CHAPTER I

SOME THEOREMS FROM METRIC SPACES

1. Metric spaces

1.1 Definition. Let M be a nonempty set. A metric d on M is a function of $M \times M$ into $\mathbb{R}$ (≥ 0) satisfying

(1) $d(x, y) = 0$ if and only if $x = y$ for all x and $y \in M$,

(2) $d(x, y) = d(y, x)$ for all x and $y \in M$,

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y and $z \in M$.

Then (M, d) is called a metric space.

2. Complete Metric spaces

2.1 Definition. In a metric space (M, d), a sequence $\{x_n\}$ is a Cauchy sequence if $\lim_{m,n \to \infty} d(x_m, x_n) = 0$; that is, given any $\varepsilon > 0$, there exists $n_o \in \mathbb{Z}$ (> 0) such that for all $m \geq n_o$, $n \geq n_o$, $d(x_m, x_n) < \varepsilon$.

2.2 Definition. A metric space is complete if every Cauchy sequence converges.

2.3 Definition. An open ball with center x and radius $r > 0$ is defined as the set

$$B(x, r) = \{ y \in M \mid d(x, y) < r \}.$$

3. Baire's Theorem

3.1 Lemma. If (M, d) is a complete metric space and if $V_1 \supset V_2 \supset \ldots$ is a sequence of nonempty closed subsets of M such that the diameters of V_n converges to 0 as $n \to \infty$, then $\bigcap_{n=1}^{\infty} V_n$ is a singleton.

Proof. Let $\{x_n\}$ be a sequence in V_1 such that $x_i \in V_i$ for $i = 1, 2, \ldots,$. Let $\varepsilon > 0$ be given. Since diameter $(V_n) \to 0$ as $n \to \infty$, there is $n_o \in \mathbb{Z}(> 0)$ such that diam $V_n < \varepsilon$ for all $n \geq n_o$. For any $m \geq n_o$, $n \geq n_o$, we may assume $m \geq n$ so that $V_m \subset V_n$ and $d(x_m, x_n) \leq \sup\{d(x, y) \mid x \in V_n, y \in V_n\}$ $=$ diam $V_n < \varepsilon$. Hence $\{x_n\}$ is a Cauchy sequence of M, which is complete. There is $x \in M$ such that $x = \lim_{n \to \infty} x_n$. And this x belongs to V_1 since V_1 is closed. Next we will show that x belongs to V_n for all n. Suppose $x \notin V_{n_o}$ for some n_o and since V_{n_o} is closed then there is $r > 0$ such that $B(x, r) \cap V_{n_o} = \phi$. For any x_n in V_{n_o}, x_n does not belong to $B(x, r)$. This implies that $d(x_n, x) \geq r$ which contradicts the fact that x is the limit of $\{x_n\}$. Finally, we will show that x is unique. Suppose there exists y in V_n for all n and y distincts from x. We have, for all n,

$$0 \neq d(x, y) \leq \sup\{d(x', y') \mid x', y' \in V_n\} = \text{diam } V_n,$$

which contradicts the fact that diam $V_n \to 0$ as $n \to \infty$. This completes the proof.

3.2 Theorem. (Baire's Theorem) If (X, d) is a complete metric space, the intersection of every countable collection of dense open subsets of X is dense in X.

Proof. Suppose $V_1, V_2, \ldots\ldots$ are dense open subsets of X. Let W be any nonempty open set in X. We have to show that $\bigcap_{n=1}^{\infty} V_n$ has a point in W. Let $\bar{B}(x, r)$ be the closure of $B(x, r)$.

Since V_1 is dense, $W \cap V_1$ is a nonempty open set. There exist x_1 and r such that $2 > r > 0$, $B(x_1, r) \subset W \cap V_1$. Let $r_1 = \frac{r}{2}$, $1 > r_1 > 0$ such that $\bar{B}(x_1, r_1) \subset B(x_1, r) \subset W \cap V_1$. If $n \geqslant 2$ and x_{n-1} and r_{n-1} are chosen, the denseness of V_n shows that $V_n \cap B(x_{n-1}, r_{n-1})$ is not empty, and we can therefore find x_n and r_n such that

$$\bar{B}(x_n, r_n) \subset B(x_{n-1}, r_{n-1}) \cap V_n \text{ and } 0 < r_n < \frac{1}{n}.$$

By induction, this process produces a non-increasing sequence of nonvoid closed sets $\bar{B}(x_n, r_n)$ in X such that diameters of $\bar{B}(x_n, r_n) \leqslant \frac{2}{n} \to 0$ as $n \to \infty$. By Lemma 3.1, $\bigcap_{n=1}^{\infty} \bar{B}(x_n, r_n)$ is a singleton, say $\{x\}$. By construction, x belongs to $W \cap V_n$ for all n and $W \cap \bigcap_{n=1}^{\infty} V_n$ is not empty. This completes the proof.

3.3 Definition. A set $E \subset X$ is nowhere dense if its closure $\bar{E}$ contains no nonempty open subset of X.

Any countable union of nowhere dense sets is called a set of the first category.

3.4 <u>Theorem</u>. The following statements are equivalent :

(1) A is nowhere dense in X.

(2) $\overline{A}$ is nowhere dense in X.

(3) $X \setminus \overline{A}$ is dense in X.

<u>Proof</u>. (1) implies (2) is clear.

(2) implies (3).

Assume $\overline{A}$ is nowhere dense. Then we have int $(\overline{A}) = \phi$ and int $(\overline{A}) = \overline{\overline{A}^c}^c$. Take complement of equality $\overline{\overline{A}^c}^c = \phi$ we have $\overline{\overline{A}^c} = X$ or $\overline{X \setminus \overline{A}} = X$. This ends the proof.

(3) implies (1).

Assume $X \setminus \overline{A}$ is dense in X then we have $\overline{\overline{A}^c} = X$. By taking the complement we have $\overline{\overline{A}^c}^c = \phi$ and int $(\overline{A}) = \overline{\overline{A}^c}^c$ so that int $(\overline{A}) = \phi$. This ends the proof.

3.5 <u>Theorem</u>. Complete metric space (X, d) is not of the first category.

<u>Proof</u>. Let $\mathcal{F}$ be any countable family of nowhere dense subsets of X. For each $A \in \mathcal{F}$, by Theorem 3.4, $(\overline{A})^c$ is a dense open subset of the complete metric space X. Since $\mathcal{F}$ is countable, by Theorem 3.2 (Baire's Theorem), there is a $p \in X$ such that $p \in (\overline{A})^c$ for every $A \in \mathcal{F}$. In particular, we have $p \notin \overline{A} \supset A$ for all $A \in \mathcal{F}$. Hence X is not the union of the family $\mathcal{F}$. The proof is complete.