CHAPTER IV

ON INVERTIBLE GRAPHS

Invertible Spaces

By the <u>n-sphere</u>, S^n, we mean a homeomorph of $\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$. J.G. Hocking and P.H. Doyle have shown that for any non-empty open subset U of S^n, there is a homeomorphism h from S^n onto itself such that $h(S^n - U)$ is a subset of U. Motivated by this property of S^n, they define a topological space $(S, \mathcal{T})$ to be <u>invertible</u> (or an invertible space) if for each non-empty open subset U of S, there exists a homeomorphism h of S onto itself such that $h(S - U)$ lies in U; and h is called an <u>inverting homeomorphism</u> for U.

Since we use the result in our thesis, we shall show that the 1-sphere, S^1, (which is a homeomorph of $\{(x, y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$) is an invertible space. Consider $\mathbb{R} \cup \{\infty\}$, a homeomorph of $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Let U be any non-empty open subset of $\mathbb{R} \cup \{\infty\}$. Let $a \in U$ and $a \neq \infty$, then there exists $r > 0$ such that $\{x \in \mathbb{R} \mid |x - a| \leq r\} \subseteq U$. Define $h : \mathbb{R} \cup \{\infty\} \longrightarrow \mathbb{R} \cup \{\infty\}$ as follows : $h(x) = -\frac{r^2}{(x-a)} + a$ if $x \in \mathbb{R} - \{a\}$, $h(a) = \infty$ and $h(\infty) = a$. It is clear that h is a one-one and onto function, $h = h^{-1}$ and h is continuous on $\mathbb{R} - \{a\}$. Now, we show that h is continuous at a and ∞. Let V be any neighborhood of ∞, then there exists $\epsilon > 0$

such that $A = \{x \in \mathbb{R} \mid |x| > \epsilon\} \cup \{\infty\} \subseteq V$. Since $B = \{x \in \mathbb{R} \mid |x-a| < \frac{r^2}{|a|+\epsilon}\}$ is a neighborhood of a such that $h(B) \subseteq A \subseteq V$, h is continuous at a. Again, let V^* be any neighborhood of a, then there exists $\epsilon^* > 0$ such that $A^* = \{x \in \mathbb{R} \mid |x - a| < \epsilon^*\} \subseteq V^*$. Since $B^* = \{x \in \mathbb{R} \mid |x| > \frac{r^2 + \epsilon^* |a|}{\epsilon^*}\} \cup \{\infty\}$ is a neighborhood of ∞ such that $h(B^*) \subseteq A^* \subseteq V^*$, h is continuous at ∞. This shows that h is a homeomorphism from S^1 onto itself. To see that h is an inverting homeomorphism for U, let x be any point in $S^1 - U$. If $x = \infty$, $h(x) \in U$. Assume $x \neq \infty$. Thus, $|x - a| > r$ and hence $|h(x) - a| = |\frac{r^2}{x-a}| < r$. That is $h(x) \in \{x \in \mathbb{R} \mid |x - a| < r\} \subseteq U$. Therefore, we have $h(S^1 - U) \subseteq U$. Hence, S^1 is an invertible space.

4.1 Theorem. Let $(S, \mathcal{T})$ be an invertible space which contains a non-empty open, connected subset U of S. If S is not connected, then U and S - U are the components of S and they are homeomorphic.

Proof. Let h be an inverting homeomorphism for U. Then $S - U \subseteq h(U)$. Assume S is not connected. Suppose S - U is a proper subset of h(U). Then $h(U) \cap U \neq \phi$; so $S = h(U) \cup U$ is connected by theorem 2.23 since h(U) and U are connected. This is a contradiction and hence $S - U = h(U)$. That is U and S - U are homeomorphic and S - U is connected. Since $S = U \cup (S - U)$ separation, we get that U is both open and closed and S - U is both open and closed. By theorem 2.30, U and S - U are the components of S. #

4.2 Theorem. Let $(S, \mathcal{T})$ be an invertible T_1-space and S contains an open connected subset U which consists of at least two points.

Then $(S, \mathcal{T})$ is connected.

Proof. Suppose S is not connected. By theorem 4.1, U and S - U are the components of S. Let $p \in U$. Since S is a T_1-space, $U - \{p\}$ is open in S. Since U contains at least two points, $U - \{p\} \neq \phi$. Let h be an inverting homeomorphism for $U - \{p\}$, so $h(S - (U - \{p\})) \subseteq U - \{p\}$ and hence $h(S - U) \subseteq U - \{p\} \subset U$. Since $h(S - U)$ is connected and both open and closed, it is a component of S by theorem 2.30. This is a contradiction since U is a component of S. Hence $(S, \mathcal{T})$ is connected. #

Invertible Graphs.

In this section we shall be concerned with $\mathbb{R}^n$, Euclidean n-space. By a zero-simplex σ^0 we mean a singleton subset of $\mathbb{R}^n$. A one-simplex σ^1 is defined to be a homeomorph of an open interval (0, 1) of real numbers such that its closure $\overline{\sigma^1}$ in $\mathbb{R}^n$ is homeomorphic to [0, 1], and $\overline{\sigma^1} - \sigma^1$ is made up of two distinct points which are homeomorphic images of 0 and 1. The two points in $\overline{\sigma^1} - \sigma^1$ are called the end points of σ^1. It is clear that they are non-cut points of the connected set $\overline{\sigma^1}$. A zero-simplex whose element is one of the end points of a one-simplex σ^1 is called a face of σ^1. Hence every one-simplex has two faces. If σ^1 is a one-simplex and σ^0 is a face of σ^1, then we say that they are incident.

A graph G is defined to be a finite collection of zero-simplexes and one-simplexes satisfying the following conditions :

1. The simplexes of G are disjoint and no two one-simplexes have the same end points.

2. If a one-simplex is in G, then both of its faces are in G.

3. There is at least one one-simplex in G.

Let G be a graph. The element of a zero-simplex in G is called a <u>vertex</u> of G. Thus, if σ is a one-simplex in G, then its end points are vertices of G. Let $|G|$ denote the point set union of all simplexes in G, i.e., $|G| = \bigcup_{\sigma \in G} \sigma$, then we call $|G|$, considered as a subspace of $\mathbb{R}^n$, the <u>topological</u> <u>realization</u> of G or the <u>1-polyhedron</u> covered by G. By the definition of a graph G, it follows that if $\sigma \in G$, then $\bar{\sigma} \subseteq |G|$ where $\bar{\sigma}$ is the closure of σ in $\mathbb{R}^n$; so we have $|G| = \bigcup_{\sigma \in G} \bar{\sigma}$. Since G is finite and $\bar{\sigma}$ is a closed and bounded subset of $\mathbb{R}^n$ for every $\sigma \in G$, $|G|$ is a closed and bounded subset of $\mathbb{R}^n$ and hence $|G|$ is a compact subspace of $\mathbb{R}^n$.

Let σ be a one-simplex of a graph G. Let $\{v_1, v_2\}$ be the set of end points of σ. Then v_1, v_2 are non-cut points of $\bar{\sigma} \subseteq |G|$. Since $\bar{\sigma}$ is an arc, by lemma 3.7, the topology of $\bar{\sigma}$ is the order topology for some linear order on $\bar{\sigma}$. Let $<$ be a linear order on $\bar{\sigma}$ which determines the topology of $\bar{\sigma}$ such that v_1, v_2 are the first and last elements of $\bar{\sigma}$, respectively. If x and y are any two distinct points in $\bar{\sigma}$ such that $x < y$, then the notations (x, y), $[x, y]$, $(x, y]$ and $[x, y)$ are defined as $\{t \in \bar{\sigma} \mid x < t < y\}$, $\{t \in \bar{\sigma} \mid x \leq t \leq y\}$, $\{t \in \bar{\sigma} \mid x < t \leq y\}$ and $\{t \in \bar{\sigma} \mid x \leq t < y\}$, respectively. It is clear that (x, y) is open in $\bar{\sigma}$ and $[x, y]$ is closed in $\bar{\sigma}$. If $x = v_1$, then $[v_1, y) = \{t \in \bar{\sigma} \mid t < y\}$ is open in $\bar{\sigma}$. If $y = v_2$, then $(x, v_2] = \{t \in \bar{\sigma} \mid x < t\}$ is open in $\bar{\sigma}$.

Now, consider $(x, v_2]$ and $[v_1, x)$ when $x \in \bar{\sigma}$ and $v_1 \neq x \neq v_2$. Since $\bar{\sigma}$ is an arc, by the proof of theorem 3.13, there exists a homeomorphism h from $[0, 1]$ onto $\bar{\sigma}$ which is order-preserving. Then $h(0) = v_1$, $h(1) = v_2$. Since $v_1 < x < v_2$, there is an r in $(0, 1)$ such that $h(r) = x$ and $h((r, 1]) = (x, v_2]$ and $h([0, r)) = [v_1, x)$. Since $(r, 1]$ and $[0, r)$ are connected, $(x, v_2]$ and $[v_1, x)$ are also connected.

A graph G is defined to be an <u>invertible graph</u> if, as a 1-polyhedron, $|G|$ is an invertible space.

<u>Example</u>.

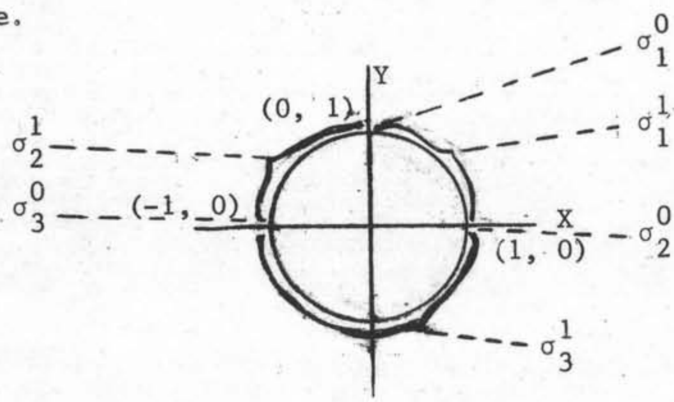
In $\mathbb{R}^2$, let $\sigma_1^0 = \{(0, 1)\}$, $\sigma_2^0 = \{(1, 0)\}$, $\sigma_3^0 = \{(-1, 0)\}$

$$\sigma_1^1 = \{(x, y) \mid x^2 + y^2 = 1 \text{ and } 0 < x, y < 1\}$$

$$\sigma_2^1 = \{(x, y) \mid x^2 + y^2 = 1 \text{ and } -1 < x < 0,\ 0 < y < 1\}$$

$$\sigma_3^1 = \{(x, y) \mid x^2 + y^2 = 1 \text{ and } -1 < x < 1,\ -1 \leq y < 0\}.$$

See the picture.



Let $G = \{\sigma_1^0, \sigma_2^0, \sigma_3^0, \sigma_1^1, \sigma_2^1, \sigma_3^1\}$. Then G is a graph and $|G| = \{(x, y) \mid x^2 + y^2 = 1\}$ which is a 1-sphere. Thus G is an invertible graph. Note that G consists of three different vertices and three different one-simplexes.

4.3 Lemma. Let G be a graph. Then $|G|$ is metrizable.

Proof. Since $\mathbb{R}^n$ is metrizable, the conclusion follows from theorem 2.12. #

4.4 Lemma. Let σ be a one-simplex in a graph G. Then σ is an open subset of $|G|$.

Proof. Let G be a graph and let σ be a one-simplex in G. Let β be any element of G such that $\beta \neq \sigma$. Then $\bar{\beta} \subseteq |G|$ and $\bar{\beta} \cap \sigma = \phi$. This implies that $|G| - \sigma = \bigcup \bar{\beta}$ for all $\beta \in G$ such that $\beta \neq \sigma$. Since $\bar{\beta}$ is closed in $|G|$ and G is finite, $|G| - \sigma$ is a finite union of closed sets in $|G|$. Thus σ is open in $|G|$. #

4.5 Lemma. If G is an invertible graph, then $|G|$ is connected.

Proof. Let G be an invertible graph. Then $|G|$ is metrizable by lemma 4.3; so $|G|$ is a T_1-space. Let σ be a one-simplex in G. Since σ is an open connected subset of $|G|$, by theorem 4.2, $|G|$ is connected. #

4.6 Lemma. If G is an invertible graph, then G has at least 3 vertices.

Proof. Let G be an invertible graph. Suppose G has only 2 vertices, say v_1, v_2. Let σ be a one-simplex in G. Then the end points of σ are vertices of G. This implies that v_1, v_2 are the end points of σ. Hence $G = \{\sigma, \{v_1\}, \{v_2\}\}$. Thus $|G| = \bar{\sigma}$ which is homeomorphic to $[0, 1]$. If $[0, 1]$ is invertible, then there is an inverting homeomorphism h for $(0, 1)$ such that $h(0), h(1) \in (0, 1)$. But 0 and 1 are non-cut points of $[0, 1]$ and the property of being a non-cut point is a topological property, so we have non-cut points in $(0, 1)$ which is impossible. #

4.7 Lemma. Let G be an invertible graph. Then for any vertex v of G there exists a one-simplex σ in G such that $\{v\}$ and σ are incident.

Proof. Suppose there exists v, a vertex of G, such that $\{v\}$ is incident with no one-simplex in G. Then for any β in G such that $\beta \neq \{v\}$, $\bar{\beta} \cap \{v\} = \phi$; hence $|G| - \{v\} = \bigcup \bar{\beta}$ for all β in G such that $\beta \neq \{v\}$. Since G is finite and $\bar{\beta}$ is closed in $|G|$, $|G| - \{v\}$ is closed in $|G|$, i.e., $\{v\}$ is open in $|G|$. But $\{v\}$ is closed in $|G|$ since $|G|$ is metrizable, so $\{v\}$ is a proper subset of $|G|$ which in both open and closed in $|G|$. Therefore, $|G|$ is not connected and we have a contradiction of lemma 4.5. #

4.8 Lemma. For any vertex v of an invertible graph G, $\{v\}$ is incident with more than one one-simplex of G.

Proof. Let G be an invertible graph. Suppose there exists a vertex v of G such that $\{v\}$ is incident with only one one-simplex σ of G. Firstly, we will show that v is a non-cut point of $|G|$. Suppose $|G| - \{v\} = A \cup B$ separation. Since $|G| - \{v\}$ is open in $|G|$, A and B are also open in $|G|$. Since σ is connected, $\sigma \subseteq A$, say. Let $\beta \in G$ such that $\sigma \neq \beta \neq \{v\}$. Then $\bar{\beta} \subseteq |G|$ and $\bar{\beta} \cap (\sigma \cup \{v\}) = \phi$. Hence $|G| - (\sigma \cup \{v\}) = \cup\, \bar{\beta}$ for all β in G such that $\sigma \neq \beta \neq \{v\}$; so $\sigma \cup \{v\}$ is open in $|G|$. Since $\sigma \subseteq A$, $(\sigma \cup \{v\}) \cup A = A \cup \{v\}$ being the union of open sets is open in $|G|$. Therefore, $|G| = (A \cup \{v\}) \cup B$ separation which contradicts lemma 4.5; so $|G| - \{v\}$ is connected. That is v is a non-cut point of $|G|$. Secondly, we will show that for any x in σ, x is a cut point of $|G|$. Let x be any point in σ. Let $<$ be a linear order on $\bar{\sigma}$ which determines the topology of $\bar{\sigma}$ and $<$ determines v as a last element. Then $x < v$. Therefore, $[x, v]$ is closed in $\bar{\sigma}$ and hence it is closed in $|G|$ since $\bar{\sigma}$ is closed in $|G|$. Since v is the last element of $\bar{\sigma}$, $(x, v] = \{t \in \bar{\sigma} \mid x < t\}$ is open in $\bar{\sigma}$. Then there exists an open subset U of $|G|$ such that $\bar{\sigma} \cap U = (x, v]$. Since $(x, v] \subseteq \sigma \cup \{v\} \subseteq \bar{\sigma}$, $(\sigma \cup \{v\}) \cap U = (x, v]$. This shows that $(x, v]$ is a finite intersection of open sets in $|G|$. That is $(x, v]$ is open in $|G|$. Since $|G| - \{x\}$ is open in $|G|$ and $|G| - [x, v] \subseteq |G| - \{x\}$ and $(x, v] \subseteq |G| - \{x\}$, $|G| - [x, v]$ and $(x, v]$ are open in $|G| - \{x\}$. Now, we have $|G| - \{x\} = (|G| - [x, v]) \cup (x, v]$ separation. This proves that x is a cut point of $|G|$. That is σ which we have proved to be an open subset of $|G|$ contains only cut points of $|G|$.

By an invertibility of G, there exists an inverting homeomorphism h for σ such that $h(v) \in \sigma$. Since v is a non-cut point of $|G|$ as we have proved above, it implies that there is a non-cut point in σ which is a contradiction and the lemma is proved. #

Following from lemma 4.6 and lemma 4.8 we have :

4.9 <u>Lemma</u>. Every invertible graph G has at least 3 distinct one-simplexes. #

4.10 <u>Lemma</u>. Let G be an invertible graph. Then there exist $\{\sigma_i \mid i = 1, 2, \ldots, n\}$ a set of one-simplexes of G and a set $\{v_i \mid i = 1, 2, \ldots, n\}$ of vertices of G for some natural $n \geq 3$ such that σ_j is incident with $\{v_j\}$, $\{v_{j+1}\}$ where $j = 1, 2, \ldots, n-1$, and σ_n is incident with $\{v_1\}$, $\{v_n\}$.

<u>Proof</u>. Let m be the number of all one-simplexes of an invertible graph G. Then $m \geq 3$. Let σ_1 be a one-simplex in G. Then there exist v_1, v_2 the vertices of G such that v_1, v_2 are end points of σ_1, i.e., σ_1 is incident with $\{v_1\}$, $\{v_2\}$. By lemma 4.8, there is a one-simplex of G different from σ_1, say σ_2, such that σ_2 is incident with $\{v_2\}$. Let v_3 be the other end point of σ_2, i.e., v_3 is a vertex of G such that $\{v_3\}$ is incident with σ_2. It is clear that $v_3 \notin \{v_1, v_2\}$. Now, we have found a set $\{v_1, v_2, v_3\}$ of distinct vertices of G and a set $\{\sigma_1, \sigma_2\}$ of distinct one-simplexes of G such that σ_j is incident with $\{v_j\}$, $\{v_{j+1}\}$ for $j = 1, 2$.

Let B be the set of natural numbers i where $3 \le i$ such that there exist a set $\{\sigma_j \mid j = 1, 2, \ldots, i\text{-}1\}$ of distinct one-simplexes of G and a set $\{v_j \mid j = 1, 2, \ldots, i\}$ of distinct vertices of G such that σ_j is incident with $\{v_j\}$, $\{v_{j+1}\}$ where $j = 1, 2, \ldots, i\text{-}1$. It is clear that $B \neq \phi$ since $3 \in B$; and since there are only m one-simplexes in G, it follows that $i \le m$ for any $i \in B$. Since B is finite, B has a maximum element. Let i_0 be the maximum element of B. If $i_0 = m$, then by lemma 4.8, there exists a one-simplex say σ_m such that $\sigma_m \neq \sigma_{m-1}$ and σ_m is incident with $\{v_m\}$. Since $v_m \neq v_j$ where $j = 1, 2, 3, \ldots, m\text{-}1$, $\sigma_m \notin \{\sigma_j \mid j = 1, 2, 3, \ldots, m\text{-}1\}$. Let v be the other end point of σ_m. Since $\{\sigma_j \mid j = 1, 2, \ldots, m\}$ is the set of all one-simplexes of G and $\{v_1\}$ is not incident with an element of $\{\sigma_j \mid j = 2, \ldots, m\text{-}1\}$, by lemma 4.8, $\{v_1\}$ must be incident with σ_m.

That is $v_1 = v$. Consider $i_0 < m$. By lemma 4.8 and the same reason as above case, there exists a one-simplex $\sigma_{i_0} \notin \{\sigma_j \mid j = 1, 2, \ldots, i_0\text{-}1\}$ such that σ_{i_0} is incident with $\{v_{i_0}\}$. Let v be the other end point of σ_{i_0}. If $v \notin \{v_j \mid j = 1, 2, \ldots, i_0\}$, then i_0 is not a maximum element of B which is a contradiction. Thus v must be in $\{v_j \mid j = 1, 2, \ldots, i_0\}$, say $v = v_t$. Since $v_{i_0-1} \neq v \neq v_{i_0}$, $1 \le t \le i_0\text{-}2$. Therefore, we have a set of one-simplexes $\{\sigma_j \mid j = t, t\text{+}1, \ldots, i_0\text{-}1, i_0\}$ and a set $\{v_j \mid j = t, t\text{+}1, \ldots, i_0\text{-}1, i_0\}$ of vertices of G such that σ_j is incident with $\{v_j\}$, $\{v_{j+1}\}$ where $j = t, t\text{+}1, \ldots, i_0\text{-}1$ and σ_{i_0} is incident with $\{v_{i_0}\}$, $\{v_t\}$, For any $j \in \{t, t+1, \ldots, i_0\}$, let $s(j) = j - t + 1$ and let σ_j, v_j be denoted by $\beta_{s(j)}$, $U_{s(j)}$, respectively. Then $\{\sigma_j \mid j = t, t+1, \ldots, i_0\} = \{\beta_1, \beta_2, \ldots, \beta_{i_0-t+1}\}$ and $\{v_j \mid j = t, t+1, \ldots, i_0\} = \{U_1, U_2, \ldots, U_{i_0-t+1}\}$. Since $1 \leq t \leq i_0 - 2$, $i_0 - t + 1 \geq 3$. Let $n = i_0 - t + 1$. Hence we have a set $\{\beta_1, \beta_2, \ldots, \beta_n\}$ of one-simplexes of G and a set $\{U_1, U_2, \ldots, U_n\}$ of vertices of G for some $n \geq 3$ such that β_j is incident with $\{U_j\}$, $\{U_{j+1}\}$ where $j = 1, 2, \ldots, n-1$, and β_n is incident with $\{U_n\}, \{U_1\}$. #

4.11 <u>Lemma</u>. Let G be an invertible graph. Let $\{\sigma_i \mid i = 1, 2, \ldots, n\}$ be a set of one-simplexes of G as stated in lemma 4.10. Then for any x and any i such that $x \in \sigma_i$, x is a non-cut point of $|G|$.

<u>Proof</u>. Let $j \in \{1, 2, \ldots, n\}$. Let x be any point in $|G|$ such that $x \in \sigma_j$. Suppose x is a cut point of $|G|$. Let $|G| - \{x\} = A \cup B$ separation. Since $|G| - \{x\}$ is open in $|G|$, A and B are open in $|G|$. Let $\{v_i \mid i = 1, 2, \ldots, n\}$ be the set of vertices of G as stated in lemma 4.10. Then we have v_j, v_{j+1} are end points of σ_j (note $v_{j+1} = v_1$ if $j = n$). Let $<$ be a linear order on $\bar{\sigma}_j$ which determines the topology of $\bar{\sigma}_j$. Assume $v_j < v_{j+1}$. Then $v_j < x < v_{j+1}$ and $[v_j, x)$, $(x, v_{j+1}]$ are connected subsets of $\bar{\sigma}_j$. By virtue of theorem 2.10, $[v_j, x)$ and $(x, v_{j+1}]$ are also connected in $|G|$. Since $\bar{\sigma}_{j-1} \cap \bar{\sigma}_j = \{v_j\}$, $\bar{\sigma}_j \cap \bar{\sigma}_{j+1} = \{v_{j+1}\}$ and $(x, v_{j+1}] \subseteq \bar{\sigma}_j$ and $[v_j, x) \subseteq \bar{\sigma}_j$, it follows that $\bar{\sigma}_{j-1} \cap [v_j, x) = \{v_j\}$ and $(x, v_{j+1}] \cap \bar{\sigma}_{j+1} = \{v_{j+1}\}$. (note: $\sigma_{j-1} = \sigma_n$ if $j = 1$,

$\sigma_{j+1} = \sigma_1$ if $j = n$). Since $\bar{\sigma}_i \cap \bar{\sigma}_{i+1} = \{v_{i+1}\}$ and $\bar{\sigma}_i$ is connected in $|G|$ for all $i = 1, 2, \ldots, n$, $\{\bar{\sigma}_i \mid i = 1, 2, \ldots, j-1, j+1, \ldots, n\}$ $\{[v_j, x), (x, v_{j+1}]\}$ is a collection of connected subsets of $|G|$ which form a bridged system (see p. 11). Since $\bar{\sigma}_j - \{x\} = [v_j, x) \cup (x, v_{j+1}]$, $\bigcup_{i=1}^{n} \bar{\sigma}_i - \{x\} = \bigcup_{i=1}^{n} \bar{\sigma}_i \cup [v_j, x) \cup (x, v_{j+1}]$. By theorem 2.22, $\bigcup_{i=1}^{n} \bar{\sigma}_i - \{x\}$ is connected in $|G|$. By virtue of theorem 2.10, $\bigcup_{i=1}^{n} \bar{\sigma}_i - \{x\}$ are also connected in $|G| - \{x\}$. Then $\bigcup_{i=1}^{n} \bar{\sigma}_i - \{x\} \subseteq A$, say. Since $x \in \sigma_j \subseteq \bigcup_{i=1}^{n} \bar{\sigma}_i \subseteq A \cup \{x\}$ and σ_j is open in $|G|$, x is an interior point of $A \cup \{x\}$. Therefore, $A \cup \{x\}$ is open in $|G|$ by theorem 2.3. That is $|G| = (A \cup \{x\}) \cup B$ separation which contradicts lemma 4.5. Hence x is a non-cut point of $|G|$. #

As a consequence of lemma 4.11, we have :

4.12 Lemma. If G is an invertible graph, then there exist at least three one-simplexes in G such that each of them contains only non-cut points of $|G|$. #

4.13 Theorem. Let G be an invertible graph. For any x in $|G|$, x is a non-cut point of $|G|$.

Proof. Let G be an invertible graph. Suppose that there exists a point x in $|G|$ such that x is a cut point of $|G|$. By lemma 4.12, there is σ, a one-simplex of G, such that $x \notin \sigma$ and σ contains only non-cut points of $|G|$. Since σ is a non-empty open set in $|G|$ and G is invertible, we have an inverting homeomorphism h for σ such that $h(x) \in \sigma$. This implies that there is a cut point in σ which is impossible. Therefore, there is no cut point in $|G|$. #

4.14 Lemma. Let $\{\sigma_i \mid i = 1, 2, \ldots, n\}$ be the set of all one-simplexes of an invertible graph G. Then, for each i, if x and y are any two distinct points in σ_i, $|G| - \{x, y\}$ is not connected.

Proof. Let $i \in \{1, 2, \ldots, n\}$. Let x and y be any two distinct points in σ_i. Let $<$ be a linear order on $\bar{\sigma}_i$ which determines the topology of $\bar{\sigma}_i$. Assume $x < y$. Then $[x, y]$ and (x, y) are closed and open in $\bar{\sigma}_i$, respectively. Since $\bar{\sigma}_i$ is closed in $|G|$, $[x, y]$ is also closed in $|G|$. Since (x, y) is open in $\bar{\sigma}$, there exists U an open subset of $|G|$ such that $(x, y) = \bar{\sigma}_i \cap U$. Since $(x, y) \subseteq \sigma_i \subseteq \bar{\sigma}_i$, $(x, y) = \sigma_i \cap U$ which is open in $|G|$. Since $|G| - \{x, y\}$ is open in $|G|$ and $|G| - [x, y]$, (x, y) are subsets of $|G| - \{x, y\}$, $|G| - [x, y]$, (x, y) are open in $|G| - \{x, y\}$. Therefore, $|G| - \{x, y\} = (|G| - [x, y]) \cup (x, y)$ separation. Hence $|G| - \{x, y\}$ is not connected. #

4.15 Theorem. Let G be an invertible graph and let x and y be any two distinct points in $|G|$. Then $|G| - \{x, y\}$ is not connected.

Proof. Let G be an invertible graph. Let x and y be any two distinct points in $|G|$. By lemma 4.12, there exists σ, a one-simplex of G, which is open in $|G|$ and $x, y \notin \sigma$. Since G is invertible, there is an inverting homeomorphism h for σ such that $h(x)$, $h(y)$ are in σ. By virtue of lemma 4.14, we have that $h(|G| - \{x, y\}) = |G| - \{h(x), h(y)\}$ is not connected. This implies that $|G| - \{x, y\}$ is not connected. #

4.16 Theorem. If G is an invertible graph, then $|G|$ is homeomorphic to S^1, the 1-sphere; i.e., $|G|$ is homeomorphic to $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$.

Proof. Let G be an invertible graph. Let x and y be any two distinct points in $|G|$. By theorem 4.15, $|G| - \{x, y\}$ is not connected. Let $|G| - \{x, y\} = C \cup D$ separation. Since $|G|-\{x, y\}$ is open in $|G|$, C and D are open subsets of $|G|$; hence $C \cup \{x, y\}$, being the complement of D in $|G|$, is closed in $|G|$ and similarly $D \cup \{x, y\}$ is closed in $|G|$. Therefore, $C \cup \{x, y\}$ and $D \cup \{x, y\}$ are compact, metrizable spaces. We must now show that they are also connected. Consider $C \cup \{x, y\}$. By theorem 4.13, y is a non-cut point; so $C \cup D \cup \{x\} = |G| - \{y\}$ is connected and hence x is a cut point of $C \cup D \cup \{x\}$. Then, by corollary 2.32, $C \cup \{x\}$ and $D \cup \{x\}$ are connected in $C \cup D \cup \{x\}$ and hence are connected in $|G|$ by virtue of theorem 2.10. Similarly, y is a cut point of $C \cup D \cup \{y\}$, so $C \cup \{y\}$ and $D \cup \{y\}$ are connected in $|G|$. Now, $(C \cup \{x\}) \cap (C \cup \{y\})$ is not empty and both are connected hence by theorem 2.23 we see that $C \cup \{x, y\}$ is connected. Similarly, $D \cup \{x, y\}$ is also connected. Thus, $C \cup \{x, y\}$ and $D \cup \{x, y\}$ are both compact, connected metrizable spaces. By theorem 2.38, then, $C \cup \{x, y\}$ and $D \cup \{x, y\}$ both have at least two non-cut points. Assume that there exists a point c in C such that c is a non-cut point of $C \cup \{x, y\}$; also assume that there exists a point d in D such that d is a non-cut point of $D \cup \{x, y\}$. Then $(C \cup \{x, y\} - \{c\})$ and $(D \cup \{x, y\} - \{d\})$ are both connected. By theorem 2.23, $|G| - \{c, d\} = (C \cup \{x, y\} - \{c\}) \cup (D \cup \{x, y\} - \{d\})$ is connected which contradicts theorem 4.15. Thus c and d as defined above can not both exist. Hence either $C \cup \{x, y\}$ or $D \cup \{x, y\}$ is an arc by theorem 3.13. Let $C \cup \{x, y\}$ be an arc, x and y must then be the two non-cut points of $C \cup \{x, y\}$. Assume now that $D \cup \{x, y\}$ is not an arc. This means that the point d as defined above exists. That is, d is a non-cut point of $D \cup \{x, y\}$. Let p be any point of C. Then, since $C \cup \{x, y\}$ is homeomorphic to an arc with end points x and y, p is a cut point of $C \cup \{x, y\}$. Hence, $C \cup \{x, y\} - \{p\}$ is the union of two connected subsets M and N one of which contains x and the other of which contains y. Therefore, $|G| - \{p, d\} = (D \cup \{x, y\} - \{d\}) \cup (M \cup N)$ is connected by theorem 2.23. This is a contradiction by theorem 4.15. Thus, $D \cup \{x, y\}$ must also be an arc with x and y as the two end points because they are non-cut points of $D \cup \{x, y\}$. Thus $|G|$ is the union of two arcs with exactly their end points in common.

Let γ_1, γ_2 be two homeomorphisms from [0, 1] onto $C \cup \{x, y\}$ and $D \cup \{x, y\}$, respectively such that $\gamma_1(0) = \gamma_2(0) = x$ and $\gamma_1(1) = \gamma_2(1) = y$. Let h be a homeomorphism from [0, 1] onto [-1, 1] defined by $h(t) = 2t-1$. Then $h(0) = -1$ and $h(1) = 1$. Let A_1 be a subspace of S^1, i.e., $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$, determined by $\{(x, y) \mid y \geq 0\}$ and let A_2 be a subspace of S^1 determined by

$\{(x, y) \mid y \leq 0\}$. The projection function P_x restricted to A_1 and A_2 yields the homeomorphisms g_1 and g_2 from A_1 and A_2, respectively, onto $[-1, 1]$. Thus if we define $f = \gamma_1 \circ h^{-1} \circ g_1$, then f_1 is a homeomorphism from A_1 onto $C \cup \{x, y\}$ with $f_1((-1, 0)) = x$ and $f_1((1, 0)) = y$. Similarly, if $f_2 = \gamma_2 \circ h^{-1} \circ g_2$, then $f_2((-1, 0)) = x$ and $f_2((1, 0)) = y$. Now f_1 and f_2 are defined on closed subspace A_1 and A_2, respectively, of S^1 and they agree on $A_1 \cap A_2 = \{(-1, 0), (1, 0)\}$. Hence by theorem 2.11, the function $f : S^1 \to |G|$ defined by $f((s, t)) = f_i((s, t))$ if $(s, t) \in A_i$; $i = 1, 2$, is continuous. By definition, f is one-one and onto. Since $|G|$ is a Hausdorff and S^1 is compact, f is a homeomorphism by corollary 2.45. #