CHAPTER IV

RUNGE'S THEOREM

Runge's Theorem is concerned with the approximation of an analytic function on an open set by a rational function on a compact subset. In this last chapter we apply the results that we have obtained so far to the proof of this theorem. The material of this chapter can be found in references [1] . The first section gives Rudin's proof of Runge's Theorem. Then we will discuss two other ways of the proving this theorem.

4.1 Runge's Theorem

Suppose K is a compact set in the extended complex plane S^2 and $\{\alpha_j\}$ is a set which contains one point in each component of $S^2 - K$. If Ω is open, $\Omega \supset K$, $f \in H(\Omega)$, and $\varepsilon > 0$, there exists a rational function R, all of whose poles lie in the prescribed set $\{\alpha_j\}$, such that

$$|f(\xi) - R(\xi)| < \varepsilon \qquad \text{for every } \xi \in K .$$

Note that $S^2 - K$ has at most countably many components.
Note also that the preassigned point in the unbounded component of $S^2 - K$ may very well be ∞ .

<u>Proof</u> : Consider the Banach space C(K) whose members are the continuous complex-valued functions on K, with the supremum norm i.e.

$$||f|| = \sup_{x \in X} |f(x)| .$$

Let M be the subspace of $C(K)$ which consists of the restriction to K of those rational functions which have all their poles in $\{\alpha_j\}$.

The theorem asserts that f is in the closure of M. i.e. $f \in \overline{M}$ **iff** for every $\varepsilon > 0$, there exists an $R \in M$ such that $|f(\xi) - R(\xi)| < \varepsilon$ for every $\xi \in K$.

By Theorem 2.56, this is equivalent to saying that every bounded linear functional on $C(K)$ which vanishes on M also vanishes at f and hence the Riesz Representation Theorem 3.2 shows that we must prove the following assertion :

If μ is a complex Borel measure on K such that $\int_K R\, d\mu = 0$ for every rational function R with poles only in the set $\{\alpha_j\}$, and if $f \in H(\Omega)$, then we also have $\int_K f\, d\mu = 0$.

So we assume that μ satisfies $\int_K R\, d\mu = 0 \quad \forall R$.

Define $$h(z) = \int_K \frac{d\mu(\xi)}{\xi - z} \qquad (z \in S^2 - K).$$

From Theorem 2.80 with $X = K$, $\varphi(\xi) = \xi$, $\Omega = S^2 - K$ we see that h is representable by a power series in $S^2 - K$, so by Theorem 2.79 $h \in H(S^2 - K)$. Claim that $h(z) \equiv 0$ for every $z \in S^2 - K$.

Let V_j be the component of $S^2 - K$ which contains α_j .

<u>Case 1</u> If $\alpha_j \neq \infty$.

Suppose that $r_1 > 0$ is sufficiently small so that

$D(\alpha_j, 2r_1) \subset V_j$ and fix z in $D(\alpha_j, r_1)$

$$\frac{1}{\xi - z} = \frac{1}{(\xi - \alpha_j) + (\alpha_j - z)}$$

$$= \frac{1}{\xi - \alpha_j}\left[\frac{1}{1 - \frac{z - \alpha_j}{\xi - \alpha_j}}\right]$$

$$= \frac{1}{\xi - \alpha_j}\left[1 + \frac{z - \alpha_j}{\xi - \alpha_j} + \left(\frac{z - \alpha_j}{\xi - \alpha_j}\right)^2 + \dots\right]$$

$$= \sum_{n=0}^{\infty} \frac{(z - \alpha_j)^n}{(\xi - \alpha_j)^{n+1}}$$

Since $\left|\frac{z - \alpha_j}{\xi - \alpha_j}\right| < \frac{|z - \alpha_j|}{2r_1} < \frac{r_1}{2r_1} = \frac{1}{2}$ for every

$z \in D(\alpha_j, r_1)$ and for every $\xi \in K$, the geometric series
$\sum_{n=0}^{\infty} \frac{(z - \alpha_j)^n}{(\xi - \alpha_j)^{n+1}} = \frac{1}{\xi - z}$ converges uniformly on K, for every

fixed $z \in D(\alpha_j, r_1)$.

$$h(z) = \int_K \frac{d\mu(\xi)}{\xi - z}$$

$$= \int_K \sum_{n=0}^{\infty} \frac{(z - \alpha_j)^n}{(\xi - \alpha_j)^{n+1}} \, d\mu(\xi)$$

$$= \sum_{n=0}^{\infty} (z - \alpha_j)^n \int_K \frac{d\mu(\xi)}{(\xi - \alpha_j)^{n+1}} \quad \text{(By Theorem 2.30)}$$

$$= 0.$$

Hence $h(z) = 0$ for all $z \in D(\alpha_j, r_1)$.

This implies that $h(z) = 0 \quad \forall \; z \in V_j$ by Theorem 2.88.

<u>Case 2</u> If $\alpha_j = \infty$.

Let $r_2 > 0$ and D_{r_2} be the set of all complex no. ξ such that $|\xi| \leq r_2$ and $D_{r_2} \supset K$.

Let D'_{r_2} be the set of all complex numbers z such that $z > 2r_2$.

$$\frac{1}{\xi - z} = - \frac{1}{z(1 - \frac{\xi}{z})}$$

$$= -\frac{1}{z}(1 + \frac{\xi}{z} + (\frac{\xi}{z})^2 + \dots)$$

$$= - \sum_{n=0}^{\infty} \frac{\xi^n}{z^{n+1}} .$$

Since $\left|\frac{\xi}{z}\right| < \frac{|\xi|}{2r_2} \leqslant \frac{r_2}{2r_2} = \frac{1}{2}$ for every $z \in D'_{r_2}$

and every $\xi \in K$, the geometric series

$$-\sum_{n=0}^{\infty} \frac{\xi^n}{z^{n+1}} = \frac{1}{\xi - z} \quad \text{converges uniformly on } K,$$

for every fixed $z \in D'_{r_2}$.

$$\begin{aligned} h(z) &= \int_K \frac{d\mu}{\xi - z} \\ &= \int_K \left(-\sum_{n=0}^{\infty} \frac{\xi^n}{z^{n+1}}\right) d\mu \\ &= -\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int_K \xi^n \, d\mu \quad \text{(By Theorem 2.30)} \\ &= 0 \end{aligned}$$

which implies again that $h(z) = 0$ in D'_{r_2} and By Theorem 2.88 $h(z) = 0$ in V_j.

Now choose the oriented line intervals $\gamma_1, \ldots, \gamma_n$ in $\Omega - K$ as in Theorem 2.93 such that the Cauchy formula

$$f(\xi) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(w)}{w - \xi} \, dw$$

holds for every $f \in H(\Omega)$ and for every $\xi \in K$.

Then $\int_K f \, d\mu = \int_K \left(\sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(w)}{w - \xi} dw \right) d\mu$

$$= \sum_{j=1}^{n} \frac{1}{2\pi i} \int_K \int_{\gamma_j} \frac{f(w)}{w - \xi} dw \, d\mu \, .$$

We shall apply Fubini's Theorem 2.74 (legitimate, since we are dealing with Borel measures and continuous functions on compact space).

First we must show that

$$\int_K \int_{\gamma_j} \left| \frac{f(w)}{w - \xi} \right| |dw| \; d|\mu| \; (\xi) < \infty \; .$$

Since γ_j is compact , $f \in H(\Omega)$

$$\exists \; M > 0 \ni |f(w)| \leq M \quad \forall \; w \in \bigcup_{j=1}^{n} \gamma_j^* \quad \text{(Def. 2.81).}$$

$$\xi \in K \;\&\; \gamma_j \cap K = \phi \quad \forall j$$

$$\exists \text{ a real number } \alpha > 0 \ni |w - \xi| \geq \alpha \quad \forall w \in \bigcup_{j=1}^{n} \gamma_j \, .$$

$$\int_K \int_{\gamma_j} \left| \frac{f(w)}{w - \xi} \right| |dw| \; d|\mu| \; (\xi) \leq \int_K \int_{\gamma_j} \frac{M}{\alpha} |dw| \; d|\mu| \; (\xi)$$

$$= \frac{M}{\alpha} \int_K \int_{\gamma_j} |dw| \; d|\mu| \; (\xi)$$

$$= \frac{M}{\alpha} L_j \, |\mu| \, (K) \quad \text{(where } L_j = \text{length of the line segment } \gamma_j\text{)}$$

$$< \infty \, .$$

$$\int_K f\,d\mu = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_K d\mu\,(\xi) \int_{\gamma_j} \frac{f(w)}{w-\xi}\,dw$$

$$= \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} dw \int_K \frac{f(w)}{w-\xi}\,d\mu\,(\xi)$$

(application of Fubini's theorem)

$$= \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} f(w)\,dw \int_K \frac{d\mu\,(\xi)}{w-\xi}$$

$$= - \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} f(w)\,h(w)\,dw$$

$$= 0 .$$

The last equality depends on the fact that each γ_j is an interval in $S^2 - K$, where h vanishes. i.e. $f \in \overline{M}$.

Then for every $\varepsilon > 0$, $\exists$ an $R \in M \ni$

$$| f(\xi) - R(\xi) | < \quad \forall \xi \in K .$$

This proof of Runge's Theorem uses the Riesz Representation Theorem for a bounded linear functional on the Banach space C(K) (Theorem 3.2) as the main theorem.

We can see from chapter III that this Riesz Representation Theorem is thus very difficult to prove in its entirety. We can prove Runge's Theorem by avoiding this Riesz Representation Theorem. We use the Theorem 3.3 in the first new proof of Runge's Theorem and use the

Riesz Representation Theorem for Hilbert space (Thm. 3.10) in the other new proof of Runge's Theorem instead of Riesz Representation Theorem 3.2. These two proofs are much easier than the one just given. So we have simplified the proof of Runge's Theorem. The proof is as follows.

4.2 Runge's Theorem

The same hypothesis as Theorem 4.1.

Note that $S^2 - K$ has at most countably many components.

Note also that the preassigned point in the unbounded component of $S^2 - K$ may very well be ∞ .

Proof : Consider the Banach space $C(K)$ whose members are the continuous complex functions on K, with the supremum norm.

Let M be the subspace of $C(K)$ which consists of the restriction to K of those rational function which have all their poles in $\{\alpha_j\}$.

Then the theorem will be proved if we can show that $f \in \overline{M}$.

Suppose on the contrary that $f \notin \overline{M}$.

By Theorem 2.55 there exists a bounded linear functional Λ on $C(K)$ such that $\Lambda f \neq 0$ but $\Lambda R = 0 \quad \forall R \in M$.

Given $z \in S^2 - K$, let u_z be the function defined by

$$u_z(\xi) = \frac{1}{\xi - z} \qquad (\xi \in K).$$

Since $z \notin K$, then $u_z \in C(K)$.

Define $h(z) = \Lambda u_z \qquad (z \in S^2 - K)$.

Claim that $h \in H(S^2 - K)$.

Given $z_o \in S^2 - K$, choose r with $0 < r < \text{dist.}(z_o, K)$.

If $|z - z_o| < r$, then $\left|\frac{z - z_o}{\xi - z_o}\right| < \frac{r}{\text{dist}(z_o, K)} < 1$

for all $\xi \in K$ and every $z \in D(z_o, r)$

$$u_z(\xi) = \frac{1}{\xi - z} = \frac{1}{(\xi - z_o) - (z - z_o)}$$

$$= \frac{1}{\xi - z_o}\left[\frac{1}{1 - \frac{z - z_o}{\xi - z_o}}\right]$$

$$= \frac{1}{\xi - z_o}\left[1 + \frac{z - z_o}{\xi - z_o} + \left(\frac{z - z_o}{\xi - z_o}\right)^2 + \dots\right]$$

$$= \sum_{n=0}^{\infty} \frac{(z - z_o)^n}{(\xi - z_o)^{n+1}}.$$

Since $\left|\frac{z - z_o}{\xi - z_o}\right| < 1$ for every $\xi \in K$ and every $z \in D(z_o, r)$, the geometric series $\sum_{n=0}^{\infty} \frac{(z - z_o)^n}{(\xi - z_o)^{n+1}}$ converges uniformly on K and for every fixed $z \in D(z_o, r)$.

$$u_z(\xi) = u_{z_o}(\xi) + u_{z_o}^2(\xi)(z - z_o) + u_{z_o}^3(\xi)(z - z_o)^2 + \dots$$

$$= \lim_{n \to \infty} \sum_{n=0}^{N} u_{z_o}^{n+1}(\xi)(z - z_o)^n \quad \text{uniformly}$$

$$\forall\, \xi \in K.$$

$$\lim_{N \to \infty} \left| \sum_{n=0}^{N} u_{z_o}^{n+1} (z - z_o)^n - u_z \right| = 0.$$

Since Λ is a bounded linear functional on $C(K)$,

$$\lim_{N \to \infty} \left| \Lambda \left(\sum_{n=0}^{N} u_{z_o}^{n+1} (z - z_o)^n \right) - \Lambda u_z \right| = 0$$

$$\lim_{N \to \infty} \left| \sum_{n=0}^{N} u_{z_o}^{n+1} (z - z_o)^n - h(z) \right| = 0 \quad \forall z \in D(z_o, r)$$

Hence $h(z) = \Lambda u_{z_o} + \Lambda u_{z_o}^2 (z - z_o) + \Lambda u_{z_o}^3 (z - z_o)^2 + \dots$.

This shows that h is representable by power series in $S^2 - K$.

Then by Theorem 2.79 $h \in H(S^2 - K)$.

Let V_j be the component of $S^2 - K$ which contains α_j.

<u>Case 1</u> If $\alpha_j \neq \infty$.

Let $\rho > 0$, suppose $D(\alpha_j, \rho) \subset V_j$ and z is fixed in $D(\alpha_j, \rho)$

$$\frac{1}{\xi - z} = \sum_{n=0}^{\infty} \frac{(z - \alpha_j)^n}{(\xi - \alpha_j)^{n+1}} \qquad \text{(see page 76)}$$

Since $\left| \frac{z - \alpha_j}{\xi - \alpha_j} \right| < \frac{|z - \alpha_j|}{\rho} < \frac{\rho}{\rho} = 1$ for every $\xi \in K$

and every $z \in D(\alpha_j, \rho)$,

then the geometric series $\sum_{n=0}^{\infty} \frac{(z - \alpha_j)^n}{(\xi - \alpha_j)^{n+1}}$ converges uniformly on K.

$$\lim_{N \to \infty} \left| \frac{1}{\xi - z} - \sum_{n=0}^{N} \frac{(z - \alpha_j)^n}{(\xi - \alpha_j)^{n+1}} \right| = 0$$

$$\lim_{N \to \infty} | u_z - Q_n | = 0 \quad \text{where } Q_N(\xi) = \sum_{n=0}^{N} \frac{(z - \alpha_j)^n}{(\xi - \alpha_j)^{n+1}}$$

Note that $Q_N(\xi) \in M$.

By the continuity of Λ

$$\lim_{N \to \infty} | \Lambda u_z - \Lambda Q_N | = 0$$

$$\lim_{N \to \infty} | h(z) - \Lambda Q_N | = 0$$

$$h(z) = \lim_{N \to \infty} \Lambda Q_N$$

$$= 0 \quad (\text{for } z \in D(\alpha_j, \rho)).$$

By Theorem 2.88 $h \equiv 0$ in V_j.

<u>Case 2</u> If $\alpha_j = \infty$.

Let D_{ρ_1} be the set of all complex numbers $\xi \ni |\xi| \leq \rho_1$ and $D_{\rho_1} \supset K$. Let D'_{ρ_1} be the set of all complex numbers $z \ni |z| > \rho_1$.

$$\frac{1}{\xi - z} = - \sum_{n=0}^{\infty} \frac{\xi^n}{z^{n+1}}$$ (see page 71).

Since $\left|\frac{\xi}{z}\right| < \frac{|\xi|}{\rho_1} \leq \frac{\rho_1}{\rho_1} = 1$ for every $\xi \in K$ and every z in

D'_{ρ_1}, the geometric series $-\sum_{n=0}^{\infty} \frac{\xi^n}{z^{n+1}}$ converges uniformly on K.

$\lim_{N \to \infty} |u_z - P_N| = 0$ where $P_N = -\sum_{n=0}^{N} \frac{\xi^n}{z^{n+1}}$ and $P_N \in M$.

By the continuity of Λ

$$\lim_{N \to \infty} |\Lambda u_z - \Lambda P_N| = 0$$

$$\lim_{N \to \infty} |h(z) - \Lambda P_N| = 0$$

$$\text{Thus } h(z) = \lim_{N \to \infty} \Lambda P_N$$

$$= 0 \qquad \forall z \in D'_{\rho_1}.$$

Similarly as before by Theorem 2.88, $h(z) \equiv 0$ in V_j.

We conclude that

$$\Lambda u_z = h(z) = 0 \qquad \forall z \text{ in } S^2 - K.$$

Now choose the oriented line intervals $\gamma_1, \ldots \gamma_n$ in $\Omega - K$ as in Theorem 2.93 such that the Cauchy formula $f(\xi) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(w)}{w - \xi} dw$ holds for every $f \in H(\Omega)$ and every $\xi \in K$.

This together with Theorem 3.3 gives that for any $\varepsilon > 0$ we can find complex numbers $\beta_1, \ldots \beta_m$ and points $z_1, \ldots, z_m$ in $\bigcup_{j=1}^{n} \gamma_j^*$

such that

$$|f(\xi) - \sum_{k=1}^{m} \frac{\beta_k}{z_k - \xi}| < \varepsilon' \quad \forall\, \xi \in K .$$

$$|f(\xi) - \sum_{k=1}^{m} \frac{-\beta_k}{\xi - z_k}| < \varepsilon'$$

$$|f(\xi) - \sum_{k=1}^{m} (-\beta_k)\, u_{z_k}| < \varepsilon' .$$

Since $z_k \in \bigcup_{j=1}^{n} \gamma_j^*$, $\quad z_k \in S^2 - K$,

$$h(z_k) = 0 .$$

i.e. $\Lambda u_{z_k} = 0 \quad \forall\, k = 1, \ldots m$.

Consider
$$|\Lambda f| = |\Lambda f - \Lambda\left(\sum_{k=1}^{m} (-\beta_k)\, u_{z_k}\right)|$$

$$= |\Lambda f - \left(\sum_{k=1}^{m} (-\beta_k)\, u_{z_k}\right)|$$

$$\leq ||\Lambda||\,||f - \sum_{k=1}^{m} (-\beta_k)\, u_{z_k}||$$

$$\leq ||\Lambda||\, \varepsilon' .$$

Since $\varepsilon' > 0$ is arbitrary;, then $|\Lambda f| = 0$.

Therefore $\Lambda f = 0$.

But we have $\Lambda f \neq 0$.

Thus $f \in \overline{M}$.

i.e. for every $\varepsilon > 0$, there exists an $R \in M \ni$

$$|f(\xi) - R(\xi)| < \varepsilon \qquad \forall \xi \in K.$$

This proof is different from the proof of Theorem 4.1. We do not use the Riesz Representation Theorem 3.2. Besides those theorems which we used in Theorem 4.1, we only use Theorem 3.3. Instead of using the difficult Riesz Representation Theorem 3.2, we use Theorem 3.3 which we can see is much easier.

The next method of proof of Runge's Theorem is a specail case and similar to the first proof in Theorem 4.1 but we do not use the Theorem 3.2. Before we show the proof, we will discuss the space L^2.

4.3 <u>Definition</u> If f is a complex measurable function on X, define $||f||_2 = \{ \int_X |f|^2 \, d\mu \}^{\frac{1}{2}}$ where X is an arbitrary measure space and μ is a positive measure, and let $L^2(\mu)$ consist of all f for which $||f||_2 < \infty$.

4.4 <u>Remark</u> If $f \in L^2(\mu)$ and α is a complex number then $\alpha f \in L^2(\mu)$ and $|| \alpha f||_2 = |\alpha| \, ||f||_2$.

$f \in L^2(\mu)$, $g \in L^2(\mu)$ then $f + g \in L^2(\mu)$ and

$$||f + g||_2 \leq ||f||_2 + ||g||_2 \qquad \text{.....................(1)}$$

this shows that $L^2(\mu)$ is a complex vector space.

Suppose f, g and h are in $L^2(\mu)$. Replacing f by f - g and g by g - h in (1) , we obtain

$$||f - h||_2 \leqslant ||f - g||_2 + ||g - h||_2 \quad \ldots\ldots\ldots\ldots\ldots(2)$$

This suggests that a metric may be introduced in $L^2(\mu)$ by defining the distance between f and g to be $||f - g||_2$. Call this distance $d(f,g)$. Then $0 \leqslant d(f,g) < \infty$, $d(f,f) = 0$, $d(f,g) = d(g,f)$ and (2) shows that the triangle inequality $d(f,h) \leqslant d(f,g) + d(g,h)$ is satisfied. The only other property which d should have to define a metric space is that $d(f,g) = 0$ should imply that $f = g$. In our present situation this need not be so ; we have $d(f,g) = 0$ precisely when $f(x) = g(x)$ for almost all x.

Let us write $f \sim g$ if and only if $d(f,g) = 0$. This is an equivalence relation which partition $L^2(\mu)$ into equivalence classes ; each class consists of all functions which are equivalent to each other. If F and G are two equivalence classes, choose $f \in F$ and $g \in G$ and define

$d(F,G) = d(f,g)$; note that $f \sim f_1$ and $g \sim g_1$ implies $d(f,g) = d(f_1, g_1)$, so that $d(F, G)$ is well defined. With this definition, the set of equivalence classes is now a metric space. Note that it is also a vector space, since $f \sim f_1$ and $g \sim g_1$ implies $f + g \sim f_1 + g_1$ and $\alpha f \sim \alpha f_1$.

When $L^2(\mu)$ is regarded as a metric space, then the space which is really under consideration is therefore not a space whose elements are functions, but a space whose elements are equivalence classes of functions. Then we continue to speak of $L^2(\mu)$ as a space of functions.

If $\{f_n\}_{n \in N}$ is a sequence in $L^2(\mu)$, if $f \in L^2(\mu)$, and $\lim_{n \to \infty} ||f_n - f||_2 = 0$, we say that $\{f_n\}$ converges to f in $L^2(\mu)$ (or that $\{f_n\}$ is L^2- convergent to f).

If to every $\varepsilon > 0$ there corresponds an integer N such that $||f_n - f_m||_2 < \varepsilon$ as soon as $n > N$, $m > N$ and $\varepsilon > 0$, we call $\{f_n\}$ a Cauchy sequence in $L^2(\mu)$. These definitions are exactly as in any metric space. By Theorem 2.48, we have $L^2(\mu)$ is a complete metric space. Then $L^2(\mu)$ is a Hilbert space. We define the inner product $(f,g) = \int_X f \overline{g} \, d\mu$ where $f, g \in L^2(\mu)$ and $\overline{g}$ is the complex conjugate of g.

So far we have considered $L^2(\mu)$ on any measure space. Now let X be a locally compact Hausdorff space and let μ be a measure on a δ - algebra m in X, with the properties stated in Theorem 3.1.

4.5 <u>Lemma</u> If $f : K \to C$ where K is a compact subset of X and f is measurable, belongs to $L^2(\mu)$. Then $f \in L^1(\mu)$.

<u>Proof</u> : Consider $(|f| - |g|)^2 \geq 0$ where f and $g \in L^2(\mu)$,

$$|f|^2 - 2|f||g| + |g|^2 \geq 0$$

$$|f||g| \leq \frac{1}{2}(|f|^2 + |g|^2).$$

Then $$\int_K |f||g|\, d\mu \leq \frac{1}{2}\int_K |f|^2 d\mu + \frac{1}{2}\int_K |g|^2 d\mu .$$

The two integrals on the right exist so that the integral on the left exist

Thus $|f||g| \in L^1(\mu)$.

Let $|g| = 1 \in L^2(\mu)$.

Hence $f \in L^1(\mu)$.

4.6 <u>Lemma</u> Suppose u is an analytic function in a region Ω [illegible] Then

$$u(z_o) = \frac{1}{\pi r^2}\iint_{D(z_o,r)} u\, dx\, dy \quad \text{where } \overline{D}(z_o,r) \subset \Omega \text{ and } z_o \in K.$$

<u>Proof</u> : By Cauchy's Integral Formula,

$$u(z_o) = \frac{1}{2\pi i}\int_C \frac{u(\xi)}{\xi - z_o}\, d\xi$$ where C is a closed path in an open set Ω and $z_o \in K$ is arbitrary .

Suppose $D(z_o,\rho) \subset D(z_o,r)$, then

$$u(z_o) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{u(z_o + \rho e^{i\theta})\, i\rho\, e^{i\theta}}{\rho\, e^{i\theta}}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u(z_o + \rho\, e^{i\theta})\, d\theta .$$

$$\text{Since } \frac{1}{\pi r^2} \iint_{D(z_o,\rho)} u(\xi)\,dxdy = \frac{1}{\pi r^2} \iint_D u(z_o + \rho e^{i\theta})\rho d\rho d\theta$$

$$= \frac{1}{\pi r^2} \int_0^r \rho\, d\rho \int_0^{2\pi} u(z_o + \rho e^{i\theta})\, d\theta$$

$$= \frac{1}{\pi r^2} \cdot \frac{r^2}{2} \cdot 2\pi\, u(z_o)$$

$$= u(z_o).$$

4.7 A Special Case Of Runge's Theorem.

Suppose K is a compact set in the extended complex plane S^2 and $\{\alpha_j\}$ is a set which contains one point in each component of $S^2 - K$. If Ω is open, $\Omega \supset K$, $f \in H(\Omega)$, and $\varepsilon > 0$, there exists a rational function R, all of whose poles lie in the prescribed set $\{\alpha_j\}$, such that $|f(\xi) - R(\xi)| < \varepsilon$ for every $\xi \in K$.

Assume $S^2 - K$ has at most countably many components. Note also that the preassigned point in the unbounded component of $S^2 - K$ may very well be ∞.

Proof : [illegible]

[illegible]

$\{\alpha_j\}$ is the set which contains one point in each component of $S^2 - K$.

Let $r_j = \underset{j=1,..n}{\text{dist}} (\alpha_j, K) > 0$

$s = \text{dist}(K, S^2 - \Omega) > 0.$

Then take $\delta = \min \{ \frac{r_j}{2}, \frac{s}{2} \}$.

Let $K_\delta = \{z \mid \text{dist}(z,K) \leq \delta \}$.

Then K_δ is a compact set which does not contain $\alpha_j \ \forall j$ and

$K \subset K_\delta \subset \Omega$.

Consider the Hilbert space $L^2(K_\delta)$ whose members are complex measurable functions on K_δ with an integrable square.

i.e. $L^2(K_\delta) = \{ f \mid f: K_\delta \to \mathbb{C}$, f is measurable and

$$\iint_{K_\delta} |f|^2 \, dxdy < \infty \},$$

Let H be the subspace of $L^2(K_\delta)$ which consists of the restriction to K_δ of those rational functions which have all their poles in $\{\alpha_j\}$.

Since R is a rational function which has all its poles in

$\{\alpha_j\} \subset S^2 - K_\delta$ then R is continuous on K_δ .

R is continuous on the compact set K_δ implies that R is bounded and is also integrable.

i.e. R is bounded and $R \in L^1(K_\delta)$, also since R^2 is bounded ,
$R \in L^2(K_\delta)$.

Also since the given $f \in H(\Omega)$, f is continuous on K_δ $f \in L^2(K_\delta)$.

We first show that $f \in \overline{H}$ (closure with respect to L^2-norm).

By Theorem 2.53 $f \in \overline{H}$ if and only if for all bounded linear functionals Λ on $L^2(K_\delta)$ such that

$$\Lambda(R) = 0 \quad \forall R \in H \Rightarrow \Lambda(f) = 0.$$

Theorem 3.9 shows that we must prove that if $f_o \in L^2(K_\delta)$ s.t $(R, f_o) = 0$ for every rational function R with poles only in the set $\{\alpha_j\}$ then $(f, f_o) = 0$.

Define $$h(z) = \iint_{K_\delta} \frac{\overline{f}_o(\xi)}{\xi - z}\, dxdy \quad (\xi = x + iy \in K_\delta ,\ z \in S^2 - K_\delta)$$

We will show that $h(z)$ is analytic in $S^2 - K_\delta$.

Suppose $D(a, 2\rho_1) \subset S^2 - K_\delta$ where $\rho_1 > 0$ and $a \in S^2 - K_\delta$.

Fix z in $D(a, \rho_1)$

$$\frac{1}{\xi - z} = \sum_{n=0}^{\infty} \frac{(z - a)^n}{(\xi - a)^{n+1}}$$ (see page 76)

Since $\left| \frac{z - a}{\xi - a} \right| \leq \frac{|z - a|}{2\rho_1} < \frac{\rho_1}{2\rho_1} = \frac{1}{2}$ for every $z \in D(a, \rho_1)$ and every $\xi \in K_\delta$,

the geometric serie $\sum_{n=0}^{\infty} \frac{(z - a)^n}{(\xi - a)^{n+1}}$ converges uniformly on K_δ.

$$h(z) = \iint_{K_\delta} \frac{\overline{f}_o(\xi)}{\xi - z}\, dxdy = \iint_{K_\delta} \sum_{n=0}^{\infty} \frac{(z - a)^n}{(\xi - a)^{n+1}} .\overline{f}_o(\xi) dxdy.$$

Let $$d\varphi = \overline{f}_o(\xi)\, dxdy . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$h(z) = \iint_{K_\delta} \sum_{n=0}^{\infty} \frac{(z - a)^n}{(\xi - a)^{n+1}}\, d\varphi$$

$$= \sum_{n=0}^{\infty} (z - a)^n \int_{K_\delta} \frac{d\varphi}{(\xi - a)^{n+1}} \quad \text{(by Theorem 2.30)}$$

$$= \sum_{n=0}^{\infty} (z - a)^n \iint_{K_\delta} \frac{\bar{f}_o(\xi)}{(\xi - a)^{n+1}} \, dx \, dy \; .$$

Consider $$|c_n| = \left| \iint_{K_\delta} \frac{\bar{f}_o(\xi)}{(\xi - a)^{n+1}} \, dx \, dy \right|$$

$$\leq \iint_{K_\delta} \frac{|\bar{f}_o(\xi)|}{|\xi - a|^{n+1}} \, dx \, dy$$

$$< \iint_{K_\delta} \frac{|\bar{f}_o(\xi)|}{(2\rho_1)^{n+1}} \, dx \, dy$$

$$= \frac{1}{(2\rho_1)^{n+1}} \iint_{K_\delta} |\bar{f}_o(\xi)| \, dx \, dy \, .$$

Since $\iint_{K_\delta} |\bar{f}_o(\xi)| \, dx \, dy = M < \infty$, then $|c_n| < \infty$.

$$h(z) = \sum_{n=0}^{\infty} c_n \, (z - a)^n .$$

Claim that $\sum_{n=0}^{\infty} c_n \, (z - a)^n$ converges.

$$c_n(z - a)^n = (z - a)^n \iint_{K_\delta} \frac{\bar{f}_o(\xi)}{(\xi - a)^{n+1}} \, dx \, dy$$

$$|c_n \, (z - a)^n| \leq |z - a|^n \iint_{K_\delta} \frac{|\bar{f}_o(\xi)|}{|\xi - a|^{n+1}} \, dx \, dy$$

$$< \rho_1^n \iint_{K_\delta} \frac{|\bar{f}_o(\xi)|}{|\xi - a|^{n+1}} \, dx \, dy$$

$$< \frac{\rho_1^n}{(2\rho_1)^{n+1}} \iint_{K_\delta} |\bar{f}_o(\xi)| \, dx \, dy$$

$$= \frac{M}{2^{n+1}\rho_1} \; .$$

Since the geometric series $\sum_{n=0}^{\infty} \frac{M}{\rho_1 2^{n+1}}$ converges ,

$\sum_{n=0}^{\infty} c_n \; (z - a)^n$ converges by Weierstrass M. Test i.e. h(z) can be representable by a power series in $S^2 - K_\delta$.

By Theorem 2.79 $h \in H(S^2 - K_\delta)$ . ...(2)

Claim that $h(z) \equiv 0 \quad \forall z \in S^2 - K_\delta$ . ...(3)

Let V_j be the component of $S^2 - K_\delta$ which contain α_j .

<u>Case 1</u> If $\alpha_j \neq \infty$.

Choose $\rho_2 > 0$ such that $D(\alpha_j, \rho_2) \subset V_j$.

If z is fixed in $D(\alpha_j , \rho_2)$ then for any $\xi \in K_\delta$, we consider

$\frac{1}{\xi - z} = \sum_{n=0}^{\infty} \frac{(z - \alpha_j)^n}{(\xi - \alpha_j)^{n+1}}$ (see page 76) which converges uniformly since $\left| \frac{z - \alpha_j}{\xi - \alpha_j} \right| < 1 \quad \forall \xi \in K_\delta$ and for every $z \in D(\alpha_j , \rho_2)$.

$$h(z) = \iint_{K_\delta} \frac{\overline{f}_o(\xi)}{\xi - z} \, dx \, dy$$

$$= \iint_{K_\delta} \sum_{n=0}^{\infty} \frac{(z - \alpha_j)^n}{(\xi - \alpha_j)^{n+1}} \, \overline{f}_o(\xi) \, dx \, dy \; .$$

From (1)

$$h(z) = \int_{K_\delta} \sum_{n=0}^{\infty} \frac{(z - \alpha_j)^n}{(\xi - \alpha_j)^{n+1}} \, d\varphi$$

$$= \sum_{n=0}^{\infty} (z - \alpha_j)^n \iint_{K_\delta} \frac{d\varphi}{(\xi - \alpha_j)^{n+1}} \quad \text{(by Theorem 2.30)}$$

$$= \sum_{n=0}^{\infty} (z - \alpha_j)^n \iint_{K_\delta} \frac{\overline{f}_o(\xi)}{(\xi - \alpha_j)^{n+1}} \, dx\, dy$$

$$= 0 \quad \forall\ z \text{ in } D(\alpha_j\ ,\ \rho_2) \quad (\text{Since } (R,\ f_o) = 0 \quad \forall\ R\ \varepsilon\ H).$$

By Theorem 2.88 $h(z) = 0 \quad \forall\ z\ \varepsilon\ V_j$.

<u>Case 2</u> $\alpha_j = \infty$.

Let $\rho_3 > 0$ and D_{ρ_3} be the set of all complex number ξ such that

$|\xi| \leq \rho_3$ and $D_{\rho_3} \supset K_\delta$.

Let D'_{ρ_3} be the set of all complex number z such that $|z| > \rho_3$.

Since $\frac{1}{\xi - z} = - \sum_{n=0}^{\infty} \frac{\xi^n}{z^{n+1}}$ (see page 71)

$$\left|\frac{\xi}{z}\right| < \frac{|\xi|}{\rho_3} \leq \frac{\rho_3}{\rho_3} = 1$$ for every ξ in D_{ρ_3} and

every z in D'_{ρ_3} ,

Then the geometric series $- \sum_{n=0}^{\infty} \frac{\xi^n}{z^{n+1}}$ converges uniformly

on K_δ for $\xi\ \varepsilon\ D_{\rho_3}$.

$$h(z) = \iint_{K_\delta} \frac{\overline{f}_o(\xi)}{\xi - z}\, dx\, dy = - \int_{K_\delta} \sum_{n=0}^{\infty} \frac{\xi^n}{z^{n+1}}\, d\varphi$$

$$= - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \int_{K_\delta} \xi^n\, d\varphi \text{ (by Theorem 2.30)}$$

$$= - \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \iint_{K_\delta} \xi^n\, \overline{f}_o(\xi)\, dx\, dy$$

$$= 0 \qquad \forall\, z \in D'_{\rho_3} .$$

Again by Theorem 2.88 $h(z) = 0 \quad \forall\, z \in V_j$.

Now choose the oriented line intervals $\gamma_1 , \dots \gamma_n$ in $\Omega - K_\delta$ as in Theorem 2.93 such that the Cauchy formula

$$f(\xi) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(w)}{w - \xi}\, dw \text{ holds.}$$

$$(f, f_o) = \iint_{K_\delta} f(\xi)\, \overline{f}(\xi)\, dx\, dy$$

$$= \iint_{K_\delta} \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(w)}{w - \xi}\, dw \,.\, \overline{f}_o(\xi)\, dx\, dy$$

$$= \sum_{j=1}^{n} \frac{1}{2\pi i} \iint_{K_\delta} \int_{\gamma_j} \frac{f(w)}{w - \xi} \,.\, \overline{f}_o(\xi)\, dw\, dx\, dy ,$$

We will use the application of Fubini's Theorem 2.74. First we show that

$$\iint_{K_\delta} \int_{\gamma_j} \left| \frac{f(w)}{w - \xi} \right| \left| \overline{f}_o(\xi) \right| |dw|\, dx\, dy < \infty .$$

Since γ_j is compact, $f \in H(\Omega)$,

$$\exists \ \beta > 0 \ \ni \ |f(w)| \leq \beta \ \forall \ w \in \bigcup_{j=1}^{n} \gamma_j^*.$$

$$\xi \in K_\delta \quad \text{and} \quad \gamma_j \cap K = \phi \ \forall_j$$

$$\exists \text{ a real number } c > 0 \ \ni \ |w - \xi| \geq c \quad \forall \ w \in \bigcup_{j=1}^{n} \gamma_j^*.$$

$$\iint_{K_\delta} \int_{\gamma_j} \left| \frac{f(w)}{w - \xi} \right| \ |\bar{f}_o(\xi)| \ |dw| \ dx \ dy$$

$$\leq \frac{\beta}{c} \iint_{K_\delta} \int_{\gamma_j} |\bar{f}_o(\xi)| \ |dw| \ dx \ dy$$

$$= \frac{\beta}{c} \iint_{K_\delta} |\bar{f}_o(\xi)| \int_{\gamma_j} |dw| \ dx \ dy$$

$$= \frac{\beta}{c} L_j \iint_{K_\delta} |\bar{f}_o(\xi)| \ dx \ dy \quad (\text{where } L_j = \text{length of } \gamma_j)$$

$$< \infty .$$

Then $$(f, f_o) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} dw \iint_{K_\delta} \frac{f(w)}{w - \xi} \bar{f}_o(\xi) \ dx \ dy$$

$$= \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} f(w) \ dw \iint_{K_\delta} \frac{\bar{f}_o(\xi)}{w - \xi} \ dx \ dy$$

$$= - \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} f(w) \ h(w) \ dx \ dy$$

$= 0$ since each γ_j is an interval in $S^2 - K_\delta$ where h vanishes.

$$\text{i.e.} \quad f \in \bar{H} . \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Then for any $\varepsilon_1 > 0$, $\exists$ $R \in H$ such that

$$||f - R||_2 < \varepsilon_1 .$$

$$\text{i.e.} \quad \sqrt{\iint_{K_\delta} |f - R|^2 \, dx\, dy} < \varepsilon_1 .$$

Let $0<\rho_4 \leq \delta$ then $D(z_o , \rho_4) \subset K_\delta$. For every $z_o \in K$.

By lemma 4.6

$$f(z_o) = \frac{1}{\pi\rho_4^2} \iint_{D(z_o,\rho_4)} f(\xi)\, dx\, dy$$

$$|f(z_o)| = \frac{1}{\pi\rho_4^2} \left| \iint_D f(\xi)\, dx\, dy \right|$$

$$\leq \frac{1}{\pi \rho_4^2} \sqrt{\iint_D |f(\xi)|^2 \, dx\, dy} \sqrt{\iint_D 1^2 \, dx\, dy}$$

$$\leq \frac{1}{\pi^2\rho_4^4} \sqrt{\iint_D |f(\xi)|^2 \, dx\, dy} \sqrt{\pi\rho_4^2}$$

$$\leq \frac{1}{\sqrt{\pi \rho_4^2}} \sqrt{\iint_{K_\delta} |f(\xi)|^2 \, dx\, dy}$$

$$\text{i.e} \quad |f(z_o)| \leq \frac{1}{\sqrt{\pi\rho_4^2}} \sqrt{\iint_{K_\delta} |f(\xi)|^2 \, dx\, dy}$$

Apply the above inequality to $f - R'$, then

$$|f(z_o) - R(z_o)| \leq \frac{1}{\sqrt{\pi\rho_4^2}} \sqrt{\iint_{K_\delta} |f(\xi) - R(\xi)|^2 \, dx \, dy}$$

$$< \frac{\varepsilon_1}{\sqrt{\pi \rho_4^2}} \cdot$$

Then given $\varepsilon > 0$ and choose ε_1 be such that $\frac{\varepsilon_1}{\sqrt{\pi\rho_4^2}} < \varepsilon$.

i.e. $|f(\xi) - R(\xi)| < \varepsilon$ for every $\xi \in K$.

Q.E.D.