$ converges to an element $f(x) \in Y$. This is also true for any $x \in X$ since we can write $x = \lambda y$ with $\|y\| = 1$ and $\lambda = \|x\|$, hence $f_n(x) = \lambda f_n(y)$ tends to a limit $f(x) = \lambda f(y)$.

The linearity of f follows since

$$f(x+y) = \lim_{n\to\infty} (f_n(x+y)) = \lim_{n\to\infty} (f_n(x) + f_n(y))$$

$$= \lim_{n\to\infty} f_n(x) + \lim_{n\to\infty} f_n(y)$$

$$= f(x) + f(y),$$

and $f(\alpha x) = \lim_{n\to\infty} (f_n(\alpha x)) = \lim_{n\to\infty} \alpha f_n(x)$

$$= \alpha \lim_{n\to\infty} f_n(x) = \alpha f(x).$$

The boundedness of f can be proved as follows. Since $\{f_n(x)\}$ converges to $f(x)$, there is $n_1 \in \mathbb{Z}\ (>0)$ such that for all $n \geq n_1$, $\|f_n(x) - f(x)\| < \frac{\varepsilon}{3}$. For any $x \in X$ such that $\|x\| = 1$, let $\varepsilon = 3$, there is $n_2 \in \mathbb{Z}\ (>0)$ such that $\|f_{n_2}(x) - f(x)\| < 1$, hence $\|f(x)\| < 1 + \|f_{n_2}(x)\| \leq 1 + \|f_{n_2}\|$, since $f_{n_2} \in B\mathcal{L}(X,Y)$; that is, $\|f(x)\| \leq (1 + \|f_{n_2}\|)\|x\|$ for all $x \in X$ such that $\|x\| = 1$. This inequality is still true for any $x \in X$ since we can write $x = \lambda y$ with $\|y\| = 1$

and $\lambda = \|x\|$, hence $\|f(x)\| = \|f(\lambda y)\| = \|\lambda f(y)\| =$

$\lambda \|f(y)\| = \|x\| \|f(y)\| \leq \|x\| \quad (1 + \|f_{n_2}\|)$ which implies that f is bounded.

We know that $\|f_n - f\| = \sup_{\substack{x \in X \\ \|x\|=1}} \|(f_n - f)(x)\|$. There is a $x_o \in X$ such that $\|x_o\| = 1$, $\|(f_n - f)(x_o)\| \geq \|f_n - f\| - \frac{\varepsilon}{3}$.

Let $n^* = \max(n_o, n_1)$. For any $n \geq n^*$,

$$\|f_n - f\| \leq \|(f_n - f)(x_o)\| + \frac{\varepsilon}{3} = \|f_n(x_o) - f(x_o)\| + \frac{\varepsilon}{3}$$

$$\leq \|f_n(x_o) - f_{n^*}(x_o)\| + \|f_{n^*}(x_o) - f(x_o)\| + \frac{\varepsilon}{3}$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

This completes the proof.

3. The Open Mapping Theorem.

3.1 Theorem. (The Open Mapping Theorem) Let U and V be the open unit balls of the Banach spaces X and Y, respectively. To every bounded linear transformation T of X onto Y there corresponds a $\delta > 0$ such that

(1) $T(U) \supset \delta V$.

Note the symbol δV stands for the set $\{\delta y : y \in V\}$; that is, the set of all $y \in Y$ with $\|y\| < \delta$.

Let us now explain the name of the theorem. Let W_1 be any open ball in X with center at x_o and radius $r > 0$; that is ; the set of all $x_o + rx$ where $x \in U$. From the linearity of T, we have $T(x_o + rx) = T(x_o) + r\,T(x)$. It follows from (1) that there is a $\delta > 0$ such that $T(U) \supset \delta V$; that is, $T(W_1) \supset \{Tx_o + r_1 y \mid y \in V,\ r_1 = r\delta > 0\}$. Hence the image of every open ball in X, with center at x_o, say, contains an open ball in Y with center at Tx_o. Thus the image under T of every open set is open; that is, T is an open mapping.

Here is another way of stating (1) : To every y with $\|y\| < \delta$ there corresponds an x with $\|x\| < 1$ so that $Tx = y$.

Proof. Given $y \in Y$, there is an $x \in X$ such that $Tx = y$; if $\|x\| < k$, it follows that $y \in T(kU)$. Hence Y is the union of the sets $T(kU)$, for k = 1, 2, ... Since Y is complete, Theorem 1.3.5 implies that Y is not a countable union of nowhere dense sets. There exists a $T(kU)$ of Y such that $T(kU)$ is not nowhere dense. By the definition of nowhere dense set, we have the closure $T(kU)$ contains a nonempty open subset W of Y.

This means that every point of W is the limit of a sequence $\{Tx_i\}$, where $x_i \in kU$; from now on, k and W are fixed.

Since W is open, we can choose $y_o \in W$ and $\eta > 0$ so that $y_o + y \in W$ if $\|y\| < \eta$. For any such y there are sequences $\{x_i'\}$, $\{x_i''\}$ in kU such that

(2) $Tx_i' \longrightarrow y_o$ and $Tx_i'' \longrightarrow y_o + y$ as $i \longrightarrow \infty$.

Setting $x_i = x_i'' - x_i'$, we have $\|x_i\| \leq \|x_i''\| + \|x_i'\| < 2k$ and $Tx_i \longrightarrow y$. Since this holds for every y with $\|y\| < \eta$, the linearity of T shows that the following is true if

$\delta = \eta / 2k$:

To each $y \in Y$ and to each $\mathcal{E} > 0$ there corresponds an $x \in X$ such that

(3) $\|x\| \leq \delta^{-1}\|y\|$ and $\|y - Tx\| < \mathcal{E}$.

This is almost the desired conclusion, as stated just before the start of the proof, except that there we had $\mathcal{E} = 0$.

Fix $y \in \delta V$, and fix $\varepsilon > 0$. By (3) there exists an x_1 with $\|x_1\| \leq \delta^{-1}\|y\| < \delta^{-1}\delta = 1$ and

(4) $\|y - Tx_1\| < \frac{1}{2}\delta\varepsilon$

which follows from (3) that

$$\left\|\frac{2x_1}{\delta}\right\| \leq \delta^{-1}\left\|\frac{2y}{\delta}\right\| \text{ and } \left\|\frac{2y}{\delta} - \frac{T2x_1}{\delta}\right\| < \mathcal{E};$$

that is, $\|y - Tx_1\| < \delta\frac{\varepsilon}{2}$.

Suppose $x_1, \ldots, x_n$ are chosen so that

(5) $\|y - Tx_1 - Tx_2 - \ldots - Tx_n\| < 2^{-n}\delta\varepsilon$.

Use (3), with y replaced by the vector on the left side of (5), to obtain an x_{n+1} so that (5) holds with $n + 1$ in place of n, and

(6) $\|x_{n+1}\| \leq \delta^{-1}\|y - Tx_1 \ldots - Tx_n\| = \delta^{-1}2^{-n}\delta\varepsilon = 2^{-n}\varepsilon$,

for $n = 1, 2, 3, \ldots$.

If we set $S_n = x_1 + \ldots + x_n$, (6) shows that $\{S_n\}$ is a Cauchy sequence in X. Since, for any $\mathcal{E}_1 > 0$, choose $n_o \in \mathbb{Z}(> 0)$ such that for any positive integer p, $2^{n_o} > \frac{\varepsilon}{\varepsilon_1}(1 + 2^{-1} \ldots + 2^{-p+1})$ and for any $n \geq n_o$,

$$\|S_{n+p} - S_n\| = \|x_{n+1} + \ldots + x_{n+p}\| \leq \|x_{n+1}\| + \ldots + \|x_{n+p}\|$$

$$\leqslant 2^{-n}\varepsilon + \dots + 2^{-(n+p-1)}\varepsilon = 2^{-n}\varepsilon(1 + 2^{-1} + \dots + 2^{-p+1})$$

$$\leqslant 2^{-n_o}\varepsilon(1 + 2^{-1} + \dots + 2^{-p+1}) < \varepsilon_1.$$

Since X is complete, there exists an $x \in X$ so that $S_n \to x$ as $n \to \infty$. The inequality $\|x_1\| < 1$, together with (6), shows that

$$\|x\| = \left\|\sum_{n=1}^{\infty} x_n\right\| \leqslant \sum_{n=1}^{\infty}\|x_n\| < 1 + 2^{-1}\varepsilon + \dots + 2^{-n}\varepsilon + \dots$$

$$= 1 + 2^{-1}\varepsilon(1 + 2^{-1} + 2^{-2} + \dots) = 1 + 2^{-1}\varepsilon\left(\frac{1}{1-\frac{1}{2}}\right) = 1 + \varepsilon .$$

Since T is continuous, $TS_n \to Tx$. By (5) $TS_n \to y$. Hence $Tx = y$. We have now proved that

(7) $T((1 + \varepsilon)U) \supset \delta V$,

or

(8) $T(U) \supset (1 + \varepsilon)^{-1}\delta V$,

for every $\varepsilon > 0$. The union of the sets on the right of (8), taken over all $\varepsilon > 0$, is δV. This proves (1).

3.2 Corollary. If X and Y are Banach spaces and if T is a bounded linear transformation of X onto Y which is also one-to-one, then there is a $\delta > 0$ such that

(1) $\|Tx\| \geqslant \delta\|x\| \quad (x \in X)$.

Proof. If δ is chosen as in (3) in the proof of the Open Mapping Theorem. In (3) of that theorem, $Tx = y$ has already been proved and T is now one-to-one, shows that

$$\|Tx\| \geqslant \delta\|x\| \quad (x \in X).$$

4. The Banach-Steinhaus Theorem or the Uniform Boundedness Principle

4.1 Definition. Let f be an extended real-valued function on a topological space. f is said to be lower semicontinuous if the set $\{x : f(x) > \alpha\}$ is open for every real α. f is said to be upper semicontinuous if the set $\{x : f(x) < \alpha\}$ is open for every real α.

4.2 Lemma. (a) An extended real-valued function f is continuous if and only if it is both upper semicontinuous and lower semicontinuous.

(b) The suppremum of any collection of lower semicontinuous functions is lower semicontinuous.

Proof of (a). Assume f is continuous. Then the set $\{x : f(x) > \alpha\}$ is open for every real α so that f is lower semicontinuous. Similarly, f is upper semicontinuous.

Conversely, let $\mathscr{C}$ be the class of all open intervals in $\widetilde{\mathbb{R}}$. Let V be any open set in $\widetilde{\mathbb{R}}$. V can be written as a countable union of open intervals in $\widetilde{\mathbb{R}}$. That is, $V = \bigcup_{i=1}^{\infty} I_i$, where $I_i \in \mathscr{C}$. Let $I = (\alpha, \beta)$ (may be $[-\infty, \beta)$, $(\alpha, \infty]$) be any open interval in $\widetilde{\mathbb{R}}$. By hypothesis, $f^{-1}[-\infty, \beta)$ and $f^{-1}(\alpha, +\infty]$ are open so that $f^{-1}(I) = f^{-1}[-\alpha, \beta) \cap f^{-1}(\alpha, \infty]$ is open. Hence $f^{-1}(V) = \bigcup \{f^{-1}(I_i) \mid i = 1, 2, \ldots, I_i \in \mathscr{C}\}$ is open. Therefore f is continuous and the proof is complete.

Proof of (b). Let $g = \sup_{n \geq 1} f_n$, where f_n is a lower semicontinuous function, n =1, 2,... . Let α be any real number. For any $x \in g^{-1}(\alpha,\infty]$, we have $\alpha < \sup_{n\geq 1} f_n(x) \leq +\infty$ and there exists $m \in \mathbb{Z}$ (>0) such that $\alpha < \sup_{n\geq 1} f_n(x) - \varepsilon \leq f_m(x) \leq +\infty$ where $\varepsilon = (\sup_{n\geq 1} f_n(x) - \alpha)/ 2$. Hence x is in $\bigcup_{n\geq 1} f_n^{-1}(\alpha,\infty]$ so that $g^{-1}(\alpha,\infty] \subset \bigcup_{n\geq 1} f_n^{-1}(\alpha,\infty]$. For any $x \in \bigcup_{n\geq 1} f_n^{-1}(\alpha,\infty]$, we have $\alpha < f_m(x) \leq \sup_n f_n(x) \leq \infty$ for some $m \in$ (>0) so that $x \in g^{-1}(\alpha,\infty]$. Hence $\bigcup_{n\geq 1} f_n^{-1}(\alpha,\infty] \subset g^{-1}(\alpha,\infty]$ and $g^{-1}(\alpha, +\infty] = \bigcup_{n\geq 1} f_n^{-1}(\alpha,\infty]$. Since the countable union of open sets is open. Then g is lower semicontinuous and the lemma is proved.

4.3 <u>Theorem</u>.(The Banach-Steinhous Theorem or the Uniform Boundedness Principle). Suppose X is a Banach space, Y is a normed linear space, and $\{T_\alpha\}$ is a collection of bounded linear transformations of X into Y, where α ranges over some index set A. Then either there exists an $M < \infty$ such that

(1) $\|T_\alpha\| \leq M$, for every $\alpha \in A$, or

(2) $\sup_{\alpha \in A} \|T_\alpha x\| = \infty$, for all x belonging to some dense G_δ set in X , where G_δ is the intersection of a countable collection of open sets of X.

<u>Proof</u>. Put $\varphi(x) = \sup_{\alpha \in A} \|T_\alpha x\|$ for all $x \in X$.

Let $V_n = \{x : \varphi(x) > n\}$ $(n = 1, 2, \ldots)$.

Since each T_α is continuous and the norm of Y is a continuous function on Y, each function $x \longmapsto \|T_\alpha x\|$ is continuous on X. By Lemma 4.2, φ is lower semicontinuous, and each V_n is open.

If one of those sets, say V_N, fails to be dense in X, then there exist an $x_0 \in X$ and an $r > 0$ such that $\|x\| \leq r$ implies $x_0 + x \notin V_N$; this means that $\varphi(x_0 + x) \leq N$, or

$$\|T_\alpha (x_0 + x)\| \leq N$$

for all $\alpha \in A$ and all x with $\|x\| \leq r$. Since $x = (x_0 + x) - x_0$, we then have

$$\|T_\alpha x\| \leq \|T_\alpha (x_0 + x)\| + \|T_\alpha x_0\| \leq 2N.$$

Hence, $\|T_\alpha\| = \sup_{\|x\|=1} \|T_\alpha x\| \leq \frac{2N}{r} = M$, for all $\alpha \in A$.

The other possibility is that every V_n is dense in X. In that case , $\cap V_n$ is a dense G_δ in X, by Baire's theorem. Moreover $\varphi(x) = \infty$ for every $x \in \cap V_n$. Hence the theorem is completely proved.

5. The Hahn-Banach Theorem

5.1 Proposition. Let V be a complex vector space.

(a) If u is the real part of a complex-linear functional f on V, then

(1) $f(x) = u(x) - iu(ix)$ for all $x \in V$.

(b) If u is a real-linear functional on V and if f is defined by (1), then f is a complex-linear functional on V.

(c) If V is a normed linear space and f and u are related as in (1), then $\|f\| = \|u\|$.

Proof. If α and β are real numbers and $z = \alpha + i\beta$, then the real part of iz is $-\beta$. Thus for all complex number z,

(2) $z = \text{Re } z - i \text{ Re } (iz)$

Since

(3) $\text{Re } (if(x)) = \text{Re } f(ix) = u(ix)$,

(1) follows from (2) with $z = f(x)$. Under the hypothesis (b), we have that $f(x + y) = u(x + y) - iu(i(x + y)) = u(x) + u(y) - iu(ix) - iu(iy) = f(x) + f(y)$ and $f(\alpha x) = u(\alpha x) - iu(i\alpha x) = \alpha(u(x) - iu(ix)) = \alpha f(x)$, for all real α and for all x, y in V.

But we also have

(4) $f(ix) = u(ix) - iu(-x) = u(ix) + iu(x) = i (f(x))$,

which proves that f is a complex-linear functional on V.

Since $|u(x)| \leq |f(x)| \leq \|f\| \|x\|$, for all $x \in X$, we have $\sup_{x \neq 0} \frac{|u(x)|}{\|x\|} \leq \|f\|$ so that $\|u\| \leq \|f\|$. On the other hand, to every $x \in V$ there corresponds a complex number α, $|\alpha| = 1$ so that $\alpha f(x) = |f(x)|$. Then

(5) $|f(x)| = f(\alpha x) = u(\alpha x) \leq \|u\| \|\alpha x\| = \|u\| \|x\|$,

so that $\|f\| \leq \|u\|$. Thus the part (c) is proved.

5.2 Definition. Let M be a subspace of a normed space X. Let F and f be bounded linear functional on X and M, respectively.

F is an extension of f if the domain of F includes the domain of f and $F(x) = f(x)$ for all x in the domain of f. In this case, f is also called a restriction of F.

The norm $\|F\|$ and $\|f\|$ are computed relative to the domains of F and f, explicitly ;

$$\|f\| = \sup\left\{\frac{|f(x)|}{\|x\|},\quad x \in M \setminus \{0\}\right\},$$

$$\|F\| = \sup\left\{\frac{|F(x)|}{\|x\|},\quad x \in X \setminus \{0\}\right\}.$$

5.3 <u>Theorem.</u> (The Hahn-Banach Theorem) If M is a subspace of a normed linear space X and if f is a bounded linear functional on M, then f can be extended to a bounded linear functional F on X so that $\|F\| = \|f\|$.

<u>Proof.</u> We first assume that X is a real normed linear space and, consequently, that f is a real-linear bounded functional on M. If $\|f\| = 0$, the desired extension is $F = 0$. We may assume that $\|f\| \neq 0$. First we shall deal with the case where $\|f\| = 1$.

Choose $x_o \in X$, $x_o \notin M$, and let M_1 be the vector space spanned by M and x_o. Then M_1 **consists of all vectors of the form** $x + \lambda x_o$, where $x \in M$ and λ is a real scalar. If we define $f_1(x + \lambda x_o) = f(x) + \lambda\alpha$, where α is any fixed real number. Then $f_1(x) = f(x)$ for all $x \in M$ and for any $\beta_1, \beta_2, \lambda_1, \lambda_2 \in \mathbb{R}$, $x_1, x_2 \in M$,

$$f_1\left[\beta_1(x_1 + \lambda_1 x_o) + \beta_2(x_2 + \lambda_2 x_o)\right] = \beta_1 f_1(x_1 + \lambda_1 x_o) + \beta_2 f_1(x_2 + \lambda_2 x_o);$$

that is, f_1 is a linear functional on M_1 extending f. The problem is then reduce to choose α so that the extended functional still has norm 1. This will be the case provided that

(1) $|f(x) + \lambda\alpha| \leq \|x + \lambda x_o\|$ $\quad (x \in M,\ \lambda$ real).

Replace x by $-\lambda x$ and divide both side of (1) by $|\lambda|$. The requirement is then that

(2) $|f(x) - \alpha| \leq \|x - x_o\|$ $(x \in M)$,

that is, that $A_x \leq \alpha \leq B_x$ for all $x \in M$, where

(3) $A_x = f(x) - \|x - x_o\|$ and $B_x = f(x) + \|x - x_o\|$.

There exists such an α if and only if all the intervals $[A_x, B_x]$ have a common point ; that is, if and only if

(4) $A_x \leq B_y$ for all x and $y \in M$.

To prove this equivalence, suppose that there exists x and $y \in M$ such that $B_y < A_x$. We have $A_y \leq B_y < A_x \leq B_x$. This implies that $[A_y, B_y] \cap [A_x, B_x] = \phi$ or not all the intervals $[A_x, B_x]$ for all $x \in M$ have a common point. Conversely, suppose that $\bigcap_{x \in M} [A_x, B_x] = \phi$, then by the finite intersection property, there exist $x_o, x_1, \ldots, x_n \in M$ such that $[A_{x_o}, B_{x_o}] \cap \bigcap_{i=1}^{n} [A_{x_i}, B_{x_i}] = \phi$. Let $a = \text{Max}\{A_{x_i} \mid i=1,\ldots,n\}$, $b = \text{Min}\{B_{x_i} \mid i=1,\ldots,n\}$. Then $a = A_{x_k}$, for some $k \in \{1,\ldots,n\}$, $b = B_{x_m}$ for some $m \in \{1,\ldots,n\}$ and $[A_{x_o}, B_{x_o}] \cap [A_{x_k}, B_{x_m}] = \phi$. This implies that $B_{x_o} < A_{x_k}$, for some x_o, x_k in M. Thus the desired equivalence is proved. Now (4) holds by virtue of (3), since

$$f(x) - f(y) = f(x - y) \leq \|x - y\| \leq \|x - x_o\| + \|y - x_o\| .$$

We have now proved that there exists a norm-preserving extension f_1 of f on M_1.

Let $\mathcal{P}$ be the collection of all ordered pairs (M', f'), where M' is a subspace of X which contains M and where f' is a real-linear extension of f to M', with $\|f'\| = 1$.

Partially order $\mathcal{P}$ by declaring $(M', f') \leq (M'', f'')$ to mean that $M' \subset M''$ and $f''(x) = f'(x)$ for all $x \in M'$. The axioms of a partial order are satisfied, $\mathcal{P}$ is not empty since it contains (M, f), and so the Hausdorff maximality theorem asserts the existence of a maximal totally ordered subcollection Ω of $\mathcal{P}$.

Let Φ be the collection of all M' such that $(M', f') \in \Omega$. Then Φ is totally ordered, by set inclusion, and therefore the union $\tilde{M}$ of all members of Φ is a subspace of X. If $x \in \tilde{M}$, then $x \in M'$ for some $M' \in \Phi$; defined $F(x) = f'(x)$, where f' is the function which occurs in the pair $(M', f') \in \Omega$. F is well-defined since, for any $x \in \tilde{M}$, suppose there exist M', M'' such that $x \in M'$ and $x \in M''$ where (M', f') and $(M'', f'') \in \Omega$. By the totally ordering of Ω, we may assume $M' \subset M''$ so that $f''(x) = f'(x) = F(x)$. F can easily be checked to be a linear functional. $\|F\| = 1$, since for any $x \in \tilde{M}$ there exists $x \in M'$ where $(M', f') \in \Omega$ such that $\|F(x)\| = \|f'(x)\| \leq \|f'\| \|x\| = \|x\|$, that is $\|F\| \leq 1$, and for any $\varepsilon > 0$, there exists $x \in M' \setminus \{0\}$ such that $\|x\|(1 - \varepsilon) \leq \|f'(x)\| = \|F(x)\| \leq \|F\|\|x\|$. This implies that $\|F\| \geq 1$ and $\|F\| = 1$. F is an extension of f on $\tilde{M}$ since we have $(M, f) \leq (M', f')$ for all $(M', f') \in \Omega$ and $(M', f') \leq (\tilde{M}, F)$ for all $(M', f') \in \Omega$. This implies that $F(x) = f(x)$ for all $x \in M$.

Suppose $\widetilde{M}$ is a proper subspace of X. Let $x_1 \in X \setminus \widetilde{M}$. As in the first part of the proof, let $\widetilde{M}_1$ be the vector space spanned by $\widetilde{M}$ and x_1. $\widetilde{M}$ is a proper subspace of $\widetilde{M}_1$. We define $F_1(\widetilde{x} + \lambda_1 x_1) = F(\widetilde{x}) + \lambda_1 \alpha_1$ where $\widetilde{x} \in \widetilde{M}$, $\lambda_1 \in \mathbb{R}$ and α is a fixed real number which is chosen so that $\| F_1 \| = \| F \| = 1$. Finally we arrived at a pair $(\widetilde{M}_1, F_1) \not\geq (\widetilde{M}, F)$ and $\Omega \cup (\widetilde{M}_1, F_1)$ is a totally ordered subset of $\mathcal{P}$ which contradicts the maximality of Ω . This shows that $\widetilde{M} = X$.

If f is a real-linear bounded functional on M such that $\| f \| = R$ where R is a positive real. Let $g = \frac{f}{R}$ so that $\| g \| = 1$, there exists a real-linear bounded functional extension G on X such that $\| G \| = \| g \| = 1$. Let $F = RG$ then F is an extended real-linear functional of f on X so that $\| F \| = \| f \|$.

If now f is a complex-linear functional on the subspace M of the complex normed linear space X, let u be a real part of f, use the real Hahn-Banach theorem to extended u to a real-linear functional U on X, with $\| U \| = \| u \|$, and define

(5) $F(x) = U(x) - i\, U(ix)$ for all $x \in X$.

By Proposition 5.1, F is a complex-linear extension of f, and

$$\| F \| = \| U \| = \| u \| = \| f \| .$$

This completes the proof.

6. Classical Banach Space $L^p(\mathbb{T})$ $(1 \leq p \leq \infty)$

Anticipating the construction of chapter III, let $\mathbb{T}$ be the (1-dimensional) torus and μ the Lebesgue measure on it. We may visualize $\mathbb{T}$ as the set $\{ z \in \mathbb{C} \mid |z| = 1 \}$.

6.1 Definition. If $1 \leq p < \infty$ and f is a complex-valued, Lebesgue measurable function on $\mathbb{T}$, define

$$\| f \|_p = \left(\int_{\mathbb{T}} |f|^p \, d\mu \right)^{1/p} .$$

Then $L^p(\mathbb{T})$ consists of all measurable complex functions f on $\mathbb{T}$ for which $\| f \|_p < \infty$ and we call $\| f \|_p$ the L^p- norm of f.

Actually, $\| \cdot \|_p$ satisfies all the axioms of a norm except that $\| f \|_p = 0$ may not implies that $f \equiv 0$.

6.2 Definition. A property is said to hold a.e. or for almost all x in $\mathbb{T}$ if it holds everywhere on $\mathbb{T}$ except on a measurable set of measure zero.

6.3 Definition. Suppose $g : \mathbb{T} \longmapsto [0, \infty]$ is measurable. Let S be the set of all real α such that

$$\mu \left(g^{-1} ((\alpha, \infty]) \right) = 0 .$$

If $S = \phi$, put $\beta = \infty$. If $S \neq \phi$, put $\beta = \inf S$. Since

$$g^{-1}((\beta, \infty]) = \bigcup_{n=1}^{\infty} g^{-1} \left((\beta + \frac{1}{n}, \infty] \right),$$

and since the union of a countable collection of sets of measure zero has measure zero, we see that $\beta \in S$. We call β the <u>essential supremum of g</u>.

If f is a complex measurable function on Ψ, we define $\|f\|_\infty$ to be the essential supremum of $|f|$, and we let <u>$L^\infty(\Psi)$ consists of all f for which $\|f\|_\infty < +\infty$</u>. The functions in $L^\infty(\Psi)$ are sometimes said to be essentially bounded.

6.4 <u>Proposition.</u> $|f(x)| \leq \lambda$ holds for almost all x if and only if $\lambda \geq \|f\|_\infty$.

<u>Proof.</u> Assume first that $|f(x)| \leq \lambda$ holds for almost all x. That is, there is a measurable set $E = |f|^{-1}(\lambda, \infty]$ so that $|f(x)| \leq \lambda$ for $x \notin E$ and $\mu(E) = 0$. By definition of $\|f\|_\infty$, we have $\|f\|_\infty \leq \lambda$.

Conversely, if $\|f\|_\infty \leq \lambda$ then $\mu(|f|^{-1}(\|f\|_\infty, \infty]) = 0$. But $|f|^{-1}(\lambda, \infty]$ is a subset of $|f|^{-1}(\|f\|_\infty, \infty]$ which implies that $\mu(|f|^{-1}(\lambda, \infty]) = 0$. Then $|f(x)| \leq \lambda$ holds for almost all x.

6.5 <u>Theorem.</u> $L^p(\Psi)$ is a complex vector space for $1 \leq p \leq \infty$.

<u>Proof.</u> We must show the following properties :

(1) If $f, g \in L^p(\Psi)$ then so is $f + g$, and

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

(2) If $f \in L^p(\Psi)$ and α is a complex number then $\alpha f \in L^p(\Psi)$. In fact, $\|\alpha f\|_p = |\alpha| \|f\|_p$.

For $1 < p < \infty$, (1) follows from Minkowski's inequality.

For $p = 1$, (1) is a consequence of the inequality

$$|f + g| \leq |f| + |g| .$$

For $p = \infty$, (1) follows from

$$|f(\dot{x}) + g(\dot{x})| \leq |f(\dot{x})| + |g(\dot{x})| \quad \text{for all } \dot{x} \text{ in } \Psi$$

$$\leq \|f\|_\infty + \|g\|_\infty \quad \text{for almost all } \dot{x} \text{ in } \Psi .$$

By Proposition 6.4 we have

$$\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty .$$

(2) follows from the equality

$$\left(\int_\Psi |\alpha f|^p \, d\mu \right)^{1/p} = |\alpha| \left(\int_\Psi |f|^p \, d\mu \right)^{1/p} \quad \text{for } 1 \leq p < \infty ,$$

and

$$|\alpha f| = |\alpha| |f| \quad \text{for } p = \infty .$$

This completes the proof.

Suppose $f, g \in L^p(\Psi)$, for $1 \leq p \leq \infty$, define

$$d(f, g) = \|f - g\|_p .$$

Then d satisfies all the axioms of a metric except that $d(f,g) = 0$ might not imply $f \equiv g$.

Let us write $f \sim g$ if and only if $d(f, g) = 0$. This is easily seen to be an equivalence relation in $L^p(\Psi)$ which partition $L^p(\Psi)$ into equivalence classes. If F and G are two equivalence classes, choose $f \in F$ and $g \in G$, and define $d(F, G) = d(f, g)$; note that $f \sim f_1$ and $g \sim g_1$ implies

$$d(f, g) \leq d(f, f_1) + d(f_1, g_1) + d(g_1, g) = d(f_1, g_1)$$

and similarly, $d(f_1,g_1) \leq d(f,g)$. Hence $d(f, g) = d(f_1,g_1)$ so that $d(F,G)$ is well defined.

The set of all equivalence classes of $L^p(\Phi)$ is now a metric space by defining $d(F,G) = d(f,g) = \|f - g\|_p$. Note that it is also a vector space, since $f \sim f_1$, $g \sim g_1$ imply $f + g \sim f_1 + g_1$ and $\alpha f \sim \alpha f_1$. From now on we shall denote the set of all equivalence classes by $L^p(\Phi)$ as well.

6.6 Theorem. $L^p(\Phi)$ is a complete metric space for $1 \leq p \leq \infty$.

Proof. Consider $1 \leq p < \infty$.

Let $\{f_n\}$ be a Cauchy sequence in $L^p(\Phi)$. Take $\varepsilon = \frac{1}{2}$, there exists $n_1 \in \mathbb{Z}\,(>0)$ such that $\|f_n - f_{n_1}\|_p < \frac{1}{2}$ for all $n \geq n_1$. Suppose we have obtained a sequence $n_1 \leq n_2 \leq \dots \leq n_k$. Then letting $\varepsilon = \frac{1}{2^k}$, there exists $n_k \geq n_{k-1}$ in $\mathbb{Z}(>0)$ such that $\|f_n - f_{n_k}\|_p < \frac{1}{2^k}$ for all $n \geq n_k$. Hence, there is a subsequence $\{f_{n_i}\}$, $n_1 \leq n_2 \leq \dots\dots$, such that

$$(*) \qquad \left\| f_{n_{i+1}} - f_{n_i} \right\| < 2^{-i} \quad \text{for } i = 1, 2, \dots .$$

Define

$$g_k = \sum_{i=1}^{k} \left| f_{n_{i+1}} - f_{n_i} \right|, \quad g = \sum_{i=1}^{\infty} \left| f_{n_{i+1}} - f_{n_i} \right| .$$

Since $(*)$ holds, the Minkowski's inequality shows that, for any $k \in \mathbb{Z}\,(>0)$,

$$\|g_k\|_p = \left(\int_{\Phi} |g_k|^p \, d\mu \right)^{1/p} = \left(\int_{\Phi} g_k^p \, d\mu \right)^{1/p} \leq \sum_{i=1}^{k} \left(\int_{\Phi} \left| f_{n_{i+1}} - f_{n_i} \right|^p d\mu \right)^{1/p}$$

$$= \sum_{i=1}^{k} \left\| f_{n_{i+1}} - f_{n_i} \right\|_p < \sum_{i=1}^{k} 2^{-i} < \sum_{i=1}^{\infty} 2^{-i} = 1.$$

Hence an application of Fatou's lemma to $\{g_k^p\}$ gives

$$\|g\|_p = \left(\int_\Phi g^p d\mu\right)^{1/p} = \left(\int_\Phi \lim_{k\to\infty} g_k^p \, d\mu\right)^{1/p} \leq \underline{\lim}_{k\to\infty} \left(\int_\Phi g_k^p \, d\mu\right)^{1/p} \leq 1.$$

And $g \in L^p(\Phi)$ implies g is finite a.e. on Φ, so that the series $\sum_{i=1}^{\infty} (f_{n_{i+1}} - f_{n_i})$ converges absolutely a.e. on Φ. Then the series

$$(**) \qquad f_{n_1}(\dot{x}) + \sum_{i=1}^{\infty} (f_{n_{i+1}}(\dot{x}) - f_{n_i}(\dot{x}))$$

converges absolutely a.e. on Φ. We denote the sum of $(**)$ by $f(\dot{x})$, for those $\dot{x}$ at which $(**)$ converges, put $f(\dot{x}) = 0$ on the remaining set of measure zero. Since

$$f_{n_1}(\dot{x}) + \sum_{i=1}^{k-1} (f_{n_{i+1}}(\dot{x}) - f_{n_i}(\dot{x})) = f_{n_k}(\dot{x}) ,$$

we see that

$$f(\dot{x}) = \lim_{k\to\infty} f_{n_k}(\dot{x}) \text{ a.e. or } f(\dot{x}) = \lim_{i\to\infty} f_{n_i}(\dot{x}) \text{ a.e. .}$$

Since $\{f_n\}$ is a Cauchy sequence in $L^p(\Phi)$. For any given $\varepsilon > 0$, there exists $N \in \mathbb{Z}\,(>0)$ such that $\int_\Phi |f_n - f_m|^p \, d\mu < \varepsilon^p$ if $n \geq N$, $m \geq N$. For some i onwards we have $n_i \geq N$ such that $\int_\Phi |f_{n_i} - f_m|^p \, d\mu < \varepsilon^p$. For every $m \geq N$, Fatou's lemma therefore shows that

$$(***) \qquad \int_\Phi |f - f_m|^p \, d\mu = \int_\Phi \lim_{i\to\infty} |f_{n_i} - f_m|^p \, d\mu$$

$$\leq \underline{\lim}_{i\to\infty} \int_\Phi |f_{n_i} - f_m|^p \, d\mu \leq \varepsilon^p .$$

We conclude from $(***)$ that $f - f_m \in L^p(\Phi)$, hence that

$f \in L^p(\Psi)$, and finally that $\|f - f_m\|_p \longrightarrow 0$ as $m \longrightarrow \infty$. This completes the proof for the case $1 \leq p < \infty$.

In $L^\infty(\Psi)$, suppose $\{f_n\}$ is a Cauchy sequence in $L^\infty(\Psi)$, let A_k and $B_{m,n}$ be the sets where $|f_k(\dot{x})| > \|f_k\|_\infty$ and $|f_n(\dot{x}) - f_m(\dot{x})| > \|f_n - f_m\|_\infty$, and let E be the union of these sets, for $k,m,n = 1,2,3,\ldots$. Then $\mu(E) = 0$, and we show that on the complement of E the sequence $\{f_n\}$ converges uniformly to a bounded function. For any $\dot{x} \in E^c$, $\{f_n(\dot{x})\}$ is a Cauchy sequence in $\mathbb{C}$, which is complete, so that $\lim_{n\to\infty} f_n(\dot{x}) = f(\dot{x})$. For any $\varepsilon > 0$, there exist $n_0, n_1 \in \mathbb{Z}(>0)$ such that for all $n \geq n_0$ $|f_n(\dot{x}) - f(\dot{x})| < \frac{\varepsilon}{3}$ and for all $m \geq n_1$, $n \geq n_1$, $\|f_n - f_m\|_\infty < \frac{\varepsilon}{3}$. Let $n' = \max(n_0, n_1)$. For any $n \geq n'$ there is a $\dot{x}_0 \in E^c$ such that

$$\begin{aligned}\sup_{\dot{x}\in E^c} |f_n(\dot{x}) - f(\dot{x})| &\leq |f_n(\dot{x}_0) - f(\dot{x}_0)| + \frac{\varepsilon}{3} \\ &\leq |f_n(\dot{x}_0) - f_{n'}(\dot{x}_0)| \\ &\quad + |f_{n'}(\dot{x}_0) - f(\dot{x}_0)| + \frac{\varepsilon}{3} \\ &< \varepsilon,\end{aligned}$$

and for any $\dot{x} \in E^c$,

$$|f(\dot{x})| \leq |f(\dot{x}) - f_{n_0}(\dot{x})| + |f_{n_0}(\dot{x})| < \varepsilon + |f_{n_0}(\dot{x})| \leq |f_{n_0}(\dot{x})| \leq \|f_{n_0}\|_\infty < \infty .$$

Define $f(\dot{x}) = 0$ for $\dot{x} \in E$. Then $f \in L^\infty(\Psi)$ and $\|f_n - f\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$.