CHAPTER II

PRELIMINARIES

In this thesis, we assume a basic knowledge of real and complex analysis. However, this chapter contains a review of some relevant definitions and facts from integration theory which we will be using. Proofs will not be given, and can be found in [1].

2.1 Definition A collection $\mathfrak{m}$ of subsets of a set X is said to be a σ - algebra in X if $\mathfrak{m}$ has the following three properties:

(a) $X \in \mathfrak{m}$

(b) If $A \in \mathfrak{m}$, then $A^c \in \mathfrak{m}$ where A^c is the complement of A relative to X.

(c) If $A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in \mathfrak{m}$ for n = 1,2,... then $A \in \mathfrak{m}$.

If $\mathfrak{m}$ is a σ - algebra in X, then X is called a measurable space and the members of $\mathfrak{m}$ are called the measurable sets in X.

2.2 Definition If X is a measurable space, Y is a topological space, and f is a mapping of X into Y, then f is said to be measurable provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

2.3 <u>Proposition</u> Let m be a σ- algebra in a set X.

(a) $\phi \in m$

(b) m is closed under finite union.

(c) m is closed under countable intersection.

m is closed under finite intersection

(d) If $A \in m$, $B \in m$ then $A - B \in m$.

2.4 <u>Theorem</u> Let Y and Z be topological spaces, and let $g: Y \to Z$ be continuous. If X is a measurable space and $f: X \to Y$ is measurable and if $h = g \circ f$, then $h: X \to Z$ is measurable.

2.5 <u>Theorem</u> Let **u** and **v** be real measurable functions on a measurable space X, let Φ be a continuous mapping of the plane into a topological space Y, and define $h(x) = \Phi(u(x), v(x))$ for $x \in X$. Then $h: X \to Y$ is measurable.

2.6 <u>Corollaries</u> Let X be a measurable space. Then we have the following results:

(a) If $f = u + iv$, where u and v are real measurable functions on X, then f is a <u>complex measurable function</u> on X.

(b) If $f = u + iv$ is a complex measurable function on X, then u, v, and $|f|$ are real measurable functions on X.

(c) If f and g are complex measurable functions on X, then so are $f + g$ and fg.

(d) If E is a measurable set in X and if

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E. \\ 0 & \text{if } x \notin E. \end{cases}$$

then χ_E is a measurable function.

We call χ_E the <u>characteristic function</u> of the set E.

(e) If f is a complex measurable function on X, there is a complex measurable function α on X such that $|\alpha| = 1$ and $f = \alpha|f|$.

2.7 <u>Theorem</u> If $\mathfrak{F}$ is any collection of subsets of X, there exists a smallest σ-algebra m^* in X such that $\mathfrak{F} \subset m^*$.

2.8 <u>Definition</u> Let X be a topological space. By theorem 2.7, there exists a smallest σ - algebra $\mathfrak{B}$ in X such that every open set in X belongs to $\mathfrak{B}$. The members of $\mathfrak{B}$ are called the <u>Borel sets</u> of X.

2.9 <u>Definition</u> If X is a Borel measurable space, Y is a topological space, and f is a mapping of X into Y. Then f is said to be <u>Borel measurable</u> provided that $f^{-1}(V)$ is a Borel set in X for every open set V in Y.

<u>Note</u> If Y is the real line or the complex plane, the Borel measurable functions will be called <u>Borel functions</u>.

2.10 <u>Definition</u> Let $\{a_n\}$ be a sequence in $[-\infty, \infty]$, and put

$b_k = \sup \{a_k, a_{k+1}, a_{k+2}, \ldots\}$ $(k = 1,2,3, \ldots)$ and

$\beta = \inf \{ b_1, b_2, b_3, \ldots \}$. We call β the <u>upper limit</u> of $\{a_n\}$, and write $\beta = \lim_{n\to\infty} \sup a_n$. The <u>lower limit</u> is defined analogously : simply interchange sup and inf.

2.11 <u>Theorem</u> If $f_n : X \to [-\infty, \infty]$ is measurable, for $n = 1,2,3, \ldots$ and

$$g = \sup_{n \geq 1} f_n , \quad h = \inf_{n \geq 1} f_n$$

$$k = \lim_{n\to\infty} \sup f_n , \; \ell = \lim_{n\to\infty} \inf f_n ,$$

then g, h, k and ℓ are measurable.

2.12 <u>Corollary</u> If f and g are measurable (with range in $[-\infty,\infty]$), then so are $\max \{f, g\}$ and $\min \{f, g\}$. In particular, this is true of the functions

$$f^+ = \max \{f, o\} \text{ and } f^- = -\min \{f, o\} .$$

The functions f^+ and f^- are called the <u>positive</u> and <u>negative</u> <u>parts</u> of f. We have $|f| = f^+ + f^-$ and $f = f^+ - f^-$.

2.13 Definition A function s on a set X whose range consists of only finitely many points in $[0,\infty)$ will be called a simple function.

If X is a measurable space, $\alpha_1, \ldots, \alpha_n$ are the distinct values of a simple function s on X, and

$A_i = \{x \mid s(x) = \alpha_i\}$ then $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ where χ_{A_i} is the characteristic function of A_i and s is measurable if and only if each of the sets A_i is measurable.

2.14 Definition A positive measure is a function μ, defined on a σ - algebra m, whose range is in $[0,\infty]$ and which is countably additive. i.e. if $\{A_i\}$ is a pairwise disjoint countable collection of members of m, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

and we shall also assume that $\mu(A) < \infty$ for at least one $A \neq \phi \in m$.

2.15 Definition A measure space is a measurable space which has a positive measure defined on the σ - algebra of its measurable sets.

2.16 Definition A complex measure on X is a complex-valued countably additive function defined on a σ - algebra m in X.

Note If $\mu(E) = 0$ for every $E \in m$, then μ is a positive measure.

2.17 Theorem Let μ be a positive measure on a σ - algebra m . Then

(a) $\mu(\phi) = 0$

(b) $\mu(A_1 \cup \ldots \cup A_n) = \mu(A_1) + \ldots + \mu(A_n)$ if $A_1, \ldots A_n$ are pairwise disjoint members of m .

(c) $A \subset B$ implies $\mu(A) \leqslant \mu(B)$ if $A \in m$, $B \in m$.

(d) $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in m$ and $A_1 \subset A_2 \subset \ldots$

(e) $\mu(A_n) \to \mu(A)$ as $n \to \infty$ if $A = \bigcap_{n=1}^{\infty} A_n$, $A_n \in m$ $A_1 \supset A_2 \supset \ldots$ and $\mu(A_1)$ is finite.

Integration of Positive Function

In this section, m will be a σ - algebra in a set X and μ will be a positive measure on m .

2.18 Definition If s is a measurable simple function on X, of the form

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

where $\alpha_1, \ldots \alpha_n$ are the distinct values of s and if $E \in m$, we define

$$\int_E s \, d\mu = \sum_{i=1}^{n} \alpha_i \, \mu(A_i \cap E).$$

The convention $0.\infty = 0$ is used here ; it may happen that $\alpha_i = 0$ for some i and that $\mu(A_i \cap E) = \infty$. If $f : X \rightarrow [0, \infty]$ is measurable, and $E \in m$, we define

$$\int_E f\, d\mu = \sup \int_E s\, d\mu \quad \ldots\ldots\ldots\ldots\ldots\ldots(*)$$

With

The supremum being taken over all simple measurable functions s such that $o \leqslant s \leqslant f$.

The left integral of (*) is called the <u>Lebesgue integral of f over E</u> with respect to the measure μ .

It is a number in $[o,\infty]$.

2.19 <u>Proposition</u> The functions and sets occuring in the following propositions are assumed to be measurable:

(a) If $o \leqslant f \leqslant g$, then $\int_E f\, d\mu \leqslant \int_E g\, d\mu$.

(b) If $A \subset B$ and $f \geqslant o$ then $\int_A f\, d\mu \leqslant \int_B f\, d\mu$.

(c) If $f \geqslant o$ and c is a constant, $o \leqslant c < \infty$, then $\int_E cf\, d\mu = c \int_E f\, d\mu$.

(d) If $f(x) = 0$ for all $x \in E$, then $\int_E f\, d\mu = 0$, even if $\mu(E) = \infty$.

(e) If $\mu(E) = 0$ then $\int_E f\, d\mu = 0$, even if $f(x) = \infty$ for every $x \in E$.

(f) If $f \geqslant 0$, then $\int_E f\, d\mu = \int_X \chi_E f\, d\mu$.

2.20 Proposition Let s and t be measurable simple functions on X. For $E \in m$, define

$$\varphi(E) = \int_E s\, d\mu .$$

Then φ is a measure on m. Also

$$\int_X (s + t)\, d\mu = \int_X s\, d\mu + \int_X t\, d\mu .$$

2.21 Lebesgue's Monotone Convergence Theorem

Let $\{f_n\}$ be a sequence of measurable functions on X and suppose that

(a) $0 \leqslant f_1(x) \leqslant f_2(x) \ldots \leqslant \infty$ for every $x \in X$,

(b) $f_n(x) \to f(x)$ as $n \to \infty$ for every $x \in X$. Then f is measurable, and

$$\int_X f_n\, d\mu \to \int_X f\, d\mu \quad \text{as } n \to \infty .$$

2.22 Theorem Suppose $f : X \to [0, \infty]$ is measurable, and

$$\varphi(E) = \int_E f\, d\mu \qquad (E \in m).$$

Then φ is a measure on m, and

$$\int_X g\, d\varphi = \int_X gf\, d\mu$$

for every measurable function g on X with range in $[0, \infty]$.

Remark The second assertion of Theorem 2.22 is sometimes written in the form

$$d\varphi = f\,d\mu\ .$$

We assign no independent meaning to the symbols $d\varphi$ and $d\mu$. This merely means that $\int_X g\,d\varphi = \int_X g\,f\,d\mu$ for every measurable $g \geq 0$.

Integration of Complex Functions

As before, μ will be a positive measure on an arbitrary measurable space X.

2.23 Definition We define $L^1(\mu)$ to be the collection of all complex measurable functions f on X for which

$$\int_X |f|\,d\mu < \infty\ .$$

The members of $L^1(\mu)$ are called Lebesgue integrable functions (with respect to μ).

2.24 Definition If $f = u + iv$ where u and v are real measurable functions on X and if $f \in L^1(\mu)$, we define

$$\int_E f\,d\mu = \int_E u^+\,d\mu - \int_E u^-\,d\mu + i\int_E v^+\,d\mu - i\int_E v^-\,d\mu$$

for every measurable set E.

Here u^+ and u^- are the positive and negative parts of u as defined in Corollary 2.12 ; v^+ and v^- are similary obtained from v. These four functions are measurable, real and nonnegative ; hence the four integrals on the right exist, by Definition 2.18. Furthermore, we have $u^+ \leq |u| \leq |f|$, etc., so that each of these four integrals is finite.

We define the integral of a measurable function f with range in $[-\infty,\infty]$ to be

$$\int_E f\, d\mu = \int_E f^+\, d\mu - \int_E f^-\, d\mu ,$$

provided that at least one of the integrals on the right is finite. The integral on the left is then a number in $[-\infty,\infty]$.

2.25 <u>Theorem</u> Suppose f and $g \in L^1(\mu)$ and α and β are complex numbers. Then $\alpha f + \beta g \in L^1(\mu)$ and

$$\int_X (\alpha f + \beta g)\, d\mu = \alpha \int_X f\, d\mu + \beta \int_X g\, d\mu .$$

2.26 <u>Theorem</u> If $f \in L^1(\mu)$, then

$$\left| \int_X f\, d\mu \right| \leq \int_X |f|\, d\mu .$$

2.27 Lebesgue's Dominated Convergence Theorem

Suppose $\{f_n\}$ is a sequence of complex measurable functions on X such that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for every $x \in X$. If there is a function $g \in L^1(\mu)$ such that $|f_n(x)| \leq g(x)$ $(n = 1,2,3, \ldots, x \in X)$, then $f \in L^1(\mu)$, $\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0$,

and $\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$.

2.28 Definition Let P be a property which a point x may or may not have. If μ is a measure on a σ - algebra m and if $E \in m$, the statement "P holds almost everywhere on E" (P holds a.e on E) means that there exists an $N \in m$ such that $\mu(N) = 0$, $N \subset E$ and P holds at every point of E - N.

The concept of a.e depends of course very strongly on the given measure.

2.29 Theorem Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Then

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu .$$

2.30 <u>Theorem</u> Let $\sum_{n=1}^{\infty} f_n$ be a uniformly convergent series of continuous complex measurable functions on a topological space X which is also measurable, f is the sum of the series and $\mu(X) < \infty$ then

$$\int_X f\, d\mu = \int_X \sum_{n=1}^{\infty} f_n\, d\mu = \sum_{n=1}^{\infty} \int_X f_n\, d\mu .$$

2.31 <u>Definition</u> A linear transformation of a vector space V into a vector space V_1 is a mapping Λ of V into V_1 such that $\Lambda(\alpha x + \beta y) = \alpha\Lambda x + \beta\Lambda y$ for all x and y $\in$ V and for all scalars α and β. In the special case in which V_1 is the field of scalars, Λ is called a <u>linear functional</u>. A linear functional is thus a complex function on V which satisfies the above equality.

<u>Note</u> All vector spaces occuring in this thesis will be complex, with one notable exception : the euclidean spaces R^k are vector spaces over the real field.

2.32 <u>Definition</u> Let C be the set of all continuous complex-valued functions on X and $\Lambda : C \to \mathbb{C}$ is a linear functional. Λ is called a <u>positive linear functional</u> if for all real valued f $\in$ C

1) $\Lambda f \in R$

2) $\Lambda f \geqslant 0$ whenever $f \geqslant 0$.

2.33 Theorem Suppose X is a Hausdorff space, $K \subset X$, K is compact and $p \in K^c$ (the complement of K). Then there are open sets U and W such that $p \in U$, $K \subset W$ and $U \cap W = \phi$.

2.34 Theorem Suppose U is open in a locally compact Hausdorff space X, $K \subset U$, and K is compact. Then there is an open set V with compact closure such that

$$K \subset V \subset \overline{V} \subset U .$$

2.35 Definition The support of a complex function f on a topological space X is the closure of the set

$$\{x | f(x) \neq 0\} .$$

The collection of all continuous complex functions on X whose support is compact is denoted by $C_c(X)$.

2.36 Theorem The range of any $f \in C_c(X)$ is a compact subset of the complex plane.

2.37 Notation

The notation $K \prec f$ will mean that K is a compact subset of X, that $f \in C_c(X)$ and f is real valued function, that $0 \leq f(x) \leq 1 \quad \forall x \in X$ and that $f(x) = 1 \quad \forall x \in K$.

The notation $f \prec V$ will mean that V is open that $f \in C_c(X)$ and f is real valued function, $0 \leq f \leq 1$ and that the support of f lies in V.

The notation $K \prec f \prec V$ will be used to indicate that both above cases hold.

2.38 Urysohn's Lemma

Suppose X is a locally compact Hausdorff space, V is open in X, $K \subset V$ and K is compact. Then there exists an $f \in C_c(X)$ such that $K \prec f \prec V$.

2.39 Theorem Suppose $V_1, \ldots, V_n$ are open subsets of a locally compact Hausdorff space X, K is compact and $K \subset V_1 \cup V_2 \cup \ldots \cup V_n$. Then there exist functions $h_i \prec V_i$ $(i = 1, \ldots n)$ such that

$$h_1(x) + \ldots + h_n(x) = 1 \qquad (x \in K) \quad \ldots\ldots\ldots(1)$$

Since this theorem is not so well-known, then we will show an idea of the proof.

Proof : By Theorem 2.3A, each $x \in K$ has a neighborhood W_x with compact closure $\overline{W}_x \subset V_i$ for some i (depending on x). There are points $x_1, \ldots, x_m$ such that $W_{x_1} \cup \ldots \cup W_{x_m} \supset K$. If $1 \leq i \leq n$, let H_i be the union of those $\overline{W}_{x_j}$ which lie in V_i. By Urysohn's Lemma 2.38, there are functions g_i such that $H_i \prec g_i \prec V_i$. Define

$$h_1 = g_1$$

$$h_2 = (1 - g_1)\, g_2$$

....

$$h_n = (1 - g_1)(1 - g_2) \ldots (1 - g_{n-1})\, g_n .$$

Then $hi \prec Vi$.

By induction

$$h_1 + h_2 + \ldots h_n = 1-(1 - g_1)(1 - g_2) \ldots (1 - g_n) \quad .....(2)$$

Since $K \subset H_1 \cup \ldots \cup H_n$, at least one $g_i(x) = 1$ at each point $x \in K$; hence (2) shows that (1) holds.

We shall now state a Theorem whose proof we shall give in Chapter III.

2.40 <u>Theorem</u> Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(X)$. Then there exists a σ - algebra m in X which contains all Borel sets in X, and there exists a unique positive measure μ on m which represents Λ in the sense that

(a) $\Lambda f = \int_X f \, d\mu$ for every $f \in C_c(X)$ and which has

the following additional properties :

(b) $\mu(K) < \infty$ for every compact set $K \subset X$.

(c) For every $E \in m$, we have

$$\mu(E) = \inf \{ \mu(V) : E \subset V, V \text{ open} \}.$$

(d) The relation

$\mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compact}\}$ holds for every open set E, and for every $E \in m$ with $\mu(E) < \infty$.

(e) If $E \in m$, $A \subset E$, and $\mu(E) = 0$ then $A \in m$.

2.41 Definition A measure μ defined on the σ - algebra of all Borel sets in a locally compact Hausdorff space X is called a Borel measure on X.

2.42 Definition A set E in a topological space is called σ - compact if E is a countable union of compact sets. A set E in a measure space (with measure μ) is said to have σ - finite measure if E is a countable union of sets E_i with $\mu(E_i) < \infty$.

2.43 Definition If μ is positive, a Borel set $E \subset X$ is outer regular or inner regular, respectively, if E has property (c) or (d) of Theorem 2.40. If every Borel set in X is both outer and inner regular, μ is called regular.

2.44 Theorem Suppose X is a locally compact, σ - compact Hausdorff space. If m and μ are as described in the statement of Theorem 2.40 then m and μ have the following properties :

(a) If $E \in m$ and $\varepsilon > 0$, there is a closed set F and an open set V such that $F \subset E \subset V$ and $\mu(V - F) < \varepsilon$.

(b) μ is a regular Borel measure on X.

2.45 Definition If p and q are positive real numbers such that $p + q = pq$ or $\frac{1}{p} + \frac{1}{q} = 1$ then we call p and q a pair of conjugate exponents.

2.46 Definition Let X be an arbitrary measure space with a positive measure μ. If $1 \leq p < \infty$ and if f is a complex measurable function on X, define

$$||f||_p = \{ \int_X |f|^p \, d\mu \}^{1/p}$$

and let $L^p(\mu)$ consist of all f for which $||f||_p < \infty$. We call $||f||_p$ the L^p- norm of f.

2.47 Definition Suppose $g : X \to [0, \infty]$ is measurable. Let S be the set of all real α such that

$$\mu(g^{-1}((\alpha, \infty])) = 0 .$$

If $S = \phi$, put $\beta = \infty$.

If $S \neq \phi$, put $\beta = \inf S$. Since $g^{-1}((\beta, \infty]) = \bigcup_{n=1}^{\infty} g^{-1}((\beta + \frac{1}{n}, \infty])$, and since the union of a countable collection of sets of measure 0 has measure 0, we see that $\beta \in S$. We call β the essential supremum of g.

If f is a complex measurable function on X, we define $||f||_\infty$ to be the essential supremum of $|f|$ and we let $L^\infty(\mu)$ consist of all f for which $||f||_\infty < \infty$.

2.48 Theorem $L^p(\mu)$ is a complete metric space, for $1 \leq p \leq \infty$ and for every positive measure μ.

2.49 Theorem Let S be the class of all complex, measurable, simple functions on X such that for any $s \in S$

$$\mu(\{x \mid s(x) \neq 0\}) < \infty .$$

If $1 \leq p < \infty$, then S is dense in $L^p(\mu)$.

Now let X be a locally compact Hausdorff space, and let μ be a measure on a σ - algebra m in X, with the properties stated in Theorem 2.40.

2.50 Theorem For $1 \leq p < \infty$, $C_c(X)$ is dense in $L^p(\mu)$.

2.51 Definition A complex function f on a locally compact Hausdorff space X is said to vanish at infinity if to every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $|f(x)| < \varepsilon$ for all x not in K.

The class of all continuous functions f on X which vanish at infinity is called $C_o(X)$.

Therefore $C_c(X) \subset C_o(X)$ and the two classes coincide if X is compact. In that case we write C(X) for either of them.

2.52 <u>Theorem</u> If X is a locally compact Hausdorff space, then $C_o(X)$ is the completion of $C_c(X)$, with respect to the metric defined by the supremum norm

$$||f|| = \sup_{x \in X} |f(x)| .$$

2.53 <u>Definition</u> Consider a linear transformation Λ from a normed linear space X into a normed linear space Y and define its norm by

$$||\Lambda|| = \sup \left\{ \frac{||\Lambda x||}{||x||} : x \in X, x \neq 0 \right\}$$

if $||\Lambda|| < \infty$ then Λ is called a <u>bounded linear transformation.</u>

2.54 <u>Hahn - Banach Theorem</u> If M is a subspace of a normed linear space X and if f is a bounded linear functional on M, then f can be extended to a bounded linear functional F on X so that $||F|| = ||f||$.

2.55 <u>Theorem</u> Let M be a linear subspace of a normed linear space X, and let $x_o \in X$. Then x_o is in the closure $\overline{M}$ of M iff there is no bounded linear functional f on X such that $f(x) = 0 \;\forall x \in M$ but $f(x_o) \neq 0$.

<u>Complex Measures</u>

2.56 <u>Definition</u> Let m be a σ - algebra in a set X. Call a countable collection $\{E_i\}_{i\epsilon I}$ of members of m a <u>partition</u> of E if $E_i \cap E_j = \phi$ whenever $i \neq j$ and if $E = \bigcup_{i \epsilon I} E_i$.

A <u>complex measure</u> μ on m is then a complex function on m such that

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i) \qquad (E \in m) \text{ for every partition } \{E_i\} \text{ of } E.$$

2.57 <u>Definition</u> Let μ be a complex measure on the σ - algebra m . We define a set function $|\mu|$ on m by

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)| \qquad (E \in m) ,$$

the supremum being taken over all partitions $\{E_i\}$ of E. Note that $|\mu|$ (E) $\geqslant$ $|\mu$ (E)$|$ but in general $|\mu|$ (E) is not equal to $|\mu$ (E)$|$. The set function $|\mu|$ is called the <u>total variation of μ</u> . If μ is a positive measure, then $|\mu| = \mu$.

2.58 <u>Theorem</u> The total variation $|\mu|$ of complex measure μ on m is a positive measure on m .

2.59 <u>Theorem</u> If μ is a complex measure on X, then $|\mu|$ (X) $< \infty$.

2.60 <u>Definition</u> If μ and λ are complex measures on the same σ - algebra m , we define $\mu + \lambda$ and $c\mu$ by

$$(\mu + \lambda)(E) = \mu(E) + \lambda(E)$$

$$(c\mu)(E) = c\mu\ (E)$$

where $E \in m$ and c is any complex number $\mu + \lambda$ and $c\mu$ are complex measure.

2.61 <u>Definition</u> Define $|\mu|$ as before and define

$$\mu^{+} = \frac{1}{2}(|\mu| + \mu), \quad \mu^{-} = \frac{1}{2}(|\mu| - \mu).$$

Then both μ^{+} and μ^{-} are positive measures on m, and they are bounded, by Theorem 2.58.

Also, $\mu = \mu^{+} - \mu^{-}$, $|\mu| = \mu^{+} + \mu^{-}$.

The measures μ^{+} and μ^{-} are called the <u>positive</u> and <u>negative</u> <u>variations of μ</u>, respectively. This representation of μ as the difference of the positive measures μ^{+} and μ^{-} is known as the <u>Jordan decomposition of μ</u>.

2.62 <u>Theorem</u> Let μ be a complex measure on a σ - algebra m in X. Then there is a complex valued measurable function h such that $|h(x)| = 1$ for all $x \in X$ and such that $d\mu = h\, d|\mu|$.

2.63 <u>Theorem</u> Suppose μ is a positive measure on m, $g \in L^{1}(\mu)$ and

$$\lambda(E) = \int_E g\, d\mu \qquad (E \in m).$$

Then

$$|\lambda|(E) = \int_E |g|\, d\mu \qquad (E \in m).$$

2.64 <u>Theorem</u> Suppose $1 \leqslant p < \infty$, μ is a σ - finite positive measure on X, and Φ is a bounded linear functional on $L^{p}(\mu)$. Then there is a unique $g \in L^{q}(\mu)$ where q is the exponent conjugate to p, such that

$$\Phi(f) = \int_X fg\, d\mu \qquad (f \in L^p(\mu)).$$

Moreover, if Φ and g are related as above, we have

$$||\Phi|| = ||g||_q .$$

2.65 Definition Let X be a locally compact Hausdorff space and μ is a complex Borel measure, we define integration with respect to a complex measure μ by the formula

$$\int f\, d\mu = \int f\, h\, d|\mu| .$$

2.66 Definition A complex Borel measure μ on X is regular if $|\mu|$ is regular.

Integration on Product Spaces

2.67 Definition If X, Y are two sets, we define the set $X \times Y = \{(x, y) | x \in X, y \in Y\}$. If $A \subset X$ and $B \subset Y$, it follows that $A \times B \subset X \times Y$. We call any set of the form $A \times B$ a rectangle in $X \times Y$.

Suppose (X, δ) and $(Y, \mathfrak{J})$ are measurable space. A measurable rectangle is any set of the form $A \times B$, where $A \in \delta$ and $B \in \mathfrak{J}$.

If $Q = R_1 \cup \ldots \cup R_n$, where each R_i is a measurable rectangle and $R_i \cap R_j = \phi$ for $i \neq j$, we say that $Q \in \mathscr{E}$ the class of elementary sets.

$\delta \times \Im$ is defined to be the smallest σ - algebra in $X \times Y$ which contains every measurable rectangle.

<u>A monotone class m</u> is a collection of sets with the following properties :

If $A_i \in m$, $B_i \in m$ such that $A_i \subset A_{i+1}$, $B_i \supset B_{i+1}$ for $i = 1,2,3, \ldots$ and if

$$A = \bigcup_{i=1}^{\infty} A_i \, , \, B = \bigcap_{i=1}^{\infty} B_i$$

then $A \in m$ and $B \in m$.

If $E \subset Y \times Y$, $x \in X$, $y \in Y$, we define

$$E_x = \{y | (x, y) \in E\}$$

$$E^y = \{x | (x, y) \in E\} .$$

We call E_x the <u>x - section</u> of E and E^y the <u>y - section</u> of E.

2.68 <u>Theorem</u> If $E \in \delta \times \Im$, then $E_x \in \Im$ and $E^y \in \delta$, for every $x \in X$ and $y \in Y$.

2.69 <u>Theorem</u> $\delta \times \Im$ is the smallest monotone class which contains all the elementary sets.

2.70 <u>Definition</u> Let f be an extended real valued function defined on $X \times Y$. For $x \in X$ we associate a function f_x defined on Y by $f_x(y) = f(x, y)$. Similarly for $y \in Y$, f^y is the function defined on X by $f^y(x) = f(x, y)$.

2.71 <u>Theorem</u> Let f be an $(\delta \times \mathfrak{J})$ - measurable function on $X \times Y$. Then the following hold:

(a) For each $x \in X$, f_x is a $\mathfrak{J}$ - measurable function.

(b) For each $y \in Y$, f^y is a δ - measurable function.

2.72 <u>Theorem</u> Let (X,δ,μ) and $(Y,\mathfrak{J},\lambda)$ be σ - finite measure spaces. Suppose $Q \in \delta \times \mathfrak{J}$.

If $\varphi(x) = \lambda(Q_x)$, $\psi(y) = \mu(Q^y)$ for every $x \in X$ and $y \in Y$, then φ is δ - measurable, ψ is $\mathfrak{J}$ - measurable and $\int_X \varphi d\mu = \int_Y \psi d\mu$.

2.73 <u>Definition</u> If (X,δ,μ) and $(Y,\mathfrak{J},\lambda)$ are σ - finite measure spaces and if $Q \in \delta \times \mathfrak{J}$, we define

$$(\mu \times \lambda)(Q) = \int_X \lambda(Q_x)d\mu\,(x) = \int_Y \mu(Q^y)d\lambda\,(y).$$

We call $\mu \times \lambda$ the product of the measure μ and λ. $\mu \times \lambda$ is a measure.

2.74 <u>The Fubini Theorem</u>

Let (X,δ,μ) and $(Y,\mathfrak{J},\lambda)$ be σ - finite measure spaces, and let f be an $(\delta \times \mathfrak{J})$ measurable function on $X \times Y$.

(a) If $0 \leq f \leq \infty$ and if

(1) $\varphi(x) = \int_Y f_x d\lambda$, $\psi(y) = \int_X f^y d\mu$ $(x \in X,\ y \in Y)$,

then φ is $\mathcal{S}$ - measurable, ψ is $\mathfrak{J}$ - measurable, and

$$(2)\quad \int_X \varphi \, d\mu = \int_{X\times Y} f \, d\,(\mu \times \lambda) = \int_Y \psi \, d\lambda .$$

(b) If f is complex and if

$$(3)\quad \varphi^*(x) = \int_Y |f|_x \, d\lambda \quad \text{and} \quad \int_X \varphi^* \, d\mu < \infty ,$$

then $f \in L^1(\mu \times \lambda)$.

(c) If $f \in L^1(\mu \times \lambda)$ then $f_x \in L^1(\lambda)$ for almost all $x \in X$, $f^y \in L^1(\mu)$ for almost all $y \in Y$, the functions φ and ψ, defined by (1) a.e. are in $L^1(\mu)$ and $L^1(\lambda)$ respectively and (2) holds.

Notes The first and the last integrals in (2) can also be written in the more usual form

$$(4)\quad \int_Y d\mu\,(x) \int_Y f(x, y)\, d\lambda\,(y) = \int_Y d\lambda\,(y) \int_X f(x,y)\, d\mu(x).$$

These are the so - called "**iterated integrals**" of f.

The middle integral in (2) is often referred to as a double integral.

The combination of (b) and (c) gives the following useful result :

If f is $(\mathcal{S} \times \mathfrak{J})$ - measurable and if

$$\int_X d\mu\,(x) \int_Y |f(x, y)|\, d\lambda\,(y) < \infty ,$$

then the two iterated integrals (4) are finite and equal. In other words "the order of integration may be reversed" for $(\mathcal{S} \times \mathfrak{J})$ - measurable functions f whenever $f \geqslant 0$ and also whenever one of the iterated integrals of $|f|$ is finite.

2.75 Definition If $r > 0$ and a is a complex number, $D(a, r) = \{z \mid |z - a| < r\}$ is the open circular disc with centre at a and radius r

$D'(a,r) = \{z \mid 0 < |z - a| < r\}$ is the punctured disc with center at a and radius r.

2.76 Definition A maximal connected subset of E is called a component of E.

A region is a nonempty connected open subset of the complex plane. Each component of a plane open set Ω is a region.

2.77 Definition Suppose f is a complex function define in Ω (plane open set). If the derivative of f denoted by $f'(z)$ exists for every $z \in \Omega$ then f is analytic in Ω. The class of analytic functions in Ω will be denoted by $H(\Omega)$.

2.78 Definition A function f defined in Ω is said to be representable by a power series in Ω if to every disc $D(a, r) \subset \Omega$ there corresponds a series $\sum_{n=0}^{\infty} c_n(z - a)^n$ which converges to $f(z)$ for all $z \in D(a,r)$.

2.79 Theorem If f is representable by power series in Ω then $f \in H(\Omega)$.

2.80 <u>Theorem</u> Suppose μ is a complex (finite) measure on a measurable space X, φ is a complex measurable function on X, Ω is an open set in the plane which does not intersect $\varphi(X)$, and

$$f(z) = \int_X \frac{d\mu\ (\xi)}{\varphi(\xi) - z} \qquad (z \in \Omega).$$

Then f is representable by power series in Ω .

2.81 <u>Definition</u> If X is a topological space, a <u>curve</u> in X is a continuous mapping γ of a compact interval $[\alpha, \beta] \subset R^1$ into X ; here $\alpha < \beta$.

We call $[\alpha, \beta]$ the <u>parameter interval</u> of γ and denote the <u>range of γ</u> by γ^* . Thus γ is a mapping , and γ^* is the set of all points $\gamma(t)$ for $\alpha \leq t \leq \beta$.

If the initial point $\gamma(\alpha)$ of γ coincides with its end point $\gamma(\beta)$, we call γ a <u>closed curve</u>.

2.82 <u>Definition</u> A <u>path</u> is a piecewise continuously differentiable curve in the plane. More explicitly, a path with parameter interval $[\alpha, \beta]$ is a continuous complex function γ on $[\alpha, \beta]$, such that the following holds : There are finitely many points s_j , $\alpha = s_0 < s_1 \ldots < s_n = \beta$, and the restriction of γ to each interval $[s_{j-1}, s_j]$ has a continuous derivative on $[s_{j-1}, s_j]$: however, at the points $s_1, \ldots s_{n-1}$ the left - and right - hand derivatives of γ may differ.

A <u>closed path</u> is a closed curve which is also a path. Suppose γ is a path and f is continuous function on γ^*. The integral of f over γ is defined as an integral over the parameter interval $[\alpha, \beta]$ of γ:

$$\int_\gamma f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\,\gamma'(t)\,dt.$$

2.83 <u>Definition</u> If a is a complex number and $r > 0$, the path defined by $\gamma(t) = a + re^{it}$ $(0 \leq t \leq 2\pi)$ is called the <u>positively oriented circle</u> with center at a and radius r.

2.84 <u>Definition</u> If a and b are complex numbers the path γ given by

$$\gamma(t) = a + (b - a)t \qquad (0 \leq t \leq 1)$$

is called the <u>oriented interval $[a, b]$</u>; its length is $|b - a|$.

2.85 <u>Theorem</u> Let γ be a closed path, let Ω be the complement of γ^* (relative to the plane), and define $\mathrm{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\xi}{\xi - z}$ $(z \in \Omega)$. Then Ind_γ is an integer - valued function on Ω which is constant in each component of Ω and which is 0 in the unbounded component.

2.86 <u>Theorem</u> If γ is the positively oriented circle with center at a and radius r, then

$$\mathrm{Ind}_\gamma(z) = \begin{cases} 1 & \text{if } |z - a| < r. \\ 0 & \text{if } |z - a| > r. \end{cases}$$

2.87 Theorem For every open set Ω in the plane, every $f \in H(\Omega)$ is representable by power series in Ω.

2.88 Theorem Suppose Ω is a region, $f \in H(\Omega)$, and $Z(f) = \{a \in \Omega \mid f(a) = 0\}$. Then either $Z(f) = \Omega$ or $Z(f)$ has no limit point in Ω. In the latter case there corresponds to each $a \in Z(f)$ a unique positive integer $m = m(a)$ such that $f(z) = (z - a)^m g(z) \quad (z \in \Omega)$, where $g \in H(\Omega)$ and $g(a) \neq 0$, furthermore, $Z(f)$ is at most countable.

2.89 Definition If $a \in \Omega$ and $f \in H(\Omega - \{a\})$, then f is said to have an isolated singularity at the point a. If f can be so defined at a that the extended function is analytic in Ω, the singularity is said to be removable.

2.90 Theorem If $a \in \Omega$ and $f \in H(\Omega - \{a\})$ then one of the following three cases must occur :

(a) f has a removable singularity at a.

(b) There are complex number $c_1, \ldots, c_m$, where m is a positive integer and $c_m \neq 0$, such that

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z - a)^k}$$ has a removable singularity at a.

(c) If $r > 0$ and $D(a, r) \subset \Omega$, then $f(D'(a, r))$ is dense in the plane.

2.91 Theorem Every open set Ω in the plane is the union of a sequence $\{K_n\}$, $n = 1,2,3, \ldots$ of compact sets such that

(a) K_n lies in the interior of K_{n+1} for $n = 1,2, \ldots$

(b) Every compact subset of Ω lies in some K_n .

(c) Let S^2 be the extended complex plane then every component of $S^2 - K_n$ contains a component of $S^2 - \Omega$, for $n = 1,2,3, \ldots$

2.92 Theorem Suppose a and b are complex numbers, $b \neq 0$ and γ is the path consisting of the oriented intervals $[a + i^n b , a + i^{n+1} b]$ $(n = 0,1,2,3)$. Then $\mathrm{Ind}_\gamma(z) = 1$ for every z in the interior of the square with vertices at the points $a + i^n b$ $(n = 0,1,2,3)$,.

2.93 Theorem If K is a compact subset of a plane open set Ω , there exist oriented line intervals $\gamma_1, \ldots, \gamma_n$ in $\Omega - K$ such that the Cauchy formula

$$f(z) = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(\xi)}{\xi - z} d\xi \quad \text{holds for every}$$

$f \in H(\Omega)$ and every $z \in K$.