CHAPTER III

RIESZ REPRESENTATION THEOREM

In this chapter we are going to discuss the Riesz Representation Theorem which we use to prove the Runge's Theorem. To do this we construct another theorem which is used in proving the Riesz Representation Theorem for bounded linear functionals on the Banach Space of continuous functions on a compact set. Next we study the bounded linear functionals on a Hilbert space and the Riemann sum of the integral. We will then give the proofs of the two Riesz Representation theorems. In order to obtain Runge's Theorem we can use each of these three theorems separately in three different proofs. We will now develop the theorems.

3.1 <u>Theorem</u> Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(X)$. Then there exists a σ - algebra $\mathfrak{m}$ in X which contains all Borel sets in X , and there exists a unique positive measure μ on $\mathfrak{m}$ which represents Λ in the sense that

(a) $\Lambda f = \int_X f \, d\mu$ for every $f \in C_c(X)$ and which has

the following additional properties :

(b) $\mu(K) < \infty$ for every compact set $K \subset X$.

(c) For every $E \in \mathfrak{m}$, we have

$$\mu(E) = \inf \{ \mu(V) : E \subset V , V \text{ open} \}$$

(d) The relation

$$\mu(E) = \sup \{\mu(K) : K \subset E , K \text{ compact}\}$$

holds for every open set E, and for every $E \in m$ with $\mu(E) < \infty$.

(e) If $E \in m$, $A \subset E$ and $\mu(E) = 0$ then $A \in m$.

Note that throughout the proof of this theorem, the letter K will stand for a compact subset of X and V will denote an open set in X.

Proof : We begin with proving the uniqueness of μ.

If μ satisfies (c) and (d), μ is determined on m by its values on compact sets.

It suffices to prove that $\mu_1(K) = \mu_2(K)$ for all K, whenever μ_1 and μ_2 are measures for which the theorem holds. So, fix K and $\varepsilon > 0$.

By (b) and (c), there exists a $V \supset K$ with

$$\mu_2(K) \leq \mu_2(V)$$

$$\mu_2(V) < \mu_2(K) + \varepsilon .$$

By Urysohn's Lemma 2.38, there exists an $f \in C_c(X)$ such that $K \prec f \prec V$; hence

$$\mu_1(K) = \int_X \chi_K \, d\mu_1 \leq \int_X f \, d\mu_1 = \Lambda f = \int_X f \, d\mu_2$$

$$\leq \int_X \chi_V \, d\mu_2 = \mu_2(V) < \mu_2(K) + \varepsilon$$

Thus $\mu_1(K) \leq \mu_2(K)$.

If we interchange the roles of μ_1 and μ_2 , the opposite inequality is obtained. Then $\mu_1(K) = \mu_2(K)$. Then uniqueness of μ is proved.

<u>Construction of μ and m</u>

For every open set V in X , define

$$\mu(V) = \sup \{\Lambda f \mid f \prec V\}. \qquad \text{.................(1)}$$

If $V_1 \subset V_2$, from (1)

$$\mu(V_1) = \sup \{\Lambda f \mid f \prec V_1\}$$

$$\mu(V_2) = \sup \{\Lambda f \mid f \prec V_2\}.$$

Since $\{\Lambda f \mid f \prec V_1\} \subseteq \{\Lambda f \mid f \prec V_2\}$, then

$$\sup \{\Lambda f \mid f \prec V_1\} \leq \sup \{\Lambda f \mid f \prec V_2\}.$$

$$\text{i.e.} \quad \mu(V_1) \leq \mu(V_2).$$

Hence, if E is open and $E \subset V$, $\mu(E) \leq \mu(V)$.

If $E \subset X$ is arbitrary , we shall define

$$\mu(E) = \inf \{\mu(V) \mid E \subset V , V \text{ open}\}. \qquad \text{............(2)}$$

Note that although we have defined $\mu(E)$ for every $E \subset X$, the countable additivity of μ will be proved only on a certain σ - algebra m in X.

Let m_F be the class of all $E \subset X$ which satisfy two conditions :

(i) $\mu(E) < \infty$

(ii) $\mu(E) = \sup \{ \mu(K) : K \subset E , K \text{ compact}\}$ .....(3)

Finally, let m be the class of all $E \subset X$ such that $E \cap K \in m_F$ for every compact K.

Proof that μ and m have the required properties

If $A \subset B$ then $\mu(A) \leq \mu(B)$. Thus μ is monotone.

If $\mu(E) = 0$ then $\mu(E) < \infty$ and

$$\mu(E) = \sup \{ \mu(K) : K \subset E , K \text{ compact} \} .$$

Hence $E \in m_F$.

Since $E \cap K \subset E$ implies $\mu(E \cap K) \leq \mu(E) = 0$

and $\mu(E \cap K) = 0$, $E \cap K \in m_F$.

Hence $E \in m$.

If $E \in m$, $A \subset E$ and $\mu(E) = 0$

$$\mu(A) \leq \mu(E) = 0$$

$$\mu(A) = 0 \Rightarrow A \in m_F \Rightarrow A \in m .$$

Thus (e) holds and so does (c) , by definition.

Since the proof of the other assertions is rather long, it will be convenient to devide it into several steps. The positivity of Λ implies that Λ is monotone.

$$f \leq g \text{ implies } \Lambda f \leq \Lambda g \quad \text{since}$$

$$\Lambda g = \Lambda f + \Lambda (g - f) \quad \text{and} \quad g - f \geq 0.$$

The monotonicity will be used in Steps II and X .

<u>Step I</u> If E_1 , E_2 , ... are arbitrary subsets of X , then

$$\mu \left(\bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} \mu (E_i) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

<u>Proof</u> : We first show that

$$\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

if V_1 and V_2 are open.

Choose $g \prec V_1 \cup V_2$.

By Theorem 2.38 there are functions h_1 and h_2 such that $h_i \prec V_i$ $(i = 1,2)$ and $h_1(x) + h_2(x) = 1$ for all x in the support of g.

Hence $h_i g \prec V_i$, $g = h_1 g + h_2 g$ and so

$$\Lambda g = \Lambda(h_1 g) + \Lambda(h_2 g) \leq \mu(V_1) + \mu(V_2) \qquad \ldots\ldots\ldots(6)$$

holds for every $g \prec V_1 \cup V_2$.

Since $\mu(V_1 \cup V_2) = \sup \{ \Lambda g \mid g \prec V_1 \cup V_2 \}$,

$$\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2)$$, (5) follows .

If $\mu(E_i) = \infty$ for some i, then (4) is true.

Suppose therefore that $\mu(E_i) < \infty$ for every i.

Choose $\varepsilon > 0$. By (2) there are open sets $V_i \supset E_i$ such that

$$\mu(V_i) < \mu(E_i) + 2^{-i}\varepsilon \qquad (i = 1,2,3, \dots) \text{ ....(7)}$$

Put $V = \bigcup_{i=1}^{\infty} V_i$ and choose $f \prec V$.

Since f has compact support, $f \prec V_1 \cup \dots \cup V_n$ for some n.

Applying induction to (5), we therefore obtain

$$\Lambda f \leq \mu(V_1 \cup \dots \cup V_n) \leq \mu(V_1) + \dots + \mu(V_n)$$

$$\leq \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon .$$

Since this holds for every $f \prec V$, and since $\cup E_i \subset V$, it follows that

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leq \mu(V) \leq \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon \text{ ....(8)}$$

which prove (4), since ε was arbitrary.

Step II $\mathfrak{m}_F$ contains every compact set.

This implies assertion (b) of the theorem.

Proof : If $K \prec f$, let $V = \{ x | f(x) > \frac{1}{2} \}$,

then $K \subset V$ and $g \leq 2f$ whenever $g \prec V$. Hence

$\mu(K) \leq \mu(V) = \sup \{ \Lambda g | g \prec V \} \leq \Lambda 2f < \infty$.

Since K evidently satisfies (3), $K \in \mathfrak{m}_F$.

<u>Step III</u> Every open set satisfies (3). Hence m_F contains every open set V with $\mu(V) < \infty$.

<u>Proof</u> : Let α be a real number such that $\alpha < \mu(V)$. There exists an $f \prec V$ with $\alpha < \Lambda f$.

If W is any open set which contains the support K of f, then $f \prec W$, hence $\Lambda f \leq \mu(W)$.

Thus $\Lambda f \leq \mu(K)$.

This exhibits a compact $K \subset V$ with $\alpha < \mu(K)$, so that (3) holds for V.

<u>Step IV</u> Suppose $E = \bigcup_{i=1}^{\infty} E_i$ where $E_1, E_2, E_3, \ldots$ are pairwise disjoint members of m_F. Then

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

If, in addition, $\mu(E) < \infty$, then also $E \in m_F$.

<u>Proof</u> : We first show that

$$\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2) \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

if K_1 and K_2 are disjoint compact sets.

Choose $\varepsilon > 0$. By Theorem 2.34 (with K_1 in place of K and K_2^c (complement of K_2) in place of U) there are disjoint open sets V_1 and V_2 such that $K_i \subset V_i$. By Step II, there is an open set $W \supset K_1 \cup K_2$ such that $\mu(W) < \mu(K_1 \cup K_2) + \varepsilon$ and there are functions $f_i \prec W \cap V_i$, $\Lambda f_i > \mu(W \cap V_i) - \varepsilon$ for $i = 1,2$.

Since $K_i \subset W \cap V_i$ and $f_1 + f_2 \prec W$ (it is here that $V_1 \cap V_2 = \phi$ is used), we obtain

$$\begin{aligned} \mu(K_1) + \mu(K_2) &\leq \mu(W \cap V_1) + \mu(W \cap V_2) \\ &< \Lambda f_1 + \Lambda f_2 + 2\varepsilon \\ &= \Lambda(f_1 + f_2) + 2\varepsilon \\ &\leq \mu(W) + 2\varepsilon \\ &< \mu(K_1 \cup K_2) + 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, (10) follows from step I.

If $\mu(E) = \infty$, then

from step I $\quad \mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$

$$\infty \leq \sum_{i=1}^{\infty} \mu(E_i).$$

Thus $\sum_{i=1}^{\infty} \mu(E_i) = \infty \Rightarrow \mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$ then (9) follows.

So we can assume that $\mu(E) < \infty$. Choose $\varepsilon > 0$, since $E_i \in \mathcal{m}_F$, there are compact sets $H_i \subset E_i$ such that

$$\mu(H_i) > \mu(E_i) - 2^{-i}\varepsilon \qquad (i = 1,2,3,\ldots). \qquad \ldots\ldots\ldots\ldots(11)$$

Putting $K_n = H_1 \cup H_2, \ldots \cup H_n$ and using induction on (10), we obtain

$$\mu(E) \geq \mu(K_n) = \sum_{i=1}^{n} \mu(H_i) > \sum_{i=1}^{n} \mu(E_i) - \varepsilon \qquad \ldots\ldots\ldots(12)$$

Since (12) holds for every n and every $\varepsilon > 0$, we see that

$$\mu(E) \geq \sum_{i=1}^{\infty} \mu(E_i) .$$

From step I, $\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$.

We have $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$.

But if $\mu(E) < \infty$ and $\varepsilon > 0$, (9) show that

$$\mu(E) \leq \sum_{i=1}^{N} \mu(E_i) + \varepsilon \text{ for some } N \quad \ldots\ldots\ldots\ldots\ldots(13)$$

By (12) , it follows that

$$\mu(E) \leq \mu(K_N) + 2\varepsilon$$

and this shows that E satisfies (3) , hence $E \in \mathcal{M}_F$.

Step V If $E \in \mathcal{M}_F$ and $\varepsilon > 0$, there is a compact K and an open V such that $K \subset E \subset V$ and $\mu(V - K) < \varepsilon$.

Proof : Our definitions show that there exist K and V so that

$$\mu(V) - \frac{\varepsilon}{2} < \mu(E) < \mu(K) + \frac{\varepsilon}{2}$$

Since V - K is open , $V - K \in \mathcal{M}_F$ by step III .

Hence step IV implies that

$$V = K \cup (V - K)$$

$$\mu(V) = \mu(K) + \mu(V - K) < \mu(K) + \varepsilon .$$

Step VI If $A \in m_F$ and $B \in m_F$ then $A - B$, $A \cup B$ and $A \cap B$ belong to m_F .

Proof : If $\varepsilon > 0$, step V shows that there are sets K_i and V_i such that $K_1 \subset A \subset V_1$, $K_2 \subset B \subset V_2$ and $\mu(V_i - K_i) < \varepsilon$ for $i = 1,2$.

Since $A - B \subset V_1 - K_2 \subset (V_1 - K_1) \cup (K_1 - V_2) \cup (V_2 - K_2)$

Step I shows that

$$\mu(A - B) \leq \varepsilon + \mu(K_1 - V_2) + \varepsilon \qquad \text{.....................(14)}$$

Since $K_1 - V_2$ is a compact subset of $A - B$, (14) shows that $A - B$ satisfies (3) , so that $A - B \in m_F$.

Since $A \cup B = (A - B) \cup B$, it follows by step IV that $A \cup B \in m_F$.

Since $A \cap B = A - (A - B)$, then $A \cap B \in m_F$.

Step VII m is a σ - algebra in X which contains all Borel sets.

Proof : Let K be an arbitrary compact set in X. If $A \in m$, then $A^c \cap K = K - (A \cap K)$ so that $A^c \cap K$ is a difference of two members of m_F . Hence $A^c \cap K \in m_F$, and we conclude : $A \in m$ implies $A^c \in m$. Next, suppose $A = \bigcup_{i=1}^{\infty} A_i$, where each $A_i \in m$.

Put $B_1 = A_1 \cap K$, and $B_n = (A_n \cap K) - (B_1 \cup \ldots \cup B_{n-1})$.....(15)

$(n = 2,3,4, \ldots)$.

Then $\{B_n\}$ is a disjoint sequence of members of m_F, by step VI and

$$A \cap K = \bigcup_{n=1}^{\infty} B_n .$$

It follows from step IV that $A \cap K \in m_F$.

Hence $A \in m$.

Finally, if C is closed, then $C \cap K$ is compact, hence $C \cap K \in m_F$ or $C \in m$. In particular, $X \in m$.

We have thus proved that m is a σ - algebra in X which contains all closed subsets of X. Hence m contains all Borel sets in X.

<u>Step VIII</u> m_F consists of precisely those sets $E \in m$ for which $\mu(E) < \infty$.

This implies assertion (d) of the theorem.

<u>Proof</u> : If $E \in m_F$, Steps II and VI imply that $E \cap K \in m_F$ for every compact K, hence $E \in m$. Conversely, suppose $E \in m$ and $\mu(E) < \infty$, and choose $\varepsilon > 0$. There is an open set $V \supset E$ with $\mu(V) < \infty$; by Step III and V, there is a compact $K \subset V$ with $\mu(V - K) < \varepsilon$.

Since $E \cap K \in m_F$, there is a compact $H \subset E \cap K$ with

$$\mu(E \cap K) < \mu(H) + \varepsilon .$$

Since $E \subset (E \cap K) \cup (V - K)$, it follows that

$$\mu(E) \leq \mu(E \cap K) + \mu(V - K) < \mu(H) + 2\varepsilon$$

which implies that $E \in \mathfrak{m}_F$.

<u>Step IX</u> μ is a measure on $\mathfrak{m}$.

<u>Proof</u> : From Step IV , we have

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$$ with E_i are pairwise

disjoint members of $\mathfrak{m}_F$.

From Step VIII , $\mathfrak{m}_F$ contains precisely those sets $E \in \mathfrak{m}$ for which

$\mu(E) < \infty$.

Hence $$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) , \quad E_i \in \mathfrak{m} .$$

Therefore μ is a measure on $\mathfrak{m}$.

<u>Step X</u> For every $f \in C_c(X)$, $\Lambda f = \int_X f \, d\mu$.

This proves (a) , and completes the theorem.

<u>Proof</u> : It is enough to prove this for real f.

Also, it is enough to prove the inequality

$$\Lambda f \leq \int_X f \, d\mu \qquad \text{.........................(16)}$$

for every real function $f \in C_c(X)$. For once (16) is established, the linearity of Λ shows that

$$-\Lambda f = \Lambda(-f) \leq \int_X (-f)d\mu = - \int_X f\, d\mu ,$$

hence $\Lambda f \geq \int_X f\, d\mu$

this together with (16), shows that equality holds.

Let K be the support of a real $f \in C_c(X)$, let $[a, b]$ be an interval which contains the range of f (note to Theorem 2.36).

Choose $\varepsilon > 0$, and choose y_i, for $i = 0,1, \dots n$, so that $y_i - y_{i-1} < \varepsilon$ and

$$y_o < a < y_1 < \dots < y_n = b \qquad \text{.............(17)}$$

Put $E_i = \{x | y_{i-1} < f(x) \leq y_i\} \cap K \; (i = 1, \dots n)$ ....(18)

Since f is continuous, f is Borel measurable, and the sets E_i are therefore disjoint Borel sets whose union is K.

There are open sets $V_i \supset E_i$ such that

$$\mu(V_i) < \mu(E_i) + \frac{\varepsilon}{n} \qquad (i = 1, \dots n) \qquad \text{............. (19)}$$

and such that $f(x) < y_i + \varepsilon$ for all $x \in V_i$.

By Theorem 2.39 there are functions $h_i \prec V_i$ such that $\Sigma h_i = 1$ on K. Hence $f = \Sigma h_i f$.

Since $h_i f \leq (y_i + \varepsilon)h_i$ and since $y_i - \varepsilon < f(x)$ on E_i, we have

$$\Lambda f = \sum_{i=1}^{n} \Lambda(h_i f) \leq \sum_{i=1}^{n} (y_i + \varepsilon)\Lambda h_i$$

$$\leq \sum_{i=1}^{n} (y_i + \varepsilon)\, \mu(V_i)$$

$$\Lambda f \leq \sum_{i=1}^{n} (y_i + \varepsilon)\mu(E_i) + \sum_{i=1}^{n} (y_i + \varepsilon)\frac{\varepsilon}{n}$$

$$= (y_1 + \varepsilon)\mu(E_1) + \ldots + (y_n + \varepsilon)\mu(E_n) + [(y_1 + \varepsilon) + \ldots (y_n + \varepsilon)]\ \frac{\varepsilon}{n}$$

$$= y_1\mu(E_1) + \varepsilon\mu(E_1) + \ldots + y_n\mu(E_n) + \varepsilon\mu(E_n) + [\ \sum_{i=1}^{n} y_n + n\varepsilon\ ]\ \frac{\varepsilon}{n}$$

$$\leq y_1\mu(E_1) - \varepsilon\mu(E_1) + 2\varepsilon\mu(E_1) + \ldots + y_n\mu(E_n) - \varepsilon\mu(E_n) + 2\varepsilon\mu(E_n) + (nb + n\varepsilon)\ \frac{\varepsilon}{n}$$

$$= (y_1 - \varepsilon)\mu(E_1) + \ldots (y_n - \varepsilon)\mu(E_n) + 2\varepsilon(\mu(E_1) + \ldots \mu(E_n)) + \varepsilon(b + \varepsilon)$$

$$= \sum_{i=1}^{n} (y_i - \varepsilon)\mu(E_i) + 2\varepsilon\ (\ \sum_{i=1}^{n} \mu(E_i)) + \varepsilon(b + \varepsilon))$$

$$= \sum_{i=1}^{n} (y_i - \varepsilon)\mu(E_i) + 2\varepsilon(\mu\ (\ \bigcup_{i=1}^{n} E_i)) + \varepsilon(b + \varepsilon)$$

$$\leq \sum_{i=1}^{n} \int_{E_i} f\ d\mu + 2\varepsilon\mu\ (K) + \varepsilon(b + \varepsilon)$$

$$= \int_{E_1} f\ d\mu\ + \ldots \int_{E_n} f\ d\mu + \varepsilon(2\mu\ (K) + b + \varepsilon)$$

$$= \int_X \chi_{E_1} f\ d\mu\ + \ldots\ \int_X \chi_{E_n} f\ d\mu + \varepsilon(2\mu\ (K) + b + \varepsilon)$$

$$= \int_X \sum_{i=1}^{n} \chi_{E_i} f\ d\mu\ + \varepsilon\ (2\mu\ (K) + b + \varepsilon)$$

$$= \int_X \chi_{E_1 \cup \ldots \cup E_n} f\ d\mu + \varepsilon(2\mu\ (K) + b + \varepsilon)$$

$$= \int_X \chi_K f\ d\mu\ + \varepsilon(2\mu\ (K) + b + \varepsilon)$$

$$= \int_K f\ d\mu\ + \varepsilon(2\mu\ (K) + b + \varepsilon)$$

$$= \int_X f\ d\mu\ + \varepsilon\ (2\mu\ (K) + b + \varepsilon).$$

Hence, $\Lambda f \leq \int_X f\, d\mu$ since $\varepsilon > 0$ is arbitrary, (16) is established , and the proof of the theorem is completed.

3.2 The Riesz Representation Theorem

To each bounded linear functional Φ on $C_o(X)$, where X is a locally compact Hausdorff space, there corresponds a unique complex regular Borel measure μ such that

$$\Phi(f) = \int_X f\, d\mu \qquad (f \in C_o(X)). \quad \ldots\ldots\ldots\ldots (1)$$

Moreover, if Φ and μ are related as in (1), then

$$||\Phi|| = |\mu|(X). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Proof : We first prove the uniqueness of μ .

Suppose μ is a complex Borel measure on X and

$$\int_X f\, d\mu = 0 \quad \text{for all} \quad f \in C_o(X) .$$

By Theorem 2.62, there is a complex Borel measurable function h, with $|h| = 1$ such that $d\mu = h\, d|\mu|$.

Since $|\mu|(X) = \int_X d|\mu|$, we have that for any sequence $\{f_n\}_{n \in N}$ in $C_c(X)$,

$$\begin{aligned} |\mu|(X) &= \int_X h\bar{h}\, d|\mu| - \int_X f_n\, d\mu \\ &= \int_X h\bar{h}\, d|\mu| - \int_X f_n h\, d|\mu| \\ &= \int_X (\bar{h} - f_n) h\, d|\mu| \end{aligned}$$

$$\leq \int_X |\overline{h} - f_n| \; |h| \; d|\mu|$$

$$= \int_X |\overline{h} - f_n| \; d|\mu| \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and since $C_c(X)$ is dense in $L^1(|\mu|)$ with respect to the L^1 - norm (Theorem 2.50), then we can choose $\{f_n\}$ such that the last expression in (3) tends to 0 as $n \to \infty$.

Thus $|\mu|(X) \leq 0$.

By Theorem 2.58 , $|\mu|(X) = 0$ then $\mu = 0$, i.e.

If $\int_X f \, d\mu = 0$ for all $f \in C_o(X)$ and for any complex Borel measure μ then $\mu = 0$.

Suppose $\Phi(f) = \int_X f \, d\mu_1$ and also $\Phi(f) = \int_X f \, d\mu_2$ where μ_1 and μ_2 are complex regular Borel measures on X.

$$\int_X f \, d\mu_1 = \int_X f \, d\mu_2$$

$$\int_X f \, d\mu_1 - \int_X f \, d\mu_2 = 0$$

$$\int_X f \, d(\mu_1 - \mu_2) = 0$$

$$\mu_1 - \mu_2 = 0$$

$$\mu_1 = \mu_2 .$$

This shows that at most one μ corresponds to each Φ .

Now consider a given bounded linear functional Φ on $C_o(X)$.

Assume $||\Phi|| = 1$, without loss of generality.

We shall construct a positive linear functional Λ on $C_c(X)$, such that

$$|\Phi(f)| \leq \Lambda(|f|) \leq ||f|| \quad (f \in C_c(X)) \quad ....(4)$$

where $||f||$ denotes the supremum norm.

Once we have this Λ , we associate with it a positive Borel measure λ as in Theorem 3.1. The conclusion of Theorem 3.1 shows that λ is regular if $\lambda(x) < \infty$. From the construction of λ in Theorem 3.1.

$$\lambda(V) = \sup \{\Lambda f \mid f \prec V\}$$

for every open set V in X.

$$\lambda(X) = \sup \{\Lambda f : 0 \leq f \leq 1, f \text{ real } \in C_c(X)\}$$

$$|\Lambda f| = |\int_X f \, d\lambda| \leq \int_X |f| \, d\lambda = \Lambda(|f|) \leq ||f||$$

If $||f|| \leq 1$, then $|\Lambda f| \leq 1$, we see that actually $\lambda(X) \leq 1$.

We also deduce from (4) that

$$|\Phi(f)| \leq \Lambda(|f|) = \int_X |f| d\lambda = ||f||_1 \quad (f \in C_c(X)).......(5)$$

The last norm refers to the space $L^1(\lambda)$.

Thus Φ is a linear functional on $C_c(X)$ of norm at most 1, with respect to the $L^1(\lambda)$ norm on $C_c(X)$. There is a norm-preserving extension of Φ to a linear functional on $L^1(\lambda)$. (By Hahn - Banach Theorem 2.54). By Theorem 2.64 (the case $p = 1$) there is a unique $g \in L^\infty(\lambda)$ such that

$$\Phi(f) = \int_X fg \, d\lambda \qquad (f \in L^1(\lambda)). \quad \ldots\ldots\ldots\ldots\ldots(6)$$

Moreover $\quad ||\Phi|| = ||g||_\infty = 1$.

By Definition 2.47 $\;||g||_\infty$ is the essential supremum of $|g|$.
It follows that $|g(x)| \leq 1$ a.e. on X.

Since (6) holds for all $f \in C_c(X)$, we claim that (6) holds for all $f \in C_o(X)$.

By Hahn - Banach Theorem 2.54, $C_c(X) \subset C_o(X)$ and Φ is a bounded linear functional on $C_c(X)$ (with respect to the supremum norm) then Φ can be extended to a bounded linear functional ψ on $C_o(X)$ such that $\psi|_{C_c(X)} = \Phi$ and $||\Phi|| = ||\psi||$.

Since $C_c(X)$ is dense in $C_o(X)$ (with respect to the supremum norm), for any $f \in C_o(X)$, there exists sequence $\{f_n\}_{n \in N} \in C_c(X)$ such that

$$||f_n - f|| \to 0 \text{ as } n \to \infty .$$

Then $f_n \to f$ uniformly on X and $\psi f_n \to \psi f$ since ψ is continuous on $C_o(X)$.

$$\therefore \quad \psi f = \lim_{n \to \infty} \psi f_n$$

$$= \lim_{n \to \infty} \int_X f_n \, g \, d\lambda .$$

Let $d\varphi = g \, d\lambda$.

$$\psi f = \lim_{n \to \infty} \int_X f_n \, d\varphi$$

$$= \int_X f \, d\varphi \quad \text{(By Theorem 2.29)}$$

$$= \int_X f \, g \, d\lambda \qquad \forall f \in C_o(X).$$

Hence (6) holds for all $f \in C_o(X)$ and we obtain the representation (1) with $d\mu = g \, d\lambda$.

It remains to prove that μ is regular.

Since $d\mu = g \, d\lambda$, by Theorem 2.62, we have $d\mu = h \, d|\mu|$ where h is a measurable function such that $|h| = 1$.

$$\therefore \quad h \, d|\mu| = g \, d\lambda$$

$$d|\mu| = \bar{h} \, g \, d\lambda .$$

Given $E \in m$ and $\varepsilon > 0$, there exist closed set F and open set V $\ni F \subset E \subset V$ and such that $\lambda(V - F) < \varepsilon$. i.e.

$$\int_{V - F} d\lambda < \varepsilon .$$

$$|\mu|(V-F) = \int_{V-F} d|\mu| = \int_{V-F} \bar{h} \, g \, d\lambda \leq \int_{V-F} |\bar{h} g| \, d\lambda$$

$$\leq \int_{V-F} d\lambda < \varepsilon .$$

Therefore $|\mu|$ is regular then μ is regular by definition.

Since $||\Phi|| = 1$, (6) shows that

$$|\Phi(f)| = \left|\int_X fg \, d\lambda\right| \leq \int_X |f||g| \, d\lambda \leq ||f|| \int_X |g| \, d\lambda .$$

If $||f|| \leq 1$ then $|\Phi(f)| \leq \int_X |g| d\lambda \quad \forall f \in C_o(X)$,

$$||\Phi|| \leq \int_X |g| d\lambda$$

$$1 \leq \int_X |g| d\lambda . \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

We also know that $\lambda(X) \leq 1$ and $|g| \leq 1$ a.e. on X.

Since we have

$$d\mu = g \, d\lambda ,$$

$$\int_X d\mu = \int_X g \, d\lambda$$

$$\mu(X) = \int_X g \, d\lambda .$$

By Theorem 2.63,

$$|\mu|(X) = \int_X |g| d\lambda$$

$$\leq \int_X d\lambda = \lambda(x) \leq 1 = ||\Phi|| . \quad \ldots\ldots(8)$$

i.e. $|\mu|(X) \leq ||\Phi||$.

From (7) , we have $||\Phi|| \leq \int_X |g| d\lambda$.

$$||\Phi|| \leq \int_x |g| d\lambda = |\mu|(X).$$

Hence $||\Phi|| = |\mu|(X)$ which proves (2).

So it all depends on finding a positive linear functional Λ which satisfies (4).

If $f \in C_c+(X)$ [the class of all nonnegative real members of $C_c(X)$.] , define

$$\Lambda f = \sup \{|\Phi(h)| : h \in C_c(X), |h| \leq f\} \quad \ldots\ldots(9)$$

Then $\Lambda f \geqslant 0$ and Λ satisfies (4) , since

$$\Lambda(|f|) = \sup \{ |\Phi(h)| : h \in C_c(X), |h| \leq |f| \}$$

$$|\Phi(f)| \leq \Lambda(|f|) \qquad (f \in C_c+(X)).$$

$$|\Phi(h)| \leq ||\Phi|| \; ||h|| = ||h|| \leq ||f|| \quad \forall h \in C_c(X).$$

Thus $\sup \{ |\Phi(h)| : h \in C_c(X), |h| \leq |f| \} \leq ||f||$ and hence

$$\Lambda(|f|) \leq ||f||.$$

i.e. $|\Phi(f)| \leq \Lambda(|f|) \leq ||f||$.

Suppose $0 \leq f_1 \leq f_2$, f_1 and f_2 belong to $C_c+(X)$

$$\Lambda f_1 = \sup \{|\Phi(h)| : h \in C_c(X), |h| \leq f_1\}$$

$$\Lambda f_2 = \sup \{|\Phi(h)| : h \in C_c(X), |h| \leq f_2\}.$$

Since $\{|\Phi(h)| : h \in C_c(X), |h| \leq f_1\} \subseteq \{|\Phi(h)| : h \in C_c(X), |h| \leq f_2\}$,

$$\Lambda f_1 \leq \Lambda f_2 .$$

Next we will show that $\Lambda(cf) = c \Lambda f$ where c is any positive real constant and $f \in C_c+(X)$.

$$\Lambda(cf) = \sup \{|\Phi(h)| : h \in C_c(X),\ |h| \leq cf\}.$$

Given $\varepsilon > 0$, arbitrary

$$\Lambda(cf) < |\Phi(h)| + \varepsilon \quad \text{for some } h \text{ with } |h| \leq cf.$$

$$\frac{1}{c}\Lambda(cf) < \frac{1}{c}|\Phi(h)| + \frac{\varepsilon}{c}$$

$$= |\Phi(\frac{h}{c})| + \frac{\varepsilon}{c}.$$

Since $\left|\frac{h}{c}\right| \leq f$,

$$\frac{1}{c}\Lambda(cf) < \Lambda f + \frac{\varepsilon}{c}$$

$$\Lambda(cf) < c\Lambda f + \varepsilon$$

$$\Lambda(cf) \leq c\Lambda f. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

$$\Lambda(f) = \sup \{|\Phi(h)| : h \in C_c(X),\ |h| \leq f\}$$

$$\Lambda(f) \geq |\Phi(h)| \quad \text{for all } h \text{ with } |h| \leq f.$$

$$c\Lambda(f) \geq c\,|\Phi(h)|$$

$$= |\Phi(ch)|.$$

Given $\varepsilon > 0$ arbitrary,

$$c\Lambda(f) < |\Phi(ch)| + \varepsilon \quad \text{for some } h \text{ with } |h| \leq f.$$

Since $|ch| \leq cf$,

$$c\Lambda(f) < \Lambda(cf) + \varepsilon$$

$$c\Lambda(f) \leq \Lambda(cf). \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

i.e. $\Lambda(cf) = c\Lambda f.$

Next we must show that, for f and $g \in C_{c+}(X)$

$$\Lambda(f+g) = \Lambda f + \Lambda g. \quad \text{.............................(12)}$$

If $\varepsilon > 0$, there exist h_1 and $h_2 \in C_c(X)$ such that

$|h_1| \leq f$, $|h_2| \leq g$ and

$$\Lambda f \leq |\Phi(h_1)| + \varepsilon \ , \ \Lambda g \leq |\Phi(h_2)| + \varepsilon . \quad \text{........(13)}$$

There are complex numbers α_i, $|\alpha_i| = 1$ such that

$\alpha_i \ \Phi(h_i) = |\Phi(h_i)|$, $i = 1,2$.

Then
$$\begin{aligned} \Lambda f + \Lambda g &\leq |\Phi(h_1)| + |\Phi(h_2)| + 2\varepsilon \\ &= \alpha_1 \Phi(h_1) + \alpha_2 \Phi(h_2) + 2\varepsilon \\ &= \Phi(\alpha_1 h_1) + \Phi(\alpha_2 h_2) + 2\varepsilon \\ &= \Phi(\alpha_1 h_1 + \alpha_2 h_2) + 2\varepsilon . \end{aligned}$$

Since
$$\begin{aligned} |\alpha_1 h_1 + \alpha_2 h_2| &\leq |\alpha_1||h_1| + |\alpha_2||h_2| \\ &= |h_1| + |h_2| \\ &\leq f + g, \end{aligned}$$

Hence
$$\begin{aligned} \Lambda f + \Lambda g &\leq \Lambda(|h_1| + |h_2|) + 2\varepsilon \\ &\leq \Lambda(f+g) + 2\varepsilon . \end{aligned}$$

Hence $\Lambda f + \Lambda g \leq \Lambda(f+g)$. ....(14)

Next, choose $h \in C_c(X)$, subject only to the condition $|h| \leq f + g$.

Let $V = \{x \mid f(x) + g(x) > 0\}$ and define

$$h_1(x) = \frac{f(x)\ h(x)}{f(x)+g(x)}, \quad h_2(x) = \frac{g(x)\ h(x)}{f(x)+g(x)} \qquad (x \in V)$$

$$h_1(x) = h_2(x) = 0 \qquad (x \notin V).$$

Claim that h_1 and h_2 belong to $C_c(X)$.

h_1 and h_2 are continuous on V.

If $x_o \notin V$, then $h_1(x_o) = h_2(x_o) = 0$ and since we have $h(x_o) = h_1(x_o) + h_2(x_o)$, $h(x_o) = 0$. Since h is continuous and since $|h_1(x)| \leq |h(x)|$ for all $x \in X$, then also $|h_1(x_o)| \leq |h(x_o)| = 0$. It follows that x_o is a point of continuity of h_1.

Thus h_1 is continuous on X.

We define $h_1(x) = \dfrac{f(x)\ h(x)}{f(x) + g(x)}$.

If $h(x) = 0$ then $h_1(x) = 0$. Thus the support of h_1 is a subset of the support of h which is compact. Therefore the support of h_1 is compact. Hence $h_1 \in C_c(X)$ and the same holds for h_2.

Since $h_1 + h_2 = h$ and $|h_1| \leq f$, $|h_2| \leq g$, we have

$$
\begin{aligned}
|\Phi(h)| &= |\Phi(h_1 + h_2)| \\
&\leq |\Phi(h_1)| + |\Phi(h_2)| \\
&\leq \Lambda f + \Lambda g .
\end{aligned}
$$

Hence $\Lambda(f + g) \leq \Lambda f + \Lambda g$. ....(15)

From (14) & (15) we get $\Lambda(f + g) = \Lambda f + \Lambda g$.

If f is now a real function, $f \in C_c(X)$, then $2 f^+ = |f| + f$, so that $f^+ \in C_{c+}(X)$; likewise $f^- \in C_{c+}(X)$, and since $f = f^+ - f^-$, we define

$$\Lambda f = \Lambda f^{+} - \Lambda f^{-} \qquad (f \in C_c(X)\text{ , f is a real valued function}).$$

Let f, g $\in C_c(X)$ and f and g are real valued functions. We want to show that $\Lambda(f + g) = \Lambda f + \Lambda g$.

Let

$$k = f + g$$

$$k^{+} - k^{-} = f^{+} - f^{-} + g^{+} - g^{-}$$

$$k^{+} + f^{-} + g^{-} = k^{-} + f^{+} + g^{+}$$

$$\Lambda(k^{+} + f^{-} + g^{-}) = \Lambda(k^{-} + f^{+} + g^{+})$$

$$\Lambda k^{+} + \Lambda f^{-} + \Lambda g^{-} = \Lambda k^{-} + \Lambda f^{+} + \Lambda g^{+}$$

$$\Lambda k^{+} - \Lambda k^{-} = \Lambda f^{+} - \Lambda f^{-} + \Lambda g^{+} - \Lambda g^{-}$$

$$\Lambda (k^{+} - k^{-}) = \Lambda(f^{+} - f^{-}) + \Lambda(g^{+} - g^{-})$$

$$\Lambda k = \Lambda f + \Lambda g$$

$$\Lambda(f + g) = \Lambda f + \Lambda g \quad .$$

To show $\Lambda(cf) = c\Lambda f$ where c is a real constant.

Claim that $(-f)^{+} = f^{-}$ and $(-f)^{-} = f^{+}$.

<u>Case 1</u> $f(x) \geq 0$

$$(-f)^{+} = \max \{ -f,o \} = 0 \text{ , } (-f)^{-} = -\min \{-f,o\} = -(-f) = f.$$

$$f^{-} = -\min \{f,o\} = 0 \text{ , } f^{+} = \max \{f,o\} = f \text{ .}$$

<u>Case 2</u> $f(x) < 0$

$$(-f)^+ = \max\{-f,o\} = -f\ ,\quad (-f)^- = -\min\{-f,o\} = 0$$

$$f^- = -\min\{f,o\} = -f\ ,\quad f^+ = \max\{f,o\} = 0\ .$$

For c is a positive real constant and f is a real valued function.

Claim that $cf^+ = (cf)^+$ & $cf^- = (cf)^-$.

<u>Case 1</u> $f(x) \geqslant 0$

$$cf^+ = c\max\{f\ o\} = cf\ ,\quad cf^- = -c\min\{f,o\} = 0$$

$$(cf)^+ = \max\{cf,o\} = cf\ ,\quad (cf)^- = -\min\{cf,o\} = 0$$

<u>Case 2</u> $f(x) < 0$

$$cf^+ = c\max\{f,o\} = 0\ ,\quad cf^- = -c\min\{f,o\} = -cf$$

$$(cf)^+ = \max\{cf,o\} = 0\ ,\quad (cf)^- = -\min\{cf,o\} = -cf$$

Consider for $c = -1$ and f is a real valued function.

$$\begin{aligned}\Lambda(-f) &= \Lambda[(-f)^+ - (-f)^-] \\ &= \Lambda[f^- - f^+] \\ &= \Lambda f^- - \Lambda f^+ \\ &= -[\Lambda f^+ - \Lambda f^-] \\ &= -[\Lambda(f^+ - f^-)] \\ &= -\Lambda f\ .\end{aligned}$$

For c is a positive real constant and f is a real valued function.

$$\Lambda(cf) = \Lambda\left[(cf)^+ - (cf)^-\right]$$

$$= \Lambda(cf)^+ - \Lambda(cf)^-$$

$$= \Lambda cf^+ - \Lambda cf^-$$

$$= c\Lambda f^+ - c\Lambda f^-$$

$$= c(\Lambda f^+ - \Lambda f^-)$$

$$= c\Lambda(f^+ - f^-)$$

$$= c\Lambda f .$$

For $c = 0$.

$$\Lambda(cf) = \Lambda\left[(cf)^+ - (cf)^-\right]$$

$$= \Lambda(cf)^+ - \Lambda(cf)^- = \Lambda(0) - \Lambda(0) = 0$$

$$= c\Lambda f .$$

Then we consider

$$\Lambda(-cf) = \Lambda\left[(-1)cf\right]$$

$$= -\Lambda\left[cf\right]$$

$$= -c\Lambda f .$$

Hence $\Lambda(cf) = c\Lambda f$ where c isareal constant and f is a real-valued function in $C_c(X)$.

If f is a complex-valued function, $f = u + iv$ $(u, v \text{ real})$, we define

$$\Lambda f = \Lambda(u + iv) = \Lambda u + i \Lambda v.$$

Let f_1 and $f_2 \in C_c(X)$

$$\begin{aligned}
\Lambda(f_1 + f_2) &= \Lambda(u_1 + iv_1 + u_2 + iv_2) \\
&= \Lambda[(u_1 + u_2) + i(v_1 + v_2)] \\
&= \Lambda(u_1 + u_2) + i\Lambda(v_1 + v_2) \\
&= \Lambda u_1 + \Lambda u_2 + i\Lambda v_1 + i\Lambda v_2 \\
&= (\Lambda u_1 + i\Lambda v_1) + (\Lambda u_2 + i\Lambda v_2) \\
&= \Lambda(u_1 + iv_1) + \Lambda(u_2 + iv_2) \\
&= \Lambda f_1 + \Lambda f_2 .
\end{aligned}$$

To show $\Lambda(cf) = c\Lambda f$ where c is complex constant and f is a complex-valued function.

For c is a real constant and f is a complex valued function.

$$\begin{aligned}
\Lambda(cf) &= \Lambda(c\,(u + iv)) \\
&= \Lambda(cu + icv) \\
&= \Lambda(cu) + i\Lambda(cv) \\
&= c\Lambda u + ic\Lambda v \\
&= c(\Lambda u + i\Lambda v) \\
&= c\Lambda(u + iv) \\
&= c\Lambda f .
\end{aligned}$$

Consider for $c = i$.

$$\Lambda(if) = \Lambda(i(u + iv))$$

$$= \Lambda(iu - v)$$

$$= \Lambda((-v) + iu)$$

$$= \Lambda(-v) + i\Lambda u$$

$$= -\Lambda(v) + i\Lambda u$$

$$= i(\Lambda u + i\Lambda v)$$

$$= i(\Lambda(u + iv)) = i\Lambda f .$$

If $c = \alpha + i\beta$ where α, β are real constants and f is a complex valued function

$$\Lambda(cf) = \Lambda((\alpha + i\beta)f)$$

$$= \Lambda(\alpha f + i\beta f)$$

$$= \Lambda(\alpha f) + i\Lambda(\beta f)$$

$$= \alpha\Lambda f + i\beta\Lambda f$$

$$= (\alpha + i\beta)\Lambda f$$

$$= c\Lambda f .$$

Now show that our extended functional Λ is linear on $C_c(X)$.

This completes the proof.

3.3 Theorem Suppose f is continuous on $[a,b] \times K$ where K is a compact subset in the plane, $a < b$ are real numbers, g is sectionally (piecewise) continuous on $[a,b]$. Then the integral

$$\int_a^b f(t,z)\, g(t)dt$$ can be approximated uniformly for

$z \in K$ by a Riemann sum of the form

$$\sum_{j=1}^{n} f(t_j, z)\, g(t_j)\, (t_j - t_{j-1}) \quad \text{where}$$

$a = t_o < t_1 < \ldots < t_n = b.$

Proof : Since g is sectionally continuous, there are at most finitely many points in the parameter $[a,b]$ at which g fails to be continuous. However such points g has left hand and right hand limits.

By considering each of the subintervals at a time, we may assume that g is continuous on $[a,b]$.

$\therefore$ gf is continuous on $[a,b] \times K$ which is compact.

$\therefore$ gf is uniformly continuous on $[a,b] \times K$.

Given $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$(t, z),\ (t_j, z_j) \in [a,b] \times K,$$

$$\text{dist.}\ [\,(t, z),\ (t_j, z_j)\,] < \delta$$

implies $|g(t)\, f(t, z) - g(t_j)\, f(t_j, z_j)| < \dfrac{\varepsilon}{b - a}$.

Let $\{a = t_o, t_1, \ldots t_n = b\}$ be a partition of $[a,b]$ such that $\max (t_i - t_{i-1}) < \delta$.

Then $\left| \int_a^b f(t,z)\, g(t)\, dt - \sum_{j=1}^{n} f(t_j,z)\, g(t_j)(t_j - t_{j-1}) \right|$

$$= \left| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left[f(t,z)\, g(t) - f(t_j,z)\, g(t_j)\right] dt \right|$$

$$\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left| f(t,z)\, g(t) - f(t_j,z)\, g(t_j)\right| dt < \varepsilon .$$

Then the integral can be approximated uniformly by the Riemann sum.

In the remainder of this chapter, our effort will be directed toward proving the Riesz Representation Theorem for Hilbert space. We use the proof found in [**3**] .

3.4 <u>Definition</u> A Hilbert space is a complete inner product space.

3.5 <u>Definition</u> Let x, y be any vectors in the complex vector space H. If $(x,y) = 0$, we say that x is <u>orthogonal</u> to y .

If M is a subspace of H, let $M^{\perp}$ be the set of all $y \in H$ which are orthogonal to every $x \in M$.

3.6 <u>Definition</u> A set of vectors $\mathbf{u}_\alpha$ in a Hilbert space H, where α runs through some index set A, is called <u>orthonormal</u> if it satisfies the orthogonality relations $(\mathbf{u}_\alpha, \mathbf{u}_\beta) = 0$ for all $\alpha \neq \beta$, $\alpha \in A$ and $\beta \in A$ and $||\mathbf{u}_\alpha|| = 1$ for each $\alpha \in A$.

3.7 Definition If V is a vector space, if $x_1, \dots x_k \in V$, and if $c_1, \dots c_k$ are scalars then $c_1x_1 + .. c_kx_k$ is called a linear combination of $x_1, \dots x_k$.

The set $[S]$ of all linear combinations of all finite subsets of S (also called the set of all finite linear combinations of members of S) is a vector space; $[S]$ is the smallest subspace of V which contain S; $[S]$ is called the span of S.

3.8 Definition Let $\{u_\alpha : \alpha \in A\}$ be an orthonormal set in H. $\{u_\alpha\}$ is a maximal orthonormal set in H if and only if the set S of all finite linear combinations of members of $\{u_\alpha\}$ is dense in H.

Maximal orthonormal sets are called orthonormal bases.

3.9 Theorem Let X be an inner product space and let M be a subspace of X such that $\dim M < \infty$, then we have the direct sum decomposition $X = M \oplus M^\perp$ [The notation $M \oplus N = X$ means that any vector $z \in X$ can be written uniquely as $z = x + y$ where $x \in M$ and $y \in N$, or, equivalently that every $z \in X$ can be written as $z = x + y$, where $x \in M$ and $y \in N$ and $M \cap N = \{0\}$].

Proof : First it will be shown that any vector in the space can be written as the sum of an element from M and element from $M^\perp$.

Since $\dim M < \infty$, we can choose an orthonormal basis for M : $x_1, \dots x_n$.

If $z \in X$, then $\sum_{i=1}^{n} (z, x_i)\, x_i \in M$. ....(1)

Now consider the vector

$$y = z - \sum_{i=1}^{n} (z, x_i)\, x_i .$$

Letting x_j be any one of the basis vectors in M, we have

$(y, x_j) = (z, x_j) - (z, x_j) = 0 \quad (j = 1, \ldots, n)$ or that y is orthogonal to each of the basis vectors in M.

Hence it is orthogonal to every vector in M, which implies that $y \in M^{\perp}$.

Therefore, we have the representation, for any vector $z \in X$,

$$z = \sum_{i=1}^{n} (z, x_i) x_i + y,$$

where the first term on the right is in M and the second is in $M^{\perp}$, and have completed the first part of the proof.

To prove the second part, suppose that

$$x \in M \cap M^{\perp}.$$

This immediately implies that $(x, x) = 0$ which implies that $x = 0$ and completes the proof.

3.10 The Riesz Representation Theorem for Hilbert space

If f is a bounded linear functional on a Hilbert space X, then there exists a unique vector a in X such that $||a|| = ||f||$ and $f(x) = (x, a)$.

Proof : Let $a \in X$, define $f(x) = (x, a)$, we shall show that f(x) is a bounded linear functional on X.

$$\begin{aligned} f(\alpha x_1 + \beta x_2) &= (\alpha x_1 + \beta x_2, a) \\ &= (\alpha x_1, a) + (\beta x_2, a) \\ &= \alpha(x_1, a) + \beta(x_2, a) \\ &= \alpha f(x_1) + \beta f(x_2). \end{aligned}$$

Also $|f(x)| = |(x, a)| \leq ||x|| \, ||a||$,

$$\therefore ||f|| \leq ||a||.$$

However $f(a) = (a, a) = ||a||^2$

$$\therefore ||f|| = ||a||.$$

Next we prove the existence of a vector a given f. If f = 0, take a = 0 then $f(x) = (x, 0) = 0$. Suppose $f \neq 0$ for some x.

Let $M = \ker f = \{x \mid f(x) = 0\}$.

The linearity of f shows that M is a subspace of X and the continuity of f shows that M is closed.

i.e. M is a closed subspace of X.

Let $M^{\perp} = \{ w \in X \mid (w, x) = 0 \quad \forall x \in M \}$

Suppose $M^{\perp} = \{ 0 \}$.

Since $M^{\perp\perp} = M = \{0\}^{\perp} = X$ (by theorem 3.9),

$$f(x) = 0 \qquad \forall x \in X$$

Then also take $a = 0$.

Suppose $M^{\perp}$ does not consist of 0 alone .

Then we may choose $a_1 \in M^{\perp}$ such that $f(a_1) = 1$.

since $a_1 \in M^{\perp}$, $a_1 \neq 0$ and $f(a_1) \neq 0$

$$\text{Let } f(x) = n \qquad (x \in M^{\perp})$$

$$\frac{1}{n} f(x) = 1$$

$$f(\frac{x}{n}) = 1 \qquad \text{(f is linear)}$$

then choose $a_1 = \frac{x}{n} \in M^{\perp}$.

For each $x \in X$

$$f(x - a_1 f(x)) = f(x) - f(a_1) f(x)$$

$$= 0$$

and therefore $(x - a_1 f(x), a_1) = 0$.

That is $(x, a_1) - f(x) (a_1, a_1) = 0$

and hence $f(x) = \dfrac{(x, a_1)}{(a_1, a_1)}$

$$= \frac{1}{||a_1||^2} (x, a_1)$$

$$= \left(x , \frac{a_1}{||a_1||^2} \right) .$$

Thus if $a = \dfrac{a_1}{||a_1||^2}$ then $f(x) = (x, a) \quad \forall \; x \in X.$

Now to prove the uniqueness of a.

Suppose f(x) can be represented in another form (x , a^*) then

$$(x, a) - (x, a^*) = 0$$

$$(x, a - a^*) = 0 \qquad \forall \; x \in X .$$

Let $x = a - a^*$

$$(a - a^* , a - a^*) = 0$$

$$a - a^* = 0$$

$$a = a^* .$$
