CHAPTER III

THE SCATTERING OF A PLANE WAVE FROM A CYLINDER

Suppose we have a plane wave which is incident on a circular object and is scattered by the object without change in frequency. The complete wave, satisfying the requirement that at large distances from the object it consists of a plane wave in the positive x-direction plus an outgoing radial wave or scattered wave, and that its normal gradient is zero at $r = a$, has the general form

$$\psi = A\,\psi_{inc}(r, \phi)\, e^{-i\omega t} + \psi_s(r, \phi)\, e^{-i\omega t}, \quad \ldots\ldots\ldots\ldots\ldots 1)$$

where $\psi_{inc}(r, \phi)$ is the incident wave,

$\psi_s(r, \phi)$ is the time independent scattered wave which satisfies the Helmholtz's equation,

and A is the amplitude of the incident wave.

Carrying through the straightforward separation of variables we obtain the following differential equations for the time independent scattered wave $R(r)\,\Phi(\phi)$

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left(k^2 - \frac{n^2}{r^2}\right) R(r) = 0 \quad \ldots\ldots\ldots\ldots 2)$$

and

$$\frac{d^2\Phi(\phi)}{d\phi^2} + n^2\,\Phi(\phi) = 0 \quad \ldots\ldots\ldots\ldots 3)$$

By the symmetry of the waves we shall require only solutions of 3) of the form

$$\Phi(\phi) = B_n \cos n\phi \qquad \ldots\ldots\ldots\ldots\ldots\ldots 4)$$

where n is an integer to ensure that $\Phi(2\pi - \phi) = \Phi(\phi)$.

In order to obtain solutions that represent circular waves we must use the following general solution of 2)

$$R(r) = C_n H_n^{(1)}(kr) + D_n H_n^{(2)}(kr) \qquad \ldots\ldots\ldots\ldots 5)$$

where $H_n^{(1)}(kr)$ and $H_n^{(2)}(kr)$ are the Hankel functions of the first and second kind respectively. We find by using the asymptotic form of the Hankel functions as r approaches infinity that at a large distance from the scattering circle $R(r)\,e^{-i\omega t}$ approaches

$$C_n \left(\frac{2}{\pi kr}\right)^{1/2} e^{i(kr - \frac{n\pi}{2} - \frac{\pi}{4} - \omega t)} + D_n \left(\frac{2}{\pi kr}\right)^{1/2} e^{-i(kr - \frac{n\pi}{2} - \frac{\pi}{4} + \omega t)}.$$

Since the solution must represent outgoing waves** only we put $D_n = 0$. Thus we obtain the following time independent scattered wave function

$$\psi_s(r,\phi) = \sum_{n=0}^{\infty} B_n H_n^{(1)}(kr) \cos n\phi \qquad \ldots\ldots\ldots\ldots\ldots 6)$$

by putting $C_n = 1$ in equation 5). Therefore the complete time independent wave is

$$u = \sum_{n=0}^{\infty} \left[A\,\epsilon_n i^n J_n(kr) + B_n H_n^{(1)}(kr) \right] \cos n\phi \qquad \ldots\ldots 7)$$

where $\epsilon_n = 1$ when $n = 0$

$= 2$ when $n > 0$.

** Morse Philip M., Feshbach Herman. Methods of Theoretical Physics Part II (McGraw-Hill Book Company, Inc. 1953) pp.1377.

Consider the derivative of u

$$\frac{\partial u}{\partial r} = \sum_{n=0}^{\infty} \left[A\, \epsilon_n i^n \frac{d}{dr} J_n(kr) + B_n \frac{d}{dr} H_n^{(1)}(kr) \right] \cos n\phi$$

$$= \sum_{n=0}^{\infty} \left[A\, \epsilon_n i^n \frac{1}{2} \left\{ J_{n-1}(kr) - J_{n+1}(kr) \right\} + \frac{1}{2} B_n \left\{ H_{n-1}^{(1)}(kr) - H_{n+1}^{(1)}(kr) \right\} \right] \cos n\phi.$$

We shall assume a physical boundary condition of the form $\frac{\partial u}{\partial r} = 0$ at $r = a$; therefore

$$\frac{1}{2} A\, \epsilon_n i^n \left[J_{n-1}(ka) - J_{n+1}(ka) \right] + \frac{1}{2} B_n \left[H_{n-1}^{(1)}(ka) - H_{n+1}^{(1)}(ka) \right] = 0$$

that is $$B_n = \frac{A\, \epsilon_n i^{n-2} \left[J_{n-1}(ka) - J_{n+1}(ka) \right]}{H_{n-1}^{(1)}(ka) - H_{n+1}^{(1)}(ka)} \qquad \ldots\ldots\ldots\ldots\ldots 8)$$

Therefore the complete wave has the general form

$$U = A\, e^{ikx - i\omega t} + \psi_s(r, \phi)\, e^{-i\omega t}, \qquad \ldots\ldots\ldots\ldots\ldots 9)$$

where $$\psi_s = A \sum_{n=0}^{\infty} \epsilon_n i^{n-2} \frac{\left[J_{n-1}(ka) - J_{n+1}(ka) \right] H_n^{(1)}(kr) \cos n\phi}{\left[H_{n-1}^{(1)}(ka) - H_{n+1}^{(1)}(ka) \right]} \ldots 10)$$

Numerical Computation of scattered waves

We shall illustrate the calculation of the scattered wave by using a partial sum of ψ_s.

Letting $k = 1$, $a = 6$, we have from 10)

$$\psi_s = \frac{\sum\limits_{n=0}^{\infty} A\,\epsilon_n i^{n-2} \left[J_{n-1}(6) - J_{n+1}(6) \right]}{H^{(1)}_{n-1}(6) - H^{(1)}_{n+1}(6)} \cdot H^{(1)}_n(r) \cos n\phi \ldots\ldots 11)$$

Without loss of generality we may assume the amplitude of the incident wave to be unity. Since

$$H^{(1)}_n(r) = J_n(r) + i\,Y_n(r) \quad \ldots\ldots\ldots\ldots\ldots\ldots 12)$$

and

$$J_{-n}(r) = (-1)^n J_n(r) \;, \; Y_{-n}(r) = (-1)^n Y_n(r) \ldots\ldots 13)$$

we obtain the following formula for the m-th partial sum of 11)

$$(\psi_s)_m = \sum_{n=0}^{m} \frac{\epsilon_n i^{n-2} \left[J_{n-1}(6) - J_{n+1}(6) \right] \left[J_n(r) + iY_n(r) \right] \cos n\phi}{\left[J_{n-1}(6) - J_{n+1}(6) \right] + i \left[Y_{n-1}(6) - Y_{n+1}(6) \right]}$$

$$\ldots\ldots\ldots\ldots\ldots 14)$$

Let $m = 8$. We have

$$(\psi_s)_8 = \frac{\left[i^{-2} \;\; J_{-1}(6) - J_1(6) \right] \left[J_o(r) + i\,Y_o(r) \right]}{\left[J_{-1}(6) - J_1(6) \right] + \left[Y_{-1}(6) - Y_1(6) \right]}$$

$$+ \frac{2\,i^{-1} \left[J_o(6) - J_2(6) \right] \left[J_1(r) + i\,Y_1(r) \right] \cos \phi}{\left[J_o(6) - J_2(6) \right] + i \left[Y_o(6) - Y_2(6) \right]}$$

$$+ \frac{2i^{o} \left[J_1(6) - J_3(6) \right] \left[J_2(r) + i\,Y_2(r) \right] \cos 2\phi}{\left[J_1(6) - J_3(6) \right] + i \left[Y_1(6) - Y_3(6) \right]}$$

$$+ \frac{2i^{1} \left[J_2(6) - J_4(6) \right] \left[J_3(r) + i\,Y_3(r) \right] \cos 3\phi}{\left[J_2(6) - J_4(6) \right] + i \left[Y_2(6) - Y_4(6) \right]}$$

$$+ \frac{2i^2\left[J_3(6) - J_5(6)\right]\left[J_4(r) + i\,Y_4(r)\right] \cos 4\phi}{\left[J_3(6) - J_5(6)\right] + i\left[Y_3(6) - Y_5(6)\right]}$$

$$+ \frac{2i^3\left[J_4(6) - J_6(6)\right]\left[J_5(r) + i\,Y_5(r)\right] \cos 5\phi}{\left[J_4(6) - J_6(6)\right] + i\left[Y_4(6) - Y_6(6)\right]}$$

$$+ \frac{2i^4\left[J_5(6) - J_7(6)\right]\left[J_6(r) + i\,Y_6(r)\right] \cos 6\phi}{\left[J_5(6) - J_7(6)\right] + i\left[Y_5(6) - Y_7(6)\right]}$$

$$+ \frac{2i^5\left[J_6(6) - J_8(6)\right]\left[J_7(r) + i\,Y_7(r)\right] \cos 7\phi}{\left[J_6(6) - J_8(6)\right] + i\left[Y_6(6) - Y_8(6)\right]}$$

$$+ \frac{2i^6\left[J_7(6) - J_9(6)\right]\left[J_8(r) + i\,Y_8(r)\right] \cos 8\phi}{\left[J_7(6) - J_9(6)\right] + i\left[Y_7(6) - Y_9(6)\right]}$$

$$= \frac{2(-0.2767)\left[J_o(r) + i\,Y_o(r)\right]}{\left[-2\,(-0.2767)\right] + i\left[-2\,(-0.1750)\right]}$$

$$- \frac{2i\left[0.1506 - (-0.2429)\right]\left[J_1(r) + i\,Y_1(r)\right] \cos\phi}{\left[0.1506 - (-0.2429)\right] + i\left[-0.2882 - 0.2299\right]}$$

$$+ \frac{2\left[-0.2767 - 0.1148\right]\left[J_2(r) + i\,Y_2(r)\right] \cos 2\phi}{\left[-0.2767 - 0.1148\right] + i\left[-0.1750 - 0.3283\right]}$$

$$+ \frac{2i\left[-0.2429 - 0.3576\right]\left[J_3(r) + i\,Y_3(r)\right] \cos 3\phi}{\left[-0.2429 - 0.3576\right] + i\left[0.2299 - 0.0984\right]}$$

$$- \frac{2\left[0.1148 - 0.3621\right]\left[J_4(r) + i\,Y_4(r)\right] \cos 4\phi}{\left[0.1148 - 0.3621\right] + i\left[0.3283 - (-0.1971)\right]}$$

$$- \frac{2i\,[0.3576 - 0.2458]\,[J_5(r) + i\,Y_5(r)]\,\cos 5\phi}{[0.3576 - 0.2458] + i\,[0.0984 - (-0.4268)]}$$

$$+ \frac{2\,[0.3621 - 0.1296]\,[J_6(r) + i\,Y_6(r)]\,\cos 6\phi}{[0.3621 - 0.1296] + i\,[-0.1971 - (-0.6566)]}$$

$$+ \frac{2i\,[0.2458 - 0.0565]\,[J_7(r) + i\,Y_7(r)]\,\cos 7\phi}{[0.2458 - 0.0565] + i\,[-0.4268 - (-1.1052)]}$$

$$- \frac{2\,[0.1296 - 0.0212]\,[J_8(r) + i\,Y_8(r)]\,\cos 8\phi}{[0.1296 - 0.0212] + i\,[-0.6566 - (-2.2907)]}$$

$$= - \frac{0.5534}{[0.5534 + 0.3500i]}\,[J_o(r) + i\,Y_o(r)]$$

$$- \frac{0.7870\,i}{[0.3935 - 0.5181\,i]}\,[J_1(r) + i\,Y_1(r)]\,\cos\phi$$

$$- \frac{0.7829}{[-0.3914 - 0.5033\,i]}\,[J_2(r) + i\,Y_2(r)]\,\cos 2\phi$$

$$- \frac{1.2010}{[-0.6005 + 0.1315\,i]}\,[J_3(r) + i\,Y_3(r)]\,\cos 3\phi$$

$$+ \frac{0.4946}{[-0.2473 + 0.5254\,i]}\,[J_4(r) + i\,Y_4(r)]\,\cos 4\phi$$

$$- \frac{0.2236\,i}{[0.1118 + 0.5252\,i]}\,[J_5(r) + i\,Y_5(r)]\,\cos 5\phi$$

$$+ \frac{0.4650}{[0.2325 + 0.4595\,i]}\,[J_6(r) + i\,Y_6(r)]\,\cos 6\phi$$

$$+ \frac{0.3786\ i}{0.1893 + 0.6784\ i} \left[J_7(r) + i\ Y_7(r)\right] \cos 7\phi$$

$$- \frac{0.2168}{0.1084 + 1.6341\ i} \left[J_8(r) + i\ Y_8(r)\right] \cos 8\phi$$

$$= - \frac{(0.5534 \times 0.5534) - (0.5534 \times 0.3500\ i)}{(0.5534)^2 + (0.3500)^2} \cdot \left[J_o(r) + i\ Y_o(r)\right]$$

$$- \frac{(0.7870\ i \times 0.3935) + (0.7870\ i \times 0.5181\ i)}{(0.3935)^2 + (0.5181)^2} \left[J_1(r) + iY_1(r)\right] \cos \phi$$

$$+ \frac{(0.7829 \times 0.3914) - (0.7829 \times 0.5033i)}{(0.3914)^2 + (0.5181)^2} \left[J_2(r) + i\ Y_2(r)\right] \cos 2\phi$$

$$+ \frac{(1.2010 \times 0.6005) + (1.2010 \times 0.1315i)}{(0.6005)^2 + (0.1315)^2} \left[J_3(r) + i\ Y_3(r)\right] \cos 3\phi$$

$$- \frac{(0.4946 \times 0.2473) + (0.4946 \times 0.5254i)}{(0.2473)^2 + (0.5254)^2} \left[J_4(r) + iY_4(r)\right] \cos 4\phi$$

$$- \frac{(0.2236i \times 0.1118) - 0.2236i \times 0.5252i}{(0.1118)^2 + (0.5252)^2} \left[J_5(r) + iY_5(r)\right] \cos 5\phi$$

$$+ \frac{(0.4650 \times 0.2325) - (0.4650 \times 0.4595i)}{(0.2325)^2 + (0.4595)^2} \left[J_6(r) + iY_6(r)\right] \cos 6\phi$$

$$+ \frac{(0.3786i \times 0.1893) - (0.3786i \times 0.6784i)}{(0.1893)^2 + (0.6784)^2} \left[J_7(r) + iY_7(r)\right] \cos 7\phi$$

$$- \frac{(0.2168 \times 0.1084) - (0.2168 \times 1.6341i)}{(0.1084)^2 + (0.6784)^2} \left[J_8(r) + iY_8(r)\right] \cos 8\phi$$

$$= -\frac{0.3063 - 0.1937\, i}{0.3063 + 0.1225}\left[J_o(r) + i\, Y_o(r)\right]$$

$$+ \frac{0.4077 - 0.3097\, i}{0.1548 + 0.2684}\left[J_1(r) + i\, Y_1(r)\right] \cos \phi$$

$$+ \frac{0.3064 - 0.3940\, i}{0.1532 + 0.2533}\left[J_2(r) + i\, Y_2(r)\right] \cos 2\phi$$

$$+ \frac{0.7212 + 0.1579\, i}{0.3606 + 0.0173}\left[J_3(r) + i\, Y_3(r)\right] \cos 3\phi$$

$$- \frac{0.1223 + 0.2599\, i}{0.0612 + 0.2760}\left[J_4(r) + i\, Y_4(r)\right] \cos 4\phi$$

$$- \frac{0.1174 + 0.0250\, i}{0.0125 + 0.2758}\left[J_5(r) + i\, Y_5(r)\right] \cos 5\phi$$

$$+ \frac{0.1081 - 0.2137\, i}{0.0541 + 0.2111}\left[J_6(r) + i\, Y_6(r)\right] \cos 6\phi$$

$$+ \frac{0.2568 + 0.0717\, i}{0.0358 + 0.4602}\left[J_7(r) + i\, Y_7(r)\right] \cos 7\phi$$

$$- \frac{0.0235 - 0.3543\, i}{0.0118 + 0.4602}\left[J_8(r) + i\, Y_8(r)\right] \cos 8\phi$$

$$= (-0.7143 + 0.4517\, i)(J_o(r) + i\, Y_o(r) +$$

$$+ (0.9634 - 0.7318\, i)(J_1(r) + i\, Y_1(r)) \cos \phi$$

$$+ (0.7538 - 0.9692\, i)(J_2(r) + i\, Y_2(r) \cos 2\phi$$

$$+ (1.9084 + 0.4178\, i)(J_3(r) + i\, Y_3(r) \cos 3\phi$$

$$+ (-0.3627 - 0.7708\ i)(J_4(r) + i\ Y_4(r))\ \cos 4\phi$$

$$+ (-0.4073 - 0.0867\ i)(J_5(r) + i\ Y_5(r)\ \cos 5\phi$$

$$+ (0.4077 - 0.8057\ i)(J_6(r) + i\ Y_6(r)\ \cos 6\phi$$

$$+ (0.5178 + 0.1445\ i)(J_7(r) + i\ Y_7(r)\ \cos 7\phi$$

$$+ (-0.0088 + 0.1321\ i)(J_8(r) + i\ Y_8(r)\ \cos 8\phi$$

$$= \Big[-0.7143\ J_o(r) + 0.9634\ J_1(r)\ \cos\phi + 0.7538\ J_2(r)\ \cos 2\phi$$

$$+ 1.9084\ J_3(r)\ \cos 3\phi - 0.3627\ J_4(r)\ \cos 4\phi$$

$$- 0.4073\ J_5(r)\ \cos 5\phi + 0.4077\ J_6(r)\ \cos 6\phi$$

$$+ 0.5178\ J_7(r)\ \cos 7\phi - 0.0088\ J_8(r)\ \cos 8\phi$$

$$- 0.4517\ Y_o(r) + 0.7318\ Y_1(r)\ \cos\phi + 0.9692\ Y_2(r)\ \cos 2\phi$$

$$- 0.4178\ Y_3(r)\ \cos 3\phi + 0.7708\ Y_4(r)\ \cos 4\phi$$

$$+ 0.0867\ Y_5(r)\ \cos 5\phi + 0.8057\ Y_6(r)\ \cos 6\phi$$

$$- 0.1445\ Y_7(r)\ \cos 7\phi - 0.1321\ Y_8(r)\ \cos 8\phi\Big]$$

$$+ i\Big[0.4517\ J_o(r) - 0.7318\ J_1(r)\ \cos\phi - 0.9692\ J_2(r)\ \cos 2\phi$$

$$+ 0.4178\ J_3(r)\ \cos 3\phi - 0.7708\ J_4(r)\ \cos 4\phi$$

$$- 0.0867\ J_5(r)\ \cos 5\phi - 0.8057\ J_6(r)\ \cos 6\phi$$

$$+ 0.1445\ J_7(r)\ \cos 7\phi + 0.1321\ J_8(r)\ \cos 8\phi$$

$$- 0.7143\ Y_o(r) + 0.9634\ Y_1(r)\ \cos\phi$$

$$+ 0.7538\ Y_2(r)\ \cos 2\phi + 1.9084\ Y_3(r)\ \cos 3\phi$$

$$- 0.3627\ Y_4(r)\ \cos 4\phi - 0.4073\ Y_5(r)\ \cos 5\phi$$

$$+ 0.4077\ Y_6(r) \cos 6\phi + 0.5178\ Y_7(r) \cos 7\phi$$
$$- 0.0088\ Y_8(r) \cos 8\phi \quad \ldots\ldots\ldots\ldots \quad 15)$$

Consider the scattering of water waves from a vertical cylinder dipping through the water surface. Suppose a straight wave with wavelength equal to 2π is incident on the cylinder whose diameter is twelve units, which is almost two times the wavelength of incident wave. We want to see the outgoing and reflected waves near the cylinder. We shall approximate the scattered wave ψ_s in equation 10)by using the partial sum to m terms given by equation 14).

Let the partial sum for the scattered wave contain nine terms. Suppose that the wave is represented by the real part of equation 15), that is

$$\begin{aligned} Rl\ (\psi_s)_8 = & - 0.7143\ J_0(r) + 0.9634\ J_1(r) \cos \phi + \\ & 0.7538\ J_2(r) \cos 2\phi + 1.9084\ J_3(r) \cos 3\phi \\ & - 0.3627\ J_4(r) \cos 4\phi - 0.4073\ J_5(r) \cos 5\phi \\ & + 0.4077\ J_6(r) \cos 6\phi + 0.5178\ J_7(r) \cos 7\phi \\ & - 0.0088\ J_8(r) \cos 8\phi - 0.4517\ Y_0(r) \\ & + 0.7318\ Y_1(r) \cos \phi + 0.9692\ Y_2(r) \cos 2\phi \\ & - 0.4178\ Y_3(r) \cos 3\phi + 0.7708\ Y_4(r) \cos 4\phi \\ & + 0.0867\ Y_5(r) \cos 5\phi + 0.8057\ Y_6(r) \cos 6\phi \\ & - 0.1445\ Y_7(r) \cos 7\phi - 0.1321\ Y_8(r) \cos 8\phi \end{aligned}$$

Each term (T_m) and the partial sum (S_m) for m = 1, 2,........., 11 are shown in figures 3.1 to 3.3

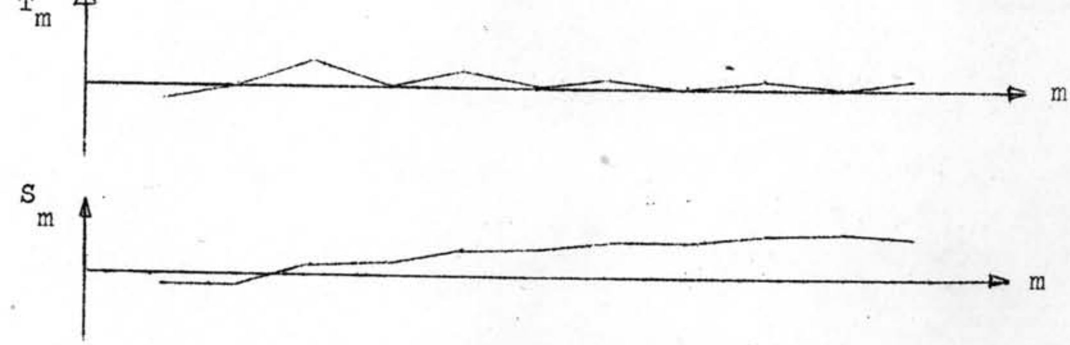

Figure 3.1 Shows the graph of m^{th} term (T_m) and partial sum of m^{th} term (S_m) of $\mathrm{Rl}(\psi_s)_m$ for m = 1,...,11, x = 0, y = 7.

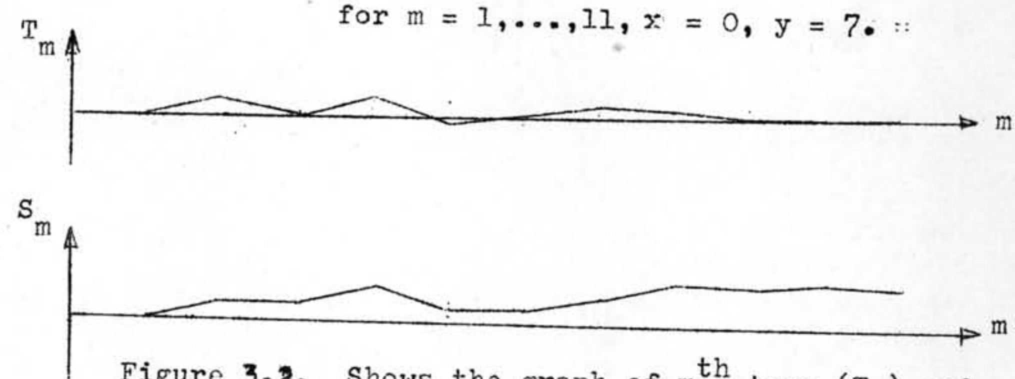

Figure 3.2. Shows the graph of m^{th} term (T_m) and partial sum of m^{th} term (S_m) of $\mathrm{Rl}(\psi_s)_m$ for m = 1,...,11, x = 6 y = 7

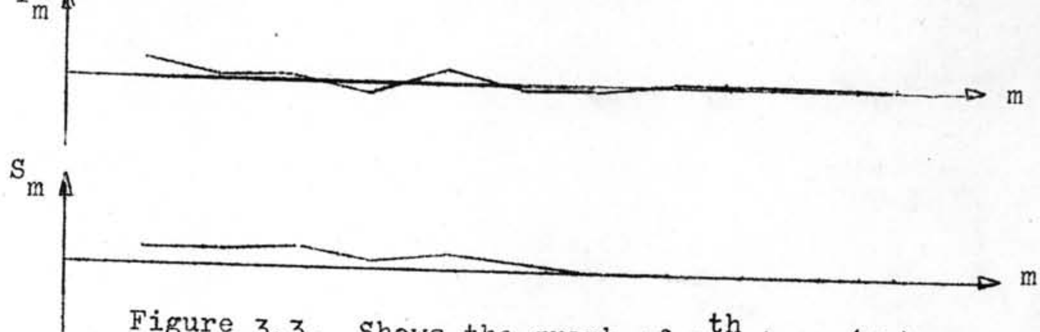

Figure 3.3. Shows the graph of m^{th} term (T_m) and partial sum of m^{th} term (S_m) of $\mathrm{Rl}(\psi_s)_m$ for m = 1,...,11, x = 8 y = 7.

For the wave at time equal to zero the approximation to the complete wave along the x axis is

$$\psi = Rl \left[e^{ikx} + (\psi_s)_8 \right]$$

$$= \cos x + Rl\ (\psi_s)_8, \quad \text{where} \quad k = 1.$$

Table 3.1 gives the values of Rl $(\psi_s)_8$ to four decimal places for $x = 0,1,2,\ldots, 9$ and $y = 0,1,2,\ldots,9$. Table 3.2 gives the values of Rl $(\psi_s)_8$ to four decimal places for $x = 0,-1,-2,\ldots, -9$ and $y = 0,1,2,\ldots,9$. Table 3.3 and 3.4 give the values of Rl$(\psi_s)_8$+ cos x to four decimal places for $y = 0,1,\ldots, 9$, $x = 0,1,2,\ldots,$ and $x = 0,-1,\ldots, -9$, respectively

The diagram for cos x + Rl $(\psi_s)_8$ is shown in figures 3.4 and 3.5. Figure 3.4 shows the complete wave near a circular object. The incident wave is little affected by the scattering along the lines $y = \pm 9$ and ψ varies from left to right with the wavelength of the incident wave. On the left side of the object facing the incident wave the incident wave is combined with a large reflected wave, while behide the object the incident and scattered waves interfere destructively reducing the amplitude and creating a "Shadow" . Figure 3.5 shows the contour curvesof cos x + Rl$(\psi_s)_8$.

Table 3.1 Values of $(\psi_s)_8$ to four decimal places.

y \ x	0	1	2	3	4	5	6	7	8	9
0						0.5779	-0.3960	-0.8773	-0.4052	0.5637
1						0.4318	-0.5111	-0.9410	-0.4636	0.4909
2						0.1726	-0.6746	-1.0373	-0.5281	0.3326
3					0.8830	0.0842	-0.7069	-1.0002	-0.5795	0.1996
4				-0.3368	0.2153	0.0553	-0.5330	-0.7244	-0.4297	0.1631
5	0.3482	-0.7956	-1.2829	-0.6790	-0.1812	0.0861	-0.1594	-0.3154	-0.1637	0.1925
6	0.3273	-0.2274	-0.4827	-0.1454	0.2597	0.6307	0.2842	0.0677	-0.1199	0.1787
7	0.4661	0.1859	0.1618	0.3978	0.6629	0.7034	0.4933	0.1916	-0.0060	0.0498
8	0.2541	0.2708	0.3574	0.5333	0.6704	0.5666	0.2860	-0.0155	-0.1705	-0.1521
9	-0.0144	0.0342	0.1224	0.2245	0.2234	0.0993	-0.1329	-0.3569	-0.3955	-0.2734

Table 3.2 Values of $(\psi_s)_8$ to four decimal places.

y \ x	0	-1	-2	-3	-4	-5	-6	-7	-8	-9
0						-0.4067	0.2346	0.3795	0.2246	0.0291
1						-0.0106	0.2955	0.2966	0.1448	-0.0205
2						0.6112	0.3536	0.0971	-0.1137	-0.1198
3					1.4584	0.7428	0.0957	-0.2768	-0.3343	-0.1808
4				0.6674	0.6895	0.1789	-0.4928	-0.7012	-0.8180	-0.1019
5	0.3482	0.6016	0.2231	0.0270	-0.4020	-0.2183	-0.9964	-0.8356	-0.3501	0.1653
6	0.3273	0.4730	0.1591	-0.2870	-0.8003	0.8672	-0.9886	-0.5387	0.0816	0.5293
7	0.4661	0.3421	0.0472	-0.3784	-0.7193	-0.7618	-0.4681	0.0950	0.5562	0.7748
8	0.2541	0.1838	-0.0016	-0.2105	-0.2670	-0.1112	0.2758	0.6821	0.8651	0.6837
9	-0.0144	-0.0140	-0.0192	0.0247	0.1568	0.4389	0.7209	0.8367	0.6763	0.2476

Table 3.3 Values of $(2\psi_s)_8$ + cos x to four decimal places.

y \ x	0	1	2	3	4	5	6	7	8	9
0						0.8646	0.5651	-0.1255	-0.5554	-0.3494
1						0.7185	0.4500	-0.1892	-0.6138	-0.4222
2						0.4593	0.2865	-0.2855	-0.6783	-0.5805
3					0.2306	0.3709	0.2542	-0.2484	-0.7297	-0.7135
4				-1.3270	-0.4371	0.3420	0.4281	0.0274	-0.5799	-0.7500
5	1.3482	-0.2553	-1.7005	-1.6692	-0.8336	0.3728	0.8017	0.4364	-0.3139	-0.7206
6	1.3273	0.3129	-0.9003	-1.1356	-0.3927	0.9174	1.2453	0.8195	-0.2701	-0.7344
7	1.4661	0.7262	-0.2558	-0.5924	0.0105	0.9901	1.4544	0.9434	-0.1562	-0.8633
8	1.2541	0.8111	-0.0602	-0.4569	0.0180	0.8533	1.2471	0.7673	-0.3207	-1.0652
9	0.9856	0.5745	-0.2952	-0.7657	-0.4290	0.3860	0.8282	0.3949	-0.5457	-1.1865

Table 3.4 Values of $(\psi_s)_8$ + cos x to four decimal places.

y \ x	0	-1	-2	-3	-4	-5	-6	-7	-8	-9
0						-0.1200	1.1957	1.1313	0.0744	-0.8840
1						0.2761	1.2566	1.0484	-0.0054	-0.9336
2						0.8979	1.3147	0.8489	-0.2639	-1.0329
3					0.8060	1.0295	1.0568	0.4750	-0.4845	-1.0939
4				-0.3228	0.0371	0.4656	0.4683	0.0506	-0.9682	-1.0150
5	1.3482	1.1419	-0.1945	-0.9632	-1.0544	0.0684	-0.0353	-0.0838	-0.5003	-0.7478
6	1.3273	1.0133	-0.2585	-1.2772	-1.4527	-0.9610	-0.0275	0.2131	-0.0686	-0.3838
7	1.4661	0.8824	-0.3704	-1.3686	-1.3717	-0.4751	0.4930	0.8468	0.4060	-0.1383
8	1.2541	0.7241	-0.4192	-1.2007	-0.9194	0.1755	1.1369	1.4339	0.7149	-0.2294
9	0.9856	0.5263	-0.4368	-0.9655	-0.4956	0.7256	1.6820	1.5885	0.5261	-0.6655

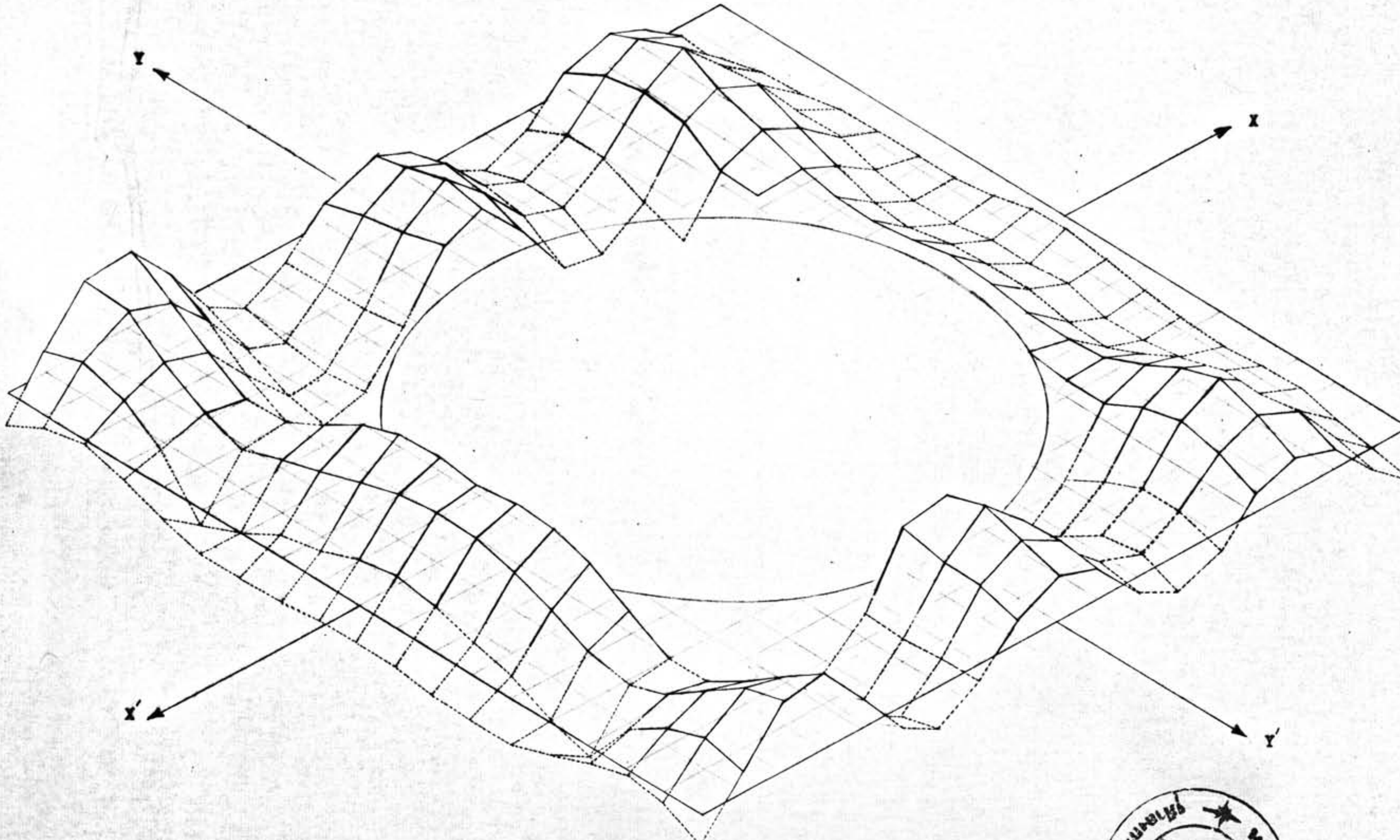

Figure 3.4

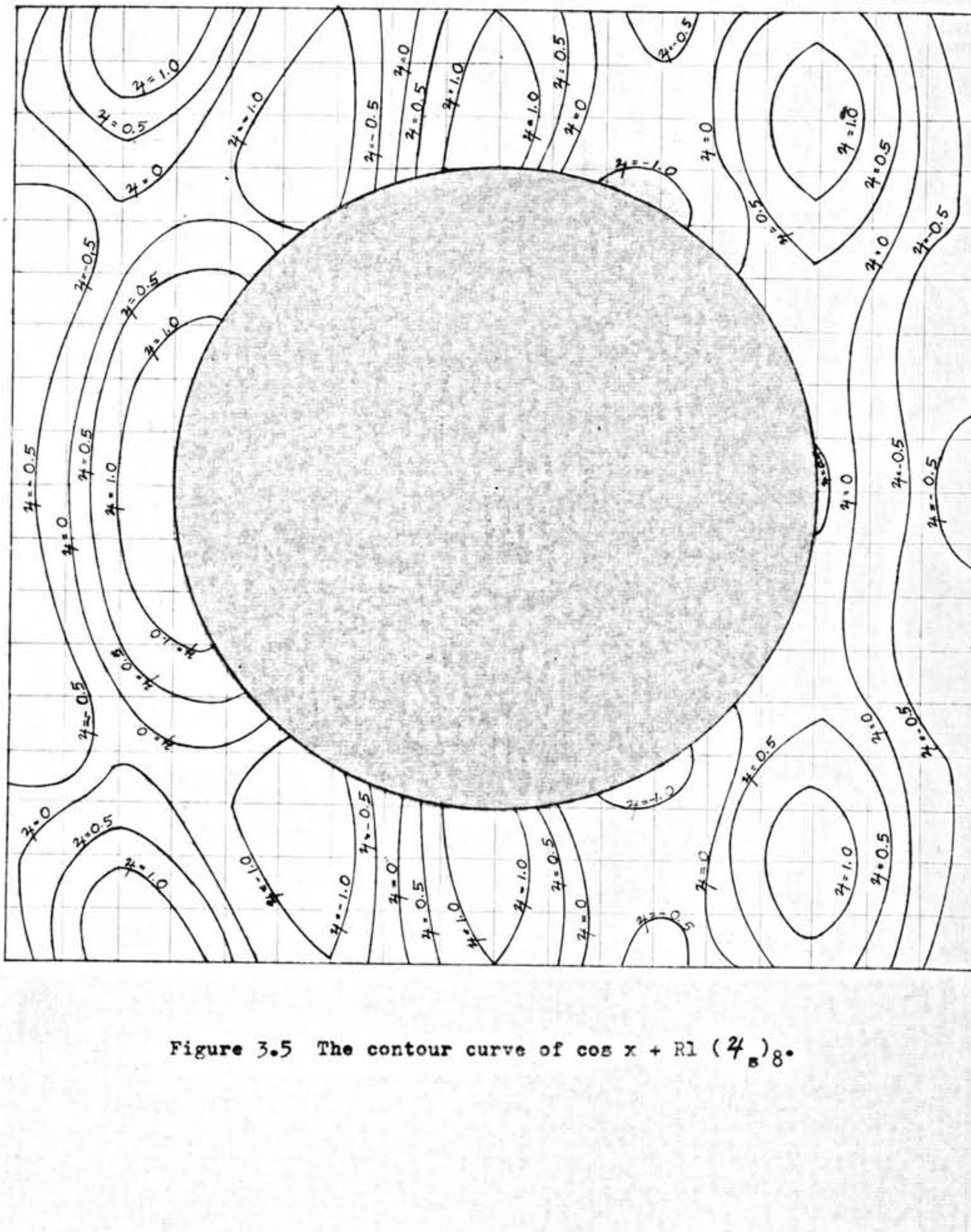

Figure 3.5 The contour curve of cos x + Rl $(\psi_s)_8$.

APPENDIX

EXPLANATION OF THE TABLES

The values in tables 2.1 to 2.5 in chapter II and tables 3.1 to 3.4 in chapter III are obtained by using tables 1 to 32.

Table 1 gives the values of r to two decimal places, where

$$r = \sqrt{x^2 + y^2}, \quad x = 0,1,2, \ldots, 9$$
$$y = 0,1,2, \ldots, 9$$

Tables 2 to 10 give the values of the Bessel functions $J_n(r)$, $n = 0, 1, 2, \ldots, 8$ to four decimal places. The values of $J_o(r)$ and $J_1(r)$ are taken from "A Treatise on Bessel Functions and Their Applications to Physics" by A. Gray and T.M. Macrobert.

The values of $J_n(r)$ for n greater than one are obtained by using the recursion formula

$$J_{n+1}(r) = \frac{2n}{r} J_n(r) - J_{n-1}(r) .$$

For example, find $J_2(1.41)$

Here $n = 1$, $r = 1.41$, $J_o(1.41) = 0.56142672$ and $J_1(1.41) = 0.54372550$.

Therefore $J_2(1.41) = \frac{2 \times 1}{1.41}(0.54372550) - 0.56142672$

$$= 0.20981512 .$$

Tables 11 to 19 give the values of the Neumann functions $Y_n(r)$ of the first kind. Table 11 and table 12 give the values of $Y_o(r)$ and $Y_1(r)$ respectively calculated by using Bessel's interpolation formula.

These values must be calculated because the tables for $Y_o(r)$ and $Y_1(r)$ from "Theory of Bessel Functions" by Watson give values only for arguments having even numbers in the second decimal place. Tables 13 to 19 are obtained from the recursion formula above by substituting $Y_n(r)$ for $J_n(r)$.

Bessel's Interpolation

Suppose we are given values of a function f(x) at equal intervals of x, say at $x_o, x_1, \ldots, x_n$ where $x_r = x_o + rh$, $r = 0, \pm 1, \pm 2, \ldots, \pm n$. Figure 1 shows the difference table for these values.

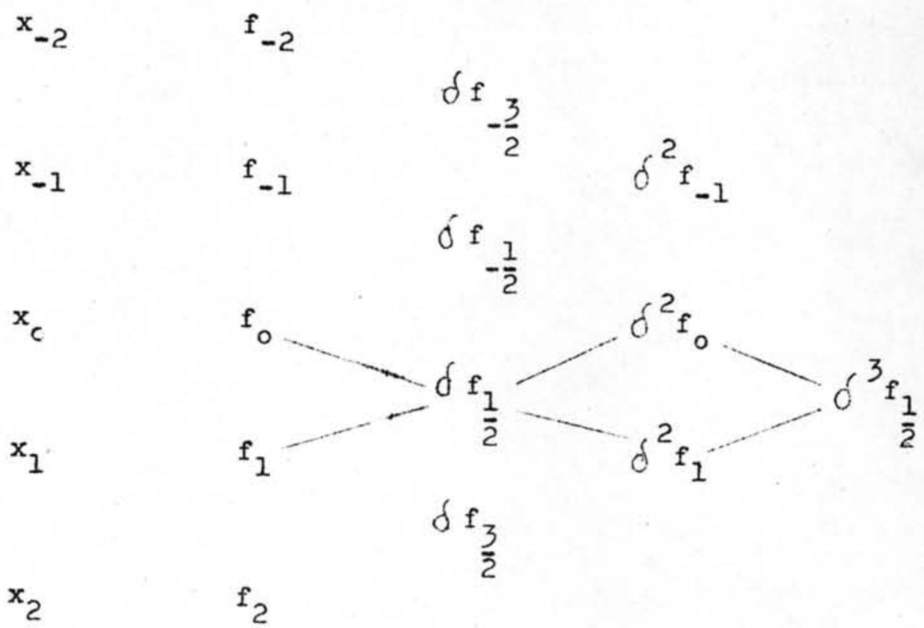

Figure 1

The interpolation at a point $\bar{x}$ between x_o and x_1 is obtained by using Bessel's formula

$$f_p = f_o + p\delta f_{\frac{1}{2}} + B_2(\delta^2 f_o + \delta^2 f_1) + B_3\delta^3 f_{\frac{1}{2}} + \cdots ,$$

where $B_{2r} = \frac{1}{2}\binom{p+r-1}{2r}$, $B_{2r+1} = \frac{p-\frac{1}{2}}{2r+1}\binom{p+r-1}{2r}$

and $p = \frac{\bar{x} - x_o}{h}$,

The array of differences involved is symmetric about a horizontal line midway between x_o and x_1.

The second differences are negligible if not greater than 4, the third differences are negligible if less than 60, and the fourth differences are negligible if less than 20, where these values refer to the smallest significant units in the calculation.

For example, find $Y_o(2.23)$.

x	f(x)	δf	$\delta^2 f$	$\delta^3 f$
2.18	5207097			
		746		
2.20	5207843		-2081	
		-1335		30
2.22	5206508		-2051	
		-3386		0
2.24	5203112		-2051	
		-5437		
2.26	5197675			

All values of f(x) and their differences are multiplied by 10^{-7}.

$$\text{Here} \quad p = \frac{2.23 - 2.22}{0.02} = 0.5$$

$$B_2 = \frac{1}{4}\, p(p - 1)$$

$$= \frac{1}{4} \times 0.5\ (0.5 - 1)$$

$$= -\ 0.062.$$

The third difference is negligible since it is less than 60. Therefore

$$Y_0(2.23) = 0.5206508 + 0.5\ (-\ 0.0003386)$$

$$+ (-\ 0.062)(-\ 0.0002051 - 0.0002051)$$

$$= 0.52050693.$$

Tables 20 to 31 give the values of $\cos n\,\phi$, $n = 1, 2, \ldots, 8$.

$$\cos n\,\phi = \frac{1}{2}\left[(2\cos\phi)^n - \frac{n}{1}(2\cos\phi)^{n-2} + \frac{n}{2}\binom{n-3}{1}(2\cos\phi)^{n-4} - \frac{n}{3}\binom{n-4}{2}(2\cos\phi)^{n-6} + \ldots\right]$$

Table 32 gives the values of $\cos x$, $x = 0, \pm 1, \pm 2, \ldots, \pm 9$.

Table 1 Values of $r = x^2 + y^2$ to two decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	0.00	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00
1	1.00	1.41	2.23	3.16	4.12	5.09	6.08	7.07	8.06	9.05
2	2.00	2.23	2.82	3.60	4.47	5.38	6.32	7.28	8.24	9.21
3	3.00	3.16	3.60	4.24	5.00	5.83	6.70	7.61	8.54	9.48
4	4.00	4.12	4.47	5.00	5.65	6.40	7.21	8.06	8.94	9.84
5	5.00	5.09	5.38	5.83	6.40	7.07	7.81	8.60	9.43	10.29
6	6.00	6.08	6.32	6.70	7.21	7.81	8.48	9.21	10.00	10.81
7	7.00	7.07	7.28	7.61	8.06	8.60	9.21	9.89	10.63	11.41
8	8.00	8.06	8.24	8.54	8.94	9.43	10.00	1063	11.31	12.04
9	9.00	9.05	9.21	9.48	9.84	10.29	10.81	11.41	12.04	12.72

Table 2. Values of $J_o(r)$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	1.0000	0.7652	0.2240	-0.2601	-0.3971	-0.1776	0.1506	0.3001	0.1717	-0.0903
1	0.7652	0.5614	0.0937	-0.3094	-0.3865	-0.1477	0.1721	0.2997	0.1573	-0.1024
2	0.2240	0.0937	-0.1932	-0.3918	-0.3274	-0.0481	0.2279	0.2898	0.1118	-0.1389
3	-0.2601	-0.3094	-0.3918	-0.3707	-0.1776	0.1010	0.2851	0.2500	0.0310	-0.1907
4	-0.3971	-0.3865	-0.3274	-0.1776	0.0436	0.2433	0.2945	0.1573	-0.0754	-0.2358
5	-0.1776	-0.1477	-0.0481	0.1010	0.2433	0.2997	0.2134	0.0146	-0.1821	-0.2480
6	0.1506	0.1721	0.2279	0.2851	0.2945	0.2134	0.0474	-0.1389	-0.2459	-0.2018
7	0.3001	0.2997	0.2898	0.2500	0.1573	0.0146	-0.1389	-0.2396	-0.2245	-0.0880
8	0.1717	0.1573	0.1118	0.0310	-0.0754	-0.1821	-0.2459	-0.2245	-0.1090	0.0566
9	-0.0903	-0.1024	-0.1389	-0.1907	-0.2358	-0.2480	-0.2018	-0.0880	0.0566	0.1792

Table 3. Values of $J_1(r)$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	0.0000	0.4401	0.5767	0.3391	-0.0660	- .3276	-0.2767	-0.0047	0.2346	0.2453
1	0.4401	0.5437	0.5515	0.2773	-0.1105	-0.3363	-0.2602	0.0163	0.2427	0.2391
2	0.5767	0.5515	0.4030	0.0955	-0.2229	-0.3457	-0.2029	0.0770	0.2614	0.2158
3	0.3391	0.2773	0.0955	-0.1522	-0.3276	-0.3065	-0.0953	0.1615	0.2733	0.1655
4	-0.0660	-0.1105	-0.2229	-0.3276	-0.3296	-0.1816	0.0572	0.2427	0.2519	0.0831
5	-0.3276	-0.3363	-0.3457	-0.3065	-0.1816	0.0163	0.2032	0.2728	0.1757	-0.0289
6	-0.2767	-0.2602	-0.2029	-0.0953	0.0572	0.2032	0.2729	0.2158	0.0435	-0.1441
7	-0.0047	0.0163	0.0770	0.1615	0.2427	0.2728	0.2158	0.0708	-0.1077	-0.2231
8	0.23 6	0.2427	0.2614	0.2733	0.2519	0.1757	0.0435	-0.1077	-0.2152	-0.2206
9	0.2453	0.2391	0.2158	0.1655	0.0831	-0.0289	-0.1441	-0.2231	-0.2206	-0.1269

Table 4. Values of $J_2(r)$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	0.0000	0.1149	0.3528	0.4861	0.3641	0.0466	-0.2429	-0.3014	-0.1130	0.1448
1	0.1149	0.2098	0.4009	0.4849	0.3329	0.0156	-0.2577	-0.2951	-0.0971	0.1553
2	0.3528	0.4009	0.4790	0.4448	0.2276	-0.0804	-0.2921	-0.2687	-0.0484	0.1858
3	0.4861	0.4849	0.4448	0.2989	0.0466	-0.2061	-0.3135	-0.2076	0.0330	0.2256
4	0.3641	0.3329	0.2276	0.0466	-0.1603	-0.3001	-0.2786	-0.0971	0.1318	0.2527
5	0.0466	0.0156	-0.0804	-0.2061	-0.3001	-0.2951	-0.1613	0.0488	0.2194	0.2424
6	-0.2429	-0.2577	-0.2921	-0.3135	-0.2786	-0.1613	0.0170	0.1858	0.2546	0.1751
7	-0.3014	-0.2951	-0.2687	-0.2076	-0.0971	0.0488	0.1858	0.2540	0.2042	0.0489
8	-0.1130	-0.0971	-0.0484	0.0330	0.1318	0.2194	0.2546	0.2042	0.0719	-0.0932
9	0.1448	0.1553	0.1858	0.2256	0.2527	0.2424	0.1751	0.0489	-0.0932	-0.1991

Table 5. Values of $J_3(r)$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	0.0000	0.0196	0.1289	0.3091	0.4302	0.3648	0.1148	-0.1676	-0.2911	-0.1809
1	0.0196	0.0515	0.1676	0.3365	0.4337	0.3485	0.0907	-0.1832	-0.2909	-0.1705
2	0.1289	0.1676	0.2764	0.3988	0.4266	0.2860	0.0180	-0.2246	-0.2849	-0.1351
3	0.3091	0.3365	0.3988	0.4342	0.3648	0.1651	-0.0918	-0.2706	-0.2578	-0.0703
4	0.4302	0.4337	0.4266	0.3648	0.2162	-0.0059	-0.2118	-0.2909	-0.1930	0.0196
5	0.3648	0.3485	0.2860	0.1651	-0.0059	-0.1832	-0.2859	-0.2501	-0.0826	0.1231
6	0.1148	0.0907	0.0180	-0.0918	-0.2118	-0.2859	-0.2649	-0.1351	0.0584	0.2089
7	-0.1676	-0.1832	-0.2246	-0.2706	-0.2909	-0.2501	-0.1351	0.0319	0.1846	0.2403
8	-0.2911	-0.2909	-0.2849	-0.2578	-0.1930	-0.0826	0.0584	0.1846	0.2406	0.1897
9	-0.1809	-0.1705	-0.1351	-0.0703	0.0196	0.1231	0.2089	0.2403	0.1897	0.0643

Talbe 6: Values of $J_4(r)$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	0.0000	0.0025	0.0340	0.1320	0.2811	0.3912	0.3576	0.1578	-0.1054	-0.2655
1	0.0025	0.0093	0.0500	0.1540	0.2987	0.3953	0.3472	0.1396	-0.1195	-0.2683
2	0.0340	0.0500	0.1091	0.2198	0.3449	0.3993	0.3092	0.0835	-0.1591	-0.2738
3	0.1320	0.1540	0.2198	0.3155	0.3912	0.3761	0.2313	-0.0058	-0.2141	-0.2700
4	0.2811	0.2987	0.3449	0.3912	0.3898	0.2945	0.1024	-0.1195	-0.2613	-0.2407
5	0.3912	0.3953	0.3993	0.3761	0.2945	0.1396	-0.0583	-0.2233	-0.2720	-0.1706
6	0.3576	0.3472	0.3092	0.2313	0.1024	-0.0583	-0.2044	-0.2738	-0.2196	-0.0592
7	0.1578	0.1396	0.0835	-0.0058	-0.1195	-0.2233	-0.2738	-0.2346	-0.1001	0.0775
8	-0.1054	-0.1195	-0.1591	-0.2141	-0.2613	-0.2720	-0.2196	-0.1001	0.0558	0.1877
9	-0.2655	-0.2683	-0.2738	-0.2700	-0.2407	-0.1706	-0.0592	0.0775	0.1877	0.2294

Table 7 : Values of $J_5(r)$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	0.0000	0.0002	0.0070	0.0430	0.1321	0.2611	0.3621	0.3479	0.1858	-0.0550
1	0.0002	0.0013	0.0116	0.0534	0.1463	0.2727	0.3662	0.3412	0.1723	-0.0667
2	0.0070	0.0116	0.0331	0.0897	0.1908	0.3078	0.3734	0.3164	0.1305	-0.1027
3	0.0430	0.0534	0.0897	0.1611	0.2611	0.3509	0.3680	0.2645	0.0572	-0.1576
4	0.1321	0.1463	0.1908	0.2611	0.3358	0.3741	0.3254	0.1723	-0.0408	-0.2153
5	0.2611	0.2727	0.3078	0.3509	0.3741	0.3412	0.2262	0.0424	-0.1481	-0.2558
6	0.3621	0.3662	0.3734	0.3680	0.3254	0.2262	0.0721	-0.1027	-0.2341	-0.2527
7	0.3479	0.3412	0.3164	0.2645	0.1723	0.0424	-0.1027	-0.2217	-0.2599	-0.1860
8	0.1858	0.1723	0.1305	0.0572	-0.0408	-0.1481	-0.2341	-0.2599	-0.2011	-0.0649
9	-0.0550	-0.0667	-0.1027	-0.1576	-0.2153	-0.2558	-0.2527	-0.1860	-0.0649	0.0800

Table 8 : Values of $J_6(r)$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	0.0000	0.0000	0.0012	0.0114	0.0491	0.1310	0.2458	0.3392	0.3376	0.2043
1	0.0000	0.0002	0.0022	0.0150	0.0564	0.1405	0.2551	0.3430	0.3333	0.1946
2	0.0012	0.0022	0.0082	0.0293	0.0818	0.1728	0.2816	0.3511	0.3174	0.1623
3	0.0114	0.0150	0.0293	0.0645	0.1310	0.2259	0.3180	0.3534	0.2812	0.1038
4	0.0491	0.0564	0.0818	0.1310	0.2045	0.2900	0.3489	0.3333	0.2156	0.0219
5	0.1310	0.1405	0.1728	0.2259	0.2900	0.3430	0.3479	0.2725	0.1149	-0.0779
6	0.2458	0.2551	0.2816	0.3180	0.3489	0.3479	0.2894	0.1623	-0.0145	-0.1745
7	0.3392	0.3430	0.3511	0.3534	0.3333	0.2725	0.1623	0.0105	-0.1444	-0.2405
8	0.3376	0.3333	0.3174	0.2812	0.2156	0.1149	-0.0145	-0.1444	-0.2336	-0.2416
9	0.2043	0.1946	0.1623	0.1038	0.0219	-0.0779	-0.1745	-0.2405	-0.2416	-0.1665

Table 9 : Values of $J_7(r)$ to four decimal places.

y \ x	0	1	2	3	4	5	6	7	8	9
0	0.0000	0.0000	0.0002	0.0025	0.0152	0.0534	0.1296	0.2336	0.3206	0.3275
1	0.0000	0.0000	0.0004	0.0036	0.0181	0.0586	0.1373	0.2409	0.3239	0.3248
2	0.0002	0.0004	0.0017	0.0080	0.0289	0.0777	0.1613	0.2623	0.3318	0.3141
3	0.0025	0.0036	0.0080	0.0214	0.0534	0.1140	0.2015	0.2927	0.3379	0.2890
4	0.0152	0.0181	0.0289	0.0534	0.0985	0.1696	0.2553	0.3239	0.3303	0.2420
5	0.0534	0.0586	0.0777	0.1140	0.1696	0.2409	0.3083	0.3379	0.2943	0.1649
6	0.1296	0.1373	0.1613	0.2015	0.2553	0.3083	0.3374	0.3141	0.2167	0.0589
7	0.2336	0.2409	0.2623	0.2927	0.3239	0.3379	0.3141	0.2344	0.0968	-0.0669
8	0.3206	0.3239	0.3318	0.3379	0.3303	0.2943	0.2167	0.0968	-0.0467	-0.1759
9	0.3275	0.3248	0.3141	0.2890	0.2420	0.1649	0.0589	-0.0669	-0.1759	-0.2371

Table 10 : Values of $J_8(r)$ to four decimal places.

y \ x	0	1	2	3	4	5	6	7	8	9
0	0.0000	0.0000	0.0000	0.0005	0.0040	0.0184	0.0565	0.1280	0.2235	0.3051
1	0.0000	0.0000	0.0001	0.0008	0.0050	0.0207	0.0610	0.1341	0.2293	0.3078
2	0.0000	0.0001	0.0003	0.0019	0.0087	0.0294	0.0757	0.1534	0.2463	0.3153
3	0.0005	0.0008	0.0019	0.0061	0.0184	0.0478	0.1031	0.1851	0.2727	0.3230
4	0.0040	0.0050	0.0087	0.0184	0.0397	0.0810	0.1469	0.2293	0.3016	0.3225
5	0.0184	0.0207	0.0294	0.0478	0.0810	0.1341	0.2048	0.2775	0.3221	0.3023
6	0.0565	0.0610	0.0757	0.1031	0.1469	0.2048	0.2677	0.3153	0.3179	0.2508
7	0.1280	0.1341	0.1534	0.1851	0.2293	0.2775	0.3153	0.3213	0.2720	0.1583
8	0.2235	0.2293	0.2463	0.2727	0.3016	0.3221	0.3179	0.2720	0.1758	0.0371
9	0.3051	0.3078	0.3153	0.3230	0.3225	0.3023	0.2508	0.1583	0.0371	-0.0945

Table 11 : Values of $Y_o(r)$ to four decimal places.

y \ x	0	1	2	3	4	5	6	7	8	9
0	$-\infty$	0.0883	0.5104	0.3769	-0.0169	-0.3085	-0.2882	-0.0259	0.2235	0.2499
1	0.0883	0.3427	0.5205	0.3217	-0.0637	-0.3204	-0.2734	-0.0048	0.2326	0.2444
2	0.5104	0.5205	0.4306	0.1477	-0.1856	-0.3399	-0.2202	0.0571	0.2542	0.2230
3	0.3769	0.3217	0.1477	-0.1083	-0.3085	-0.3140	-0.1162	0.1448	0.2710	0.1752
4	-0.0169	-0.0637	-0.1856	-0.3085	-0.3322	-0.1999	0.0368	0.2326	0.2558	0.0949
5	-0.3085	-0.3204	-0.3399	-0.3140	-0.1999	-0.0048	0.1893	0.2715	0.1851	-0.0168
6	-0.2882	-0.2734	-0.2202	-0.1162	0.0368	0.1893	0.2696	0.2230	0.0557	-0.1346
7	-0.0259	-0.0048	0.0571	0.1448	0.2326	0.2715	0.2230	0.0828	-0.0971	-0.2191
8	0.2235	0.2326	0.2542	0.2710	0.2558	0.1851	0.0557	-0.0971	-0.2101	-0.2228
9	0.2499	0.2444	0.2230	0.1752	0.0949	-0.0168	-0.1346	-0.2191	-0.2228	-0.1338

Table 12 : Values of $Y_1(r)$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	$-\infty$	-0.7812	-0.1070	0.3247	0.3979	0.1479	-0.1750	-0.3027	-0.1581	0.1043
1	-0.7812	-0.4724	0.0170	0.3627	0.3815	0.1172	-0.1950	-0.3007	-0.1432	0.1161
2	-0.1070	0.0170	0.2703	0.4154	0.3087	0.0170	-0.2459	-0.2866	-0.0967	0.1512
3	0.3247	0.3627	0.4154	0.3605	0.1479	-0.1281	-0.2945	-0.2410	-0.0152	0.2001
4	0.3979	0.3815	0.3087	0.1479	-0.0729	-0.2596	-0.2927	-0.1432	0.0898	0.2409
5	0.1479	0.1172	0.0170	-0.1281	-0.2596	-0.3007	-0.2017	0.0011	0.1922	0.2475
6	-0.1750	-0.1950	-0.2459	-0.2945	-0.2927	-0.2017	-0.0316	0.1512	0.2490	0.1958
7	-0.3027	-0.3007	-0.2866	-0.2410	-0.1432	0.0011	0.1512	0.2441	0.2202	0.0785
8	-0.1581	-0.1432	-0.0967	-0.0152	0.0898	0.1922	0.2490	0.2202	0.1008	-0.0659
9	0.1043	0.1161	0.1512	0.2001	0.2409	0.2475	0.1958	0.0785	-0.0659	-0.1846

Table 13 : Values of $Y_2(r)$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	$-\infty$	-1.6507	-0.6174	-0.1604	0.2159	0.3677	0.2299	-0.0605	-0.2630	-0.2268
1	-1.6507	-1.0127	-0.5053	-0.0921	0.2490	0.3665	0.2092	-0.0803	-0.2681	-0.2188
2	-0.6174	-0.5053	-0.2389	-0.0831	0.3237	0.3462	0.1424	-0.1358	-0.2777	-0.1902
3	-0.1604	-0.0921	-0.0831	0.2784	0.3677	0.2701	0.0283	-0.2082	-0.2746	-0.1330
4	0.2159	0.2490	0.3237	0.3677	0.3064	0.1188	-0.1180	-0.2681	-0.2357	-0.0460
5	0.3677	0.3665	0.3462	0.2701	0.1188	-0.0803	-0.2409	-0.2712	-0.1443	0.0649
6	0.2299	0.2092	0.1424	0.0283	-0.1180	-0.2409	-0.2771	-0.1902	-0.0059	0.1708
7	-0.0605	-0.0803	-0.1358	-0.2082	-0.2681	-0.2712	-0.1902	-0.0335	0.1385	0.2328
8	-0.2630	-0.2681	-0.2777	-0.2746	-0.2357	-0.1443	-0.0059	0.1385	0.2279	0.2118
9	-0.2268	-0.2188	-0.1902	-0.1330	-0.0460	0.0649	0.1708	0.2328	0.2118	0.1048

Table 14 : Values of $Y_3(r)$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	$-\infty$	-5.8215	-1.1278	-0.5385	-0.1820	0.1463	0.3282	0.3373	0.0265	-0.2051
1	-5.8215	-2.4004	-0.9233	-0.4794	-0.1398	0.1708	0.3327	0.2553	0.0102	-0.2128
2	-1.1278	-0.9233	-0.6091	-0.5077	-0.0191	0.2404	0.3360	0.2120	-0.0381	-0.2338
3	-0.5385	-0.4794	-0.5077	-0.0979	0.1463	0.3134	0.3114	0.1316	-0.1134	-0.2563
4	-0.1820	-0.1398	-0.0191	0.1463	0.2898	0.3338	0.2272	0.0102	-0.1952	-0.2596
5	0.1463	0.1708	0.2404	0.3134	0.3338	0.2553	0.0784	-0.1272	-0.2534	-0.2223
6	0.3282	0.3327	0.3360	0.3114	0.2272	0.0784	-0.0991	-0.2338	-0.2514	-0.1326
7	0.3373	0.2553	0.2120	0.1316	0.0102	-0.1272	-0.2338	-0.2577	-0.1681	0.0031
8	0.0265	0.0102	-0.0381	-0.1134	-0.1952	-0.2534	-0.2514	-0.1681	-0.0201	0.1362
9	-0.2051	-0.2128	-0.2338	-0.2563	-0.2596	-0.2223	-0.1326	0.0031	0.1362	0.2175

Table 15 : Values of $Y_4(r)$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	$-\infty$	-33.2784	-2.7659	-0.9167	-0.4889	-0.1921	0.0984	0.3496	0.2829	0.0900
1	-33.2784	-9.2020	-1.9790	-0.8181	-0.4526	-0.1651	0.1191	0.2969	0.2757	0.0777
2	-2.7659	-1.9790	-1.0572	-0.7631	-0.3493	-0.0781	0.1767	0.3105	0.2499	0.0379
3	-0.9167	-0.8181	-0.7631	-0.4169	-0.1921	0.0524	0.2505	0.3120	0.1679	-0.0292
4	-0.4889	-0.4526	-0.3493	-0.1921	0.0014	0.1941	0.3070	0.2757	0.1046	-0.1123
5	-0.1921	-0.1651	-0.0781	0.0524	0.1941	0.2969	0.3011	0.1824	-0.0169	-0.1945
6	0.0984	0.1191	0.1767	0.2505	0.3070	0.3011	0.2070	0.0379	-0.1449	-0.2444
7	0.3496	0.2969	0.3105	0.3120	0.2757	0.1824	0.0379	-0.1229	-0.2334	-0.2312
8	0.2829	0.2575	0.2499	0.1679	0.1046	-0.0169	-0.1449	-0.2334	-0.2386	-0.1439
9	0.0900	0.0777	0.0379	-0.0292	-0.1123	-0.1945	-0.2444	-0.2312	-0.1439	-0.0022

Table 16 : Values of $Y_5(r)$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	$-\infty$	-260.4059	-9.9360	-1.9059	-0.7959	-0.4537	-0.1971	0.0623	0.2564	0.2851
1	-260.4059	-49.8095	-6.1761	-1.5917	-0.7390	-0.4303	-0.1760	0.0807	0.2635	0.2815
2	-9.9360	-6.1761	-2.3899	-1.1880	-0.6061	-0.3565	-0.1124	0.1292	0.2807	0.2667
3	-1.9059	-1.5917	-1.1880	-0.6887	-0.4537	-0.2594	-0.0122	0.1963	0.2707	0.2316
4	-0.7959	-0.7390	-0.6061	-0.4537	-0.2878	-0.0912	0.1135	0.2635	0.2889	0.1683
5	-0.4537	-0.4303	-0.3565	-0.2594	-0.0912	0.0807	0.2301	0.2969	0.2390	0.0710
6	-0.1971	-0.1760	-0.1124	-0.0122	0.1135	0.2301	0.2443	0.2667	0.1354	-0.0483
7	0.0623	0.0807	0.1292	0.1963	0.2635	0.2969	0.2667	0.1583	-0.0076	-0.1652
8	0.2564	0.2635	0.2807	0.2707	0.2889	0.2390	0.1354	-0.0076	-0.1486	-0.2319
9	0.2851	0.2815	0.2667	0.2316	0.1683	0.0710	-0.0483	-0.1652	-0.2319	-0.2189

Table 17 : Values of $Y_6(r)$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	$-\infty$	-2570.7802	-46.9140	-5.4365	-1.5007	-0.7152	-0.4268	-0.2606	0.0376	0.2268
1	-2570.7802	344.0569	-25.7165	-4.2189	-1.3411	-0.6804	-0.4086	-0.1828	0.0512	0.2333
2	-46,9140	-25.7165	- 7.4176	-2.5370	-1.0066	-0.5846	-0.3545	-0.1330	0.0908	0.2517
3	-5.4365	-4.2189	-2.5370	-1.2274	-0.7152	-0.4974	-0.2688	-0.0540	0.1490	0.2735
4	-1.5007	-1.3411	-1.0066	-0.7152	-0.5109	-0.3366	-0.1497	0.0512	0.2185	0.2833
5	-0.7152	-0.6804	-0.5846	-0.4974	-0.3366	-0.1828	-0.0065	0.1628	0.2704	0.2635
6	-0.4268	-0.4086	-0.3545	-0.2688	-0.1497	-0.0065	0.1401	0.2517	0.2804	0.1997
7	-0.2606	-0.1828	-0.1330	-0.0540	0.0512	0.1628	0.2517	0.2829	0.2263	0.0864
8	0.0376	0.0512	0.0908	0.1490	0.2185	0.2704	0.2804	0.2263	0.1072	-0.0486
9	0.2268	0.2333	0.2517	0.2735	0.2833	0.2635	0.1997	0.0864	-0.0486	-0.1699

Table 18 : Values of $Y_7(r)$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	$-\infty$	-30588.9570	-271.5480	-19.8399	-3.7062	-1.2629	-0.6566	-0.5091	-0.2001	0.0172
1	-30588.9570	-2878.3347	-132.2086	-14.4295	-3.1670	-1.1736	-0.6304	-0.3910	-0.1873	0.0279
2	-271.5480	-132.2086	-29.1745	-7.2688	-2.0961	-0.9474	-0.5607	-0.3484	-0.1485	0.0612
3	-19.8399	-14.4295	-7.2688	-2.7286	-1.2629	-0.7644	-0.4691	-0.2814	-0.0613	0.1146
4	-3.7062	-3.1670	-2.0961	-1.2629	-0.7972	-0.5399	-0.3626	-0.1873	0.0044	0.1772
5	-1.2629	-1.1736	-0.9474	-0.7644	-0.5399	-0.3910	-0.2401	-0.0697	0.1051	0.2363
6	-0.6566	-0.6304	-0.5607	-0.4691	-0.3626	-0.2401	-0.0961	0.0612	0.2010	0.2700
7	-0.5091	-0.3910	-0.3484	-0.2814	-0.1873	-0.0697	0.0612	0.1850	0.2630	0.2560
8	-0.2001	-0.1873	-0.1485	-0.0613	0.0044	0.1051	0.2010	0.2630	0.2624	0.1834
9	0.0172	0.0279	0.0612	0.1146	0.1772	0.2363	0.2700	0.2560	0.1834	0.0586

Table 19 : Values of $Y_8(r)$ to four decimal places.

y \ x	0	1	2	3	4	5	6	7	8	9
0	$-\infty$	-425674.6181	-1853.9222	-87.1499	-11.4711	-2.8209	-1.1052	-0.7575	-0.3877	-0.1999
1	-425674.6181	-28235.1527	-804.0000	-59.7095	- 9.4207	-2.5477	-1.0429	-0.5914	-0.3765	-0.1902
2	-1853.9222	-804.0000	-137.0000	-25.7303	-5.5585	-1.8808	-0.8876	-0.5371	-0.3431	-0.1586
3	-87.1499	-59.7095	-25.7303	-7.8020	-2.8209	-1.3382	-0.7115	-0.4638	-0.2495	-0.1043
4	-11.4711	-9.4207	-5.5585	-2.8209	-1.4644	-0.8445	-0.5544	-0.3765	-0.2116	-0.0312
5	-2.8209	-2.5477	-1.8808	-1.3382	-0.8445	-0.5914	-0.4239	-0.2763	-0.1144	0.0579
6	-1.1052	-1.0429	-0.8876	-0.7115	-0.5544	-0.4239	-0.2987	-0.1586	0.0011	0.1499
7	-0.7575	-0.5914	-0.5371	-0.4638	-0.3765	-0.2763	-0.1586	-0.0211	0.1201	0.2278
8	-0.3877	-0.3765	-0.3431	-0.2495	-0.2116	-0.1144	0.0011	0.1201	0.2176	0.2619
9	-0.1999	-0.1902	-0.1586	-0.1043	-0.0312	0.0579	0.1499	0.2278	0.2619	0.2344

Table 20 : Values of cos Ø to four decimal places.

y \ x	0	1	2	3	4	5	6	7	8	9
0	0.0000	1.0000	1.0000	1.0000	1.0000	1.000	1.0000	1.0000	1.0000	1.0000
1	0.0000	0.7092	0.8968	0.9493	0.9708	0.9823	0.9868	0.9900	0.9925	0.9944
2	0.0000	0.4484	0.7092	0.8333	0.8948	0.9293	0.9493	0.9615	0.9708	0.9771
3	0.0000	0.3164	0.5555	0.7075	0.8000	0.8576	0.8955	0.9198	0.9367	0.9493
4	0.0000	0.2427	0.4474	0.6000	0.7079	0.7812	0.8321	0.8684	0.8948	0.9146
5	0.0000	0.1964	0.3717	0.5145	0.6250	0.7072	0.7682	0.8139	0.8483	0.8746
6	0.0000	0.1644	0.3164	0.4477	0.5547	0.6402	0.7075	0.7600	0.8000	0.8325
7	0.0000	0.1414	0.2747	0.3942	0.4962	0.5813	0.6514	0.7077	0.7525	0.7887
8	0.0000	0.1240	0.2427	0.3512	0.4474	0.5302	0.6000	0.6585	0.7073	0.7475
9	0.0000	0.1104	0.2171	0.3164	0.4065	0.4859	0.5550	0.6134	0.6644	0.7075

Table 21 : Value of cos Ø to four decimal places.

y \ x	0	-1	-2	-3	-4	-5	-6	-7	-8	-9
0	0.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000
1	0.0000	-0.7092	-0.8968	-0.9493	-0.9708	-0.9823	-0.9868	-0.9900	-0.9925	-0.9944
2	0.0000	-0.4484	-0.7092	-0.8333	-0.8948	-0.9293	-0.9493	-0.9615	-0.9708	-0.9771
3	0.0000	-0.3164	-0.5555	-0.7075	-0.8000	-0.8576	-0.8955	-0.9198	-0.9367	-0.9493
4	0.0000	-0.2427	-0.4474	-0.6000	-0.7079	-0.7812	-0.8321	-0.8684	-0.8948	-0.9146
5	0.0000	-0.1964	-0.3717	-0.5145	-0.6250	-0.7072	-0.7682	-0.8139	-0.8483	-0.8746
6	0.0000	-0.1644	-0.3164	-0.4477	-0.5547	-0.6402	-0.7075	-0.7600	-0.8000	-0.8325
7	0.0000	-0.1414	-0.2747	-0.3942	-0.4962	-0.5813	-0.6514	-0.7077	-0.7525	-0.7887
8	0.0000	-0.1240	-0.2427	-0.3512	-0.4474	-0.5302	-0.6000	-0.6585	-0.7073	-0.7475
9	0.0000	-0.1104	-0.2171	-0.3164	-0.4065	-0.4859	-0.5550	-0.6134	-0.6644	-0.7075

Table 22 : Value of cos 2 Ø to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	0.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	-1.0000	0.0000	0.6000	0.8000	0.8824	0.9231	0.9459	0.9600	0.9692	0.9756
2	-1.0000	-0.6000	0.0000	0.3846	0.6000	0.7241	0.8000	0.8491	0.8824	0.9059
3	-1.0000	-0.8000	-0.3846	0.0000	0.2800	0.4706	0.6000	0.6897	0.7534	0.8000
4	-1.0000	-0.8824	-0.6000	-0.2800	0.0000	+0.2195	0.3846	0.5077	0.6000	0.6701
5	-1.0000	-0.9231	-0.7241	-0.4706	-0.2195	0.0000	0.1803	0.3243	0.4382	0.5283
6	-1.0000	-0.9459	-0.8000	-0.6000	-0.3846	-0.1803	0.0000	0.1529	0.2800	0.3846
7	-1.0000	-0.9600	-0.8491	-0.6897	-0.5077	-0.3243	-0.1529	0.0000	0.1327	0.2462
8	-1.0000	-0.9692	-0.8824	-0.7534	-0.6000	-0.4382	-0.2800	-0.1327	0.0000	0.1172
9	-1.0000	-0.9756	-0.9059	-0.8000	-0.6701	-0.5283	-0.3846	-0.2462	-0.1172	0.0000

Table 23 : Values of cos 3 Ø to four decimal places.

y \ x	0	1	2	3	4	5	6	7	8	9
0	0.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	0.0000	-0.7071	0.1789	0.5692	0.7419	0.8297	0.8797	0.9107	0.9312	0.9454
2	0.0000	-0.9839	-0.7071	-0.1920	0.1789	0.4162	0.5692	0.6713	0.7419	0.7924
3	0.0000	-0.8222	-0.9814	-0.7071	-0.3520	-0.0505	0.1789	0.3486	0.4746	0.5692
4	0.0000	-0.6705	-0.9839	-0.9360	-0.7071	-0.4381	-0.1920	-0.0141	0.1789	0.3109
5	0.0000	-0.5582	-0.9093	-0.9987	-0.8989	-0.7071	-0.4912	-0.2859	-0.1049	0.0495
6	0.0000	-0.4754	-0.82222	-0.9839	-0.9814	-0.8711	-0.7071	-0.5270	-0.3520	-0.1920
7	0.0000	-0.4130	-0.7412	-0.9373	-0.9999	-0.9583	-0.8499	-0.7071	-0.5528	-0.4007
8	0.0000	-0.3645	-0.6705	-0.8802	-0.9839	-0.9945	-0.9360	-0.8333	-0.7071	-0.5721
9	0.0000	-0.3259	-0.6100	-0.8222	-0.9504	-0.9988	-0.9814	-0.9162	-0.8202	-0.7071

Table 24 : Values of cos 3 Ø to four decimal places.

y \ x	0	-1	-2	-3	-4	-5	-6	-7	-8	-9
0	0.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000
1	0.0000	0.7071	-0.1789	-0.5692	-0.7419	-0.8297	-0.8797	-0.9107	-0.9312	-0.9454
2	0.0000	0.9839	0.7071	0.1920	-0.1789	-0.4162	-0.5692	-0.6713	-0.7419	-0.7924
3	0.0000	0.8222	0.9814	0.7071	0.3520	0.0505	-0.1789	-0.3486	-0.4746	-0.5692
4	0.0000	0.6705	0.9839	0.9360	0.7071	0.4381	0.1920	0.0141	-0.1789	-0.3109
5	0.0000	0.5582	0.9093	0.9987	0.8989	0.7071	0.4912	0.2859	0.1049	-0.0495
6	0.0000	0.4754	0.8222	0.9839	0.9814	0.8711	0.7071	0.5270	0.3520	0.1920
7	0.0000	0.4130	0.7412	0.9373	0.9999	0.9583	0.8499	0.7071	0.5528	0.4007
8	0.0000	0.3645	0.6705	0.8802	0.9839	0.9945	0.9360	0.8333	0.7071	0.5721
9	0.0000	0.3259	0.6100	0.8222	0.9504	0.9988	0.9814	0.9162	0.8202	0.7071

Table 25 : Values of cos 4 Ø to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	-1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	1.0000	-1.0000	-0.2800	0.2800	0.5572	0.7042	0.7894	0.8432	0.8786	0.9036
2	1.0000	-0.2800	-1.0000	-0.7042	-0.2800	0.0486	0.2800	0.4420	0.5572	0.6414
3	1.0000	0.2800	-0.7042	-1.0000	-0.8432	-0.5570	-0.2800	-0.0486	0.1352	0.2800
4	1.0000	0.5572	-0.2800	-0.8432	-1.0000	-0.9036	-0.7042	-0.4844	-0.2800	-0.1020
5	1.0000	0.7042	0.0486	-0.5570	-0.9036	-1.0000	-0.9350	-0.7896	-0.6160	-0.4418
6	1.0000	0.7894	0.2800	-0.2800	-0.7042	-0.9350	-1.0000	-0.9532	-0.8432	-0.7042
7	1.0000	0.8432	0.4420	-0.0486	-0.4844	-0.7896	-0.9532	-1.0000	-0.9648	-0.8788
8	1.0000	0.8786	0.5572	0.1352	-0.2800	-0.6160	-0.8432	-0.9648	-1.0000	-0.9726
9	1.0000	0.9036	0.6414	0.2800	-0.1020	-0.4418	-0.7042	-0.8788	-0.9726	-1.0000

Table 26 : Values of cos 5 ∅ to four decimal places.

y \ x	0	1	2	3	4	5	6	7	8	9
0	0.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	0.0000	-0.7175	-0.6600	-0.0282	0.3518	0.5881	0.6871	0.7599	0.8181	0.8631
2	0.0000	0.7289	-0.7175	-0.9774	-0.6767	-0.3152	-0.0282	0.1779	0.3518	0.4783
3	0.0000	0.9992	0.1955	-0.7091	-0.9971	-0.9045	-0.6709	-0.4308	-0.2161	-0.0282
4	0.0000	0.9411	0.7327	-0.0758	-0.7111	-0.9738	-0.9797	-0.8539	-0.6767	-0.4887
5	0.0000	0.8352	0.9449	0.4255	-0.2319	-0.7076	-0.9453	-0.9991	-0.9389	-0.8193
6	0.0000	0.7351	0.9992	0.7316	0.2002	-0.3261	-0.7091	-0.9227	-0.9971	-0.9789
7	0.0000	0.6514	0.9839	0.8982	0.5189	0.0400	-0.3945	-0.7101	-0.8991	-0.9858
8	0.0000	0.5823	0.9411	0.9751	0.7327	0.3405	-0.0758	-0.4372	-0.7081	-0.8819
9	0.0000	0.5254	0.8886	0.9992	0.8667	0.5685	0.1985	-0.1595	-0.4723	-0.7091

Table 27 : Values of cos 5 Ø to four decimal places

y \ x	0	-1	-2	-3	-4	-5	-6	-7	-8	-9
0	0.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000
1	0.0000	0.7175	0.6600	0.0282	-0.3518	-0.5881	-0.6871	-0.7599	-0.8181	-0.8631
2	0.0000	-0.7289	0.7175	0.9774	0.6767	0.3152	0.0282	-0.1779	-0.3518	-0.4783
3	0.0000	-0.9992	-0.1955	0.7091	0.9971	0.9045	0.6709	0.4308	0.2161	0.0282
4	0.0000	-0.9411	-0.7327	0.0758	0.7111	0.9738	0.9797	0.8539	0.6767	0.4887
5	0.0000	-0.8352	-0.9449	-0.4255	0.2319	0.7076	0.9453	0.9991	0.9389	0.8193
6	0.0000	-0.7351	-0.9992	-0.7316	-0.2002	0.3261	0.7091	0.9227	0.9971	0.9789
7	0.0000	-0.6514	-0.9839	-0.8982	-0.5189	-0.0400	0.3945	0.7101	0.8991	0.9858
8	0.0000	-0.5823	-0.9411	-0.9751	-0.7327	-0.3405	0.0758	0.4372	0.7081	0.8819
9	0.0000	-0.5254	-0.8886	-0.9992	-0.8667	-0.5685	-0.1985	0.1595	0.4723	0.7091

Table 28 : Values of cos 6 Ø to four decimal places.

y \ x	0	1	2	3	4	5	6	7	8	9
0	0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	-1.0000	0.0000	-0.9360	-0.3520	0.1008	0.3769	0.5479	0.6589	0.7343	0.7876
2	-1.0000	0.9360	0.0000	-0.9263	-0.9360	-0.6535	-0.3520	-0.0988	0.1008	0.2559
3	-1.0000	0.3520	0.9263	0.0000	-0.7522	-0.9949	-0.9360	-0.7569	-0.5496	-0.3520
4	-1.0000	-0.1008	0.9360	0.7522	0.0000	-0.6162	-0.9263	-0.9996	-0.9360	-0.8067
5	-1.0000	-0.3769	0.6535	0.9949	0.6162	0.0000	-0.5175	-0.8365	-0.9780	-0.9951
6	-1.0000	-0.5479	0.3520	0.9360	0.9263	0.5175	0.0000	-0.4445	-0.7522	-0.9263
7	-1.0000	-0.6589	0.0988	0.7569	0.9996	0.8365	0.4445	0.0000	-0.3889	-0.6788
8	-1.0000	-0.7343	-0.1008	0.5496	0.9360	0.9780	0.7522	0.3889	0.0000	-0.3453
9	-1.0000	-0.7876	-0.2559	0.3520	0.8067	0.9951	0.9263	0.6788	0.3453	0.0000

Table 29 : Values of cos 7 [illegible] to four decimal places.

y \ x	0.	1	2	3	4	5	6	7	8	9
0	0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	0	0.6923	-0.9978	-0.6192	-0.1246	0.2492	0.4188	0.5480	0.6541	0.7377
2	0	0.1130	0.6923	-0.5746	-0.9952	-0.8806	-0.6192	-0.3690	-0.1246	0.0698
3	0	-0.7759	0.8311	0.7043	-0.2064	-0.8022	-0.9962	-0.9495	-0.8035	-0.6192
4	0	-0.9895	0.1052	0.9785	0.7015	0.0071	-0.5622	-0.8822	-0.9952	-0.9742
5	0	-0.9826	-0.4580	0.5983	0.9999	0.7065	0.1504	-0.3641	-0.7216	-0.9204
6	0	-0.9152	-0.7759	0.1075	0.8274	0.9886	0.7043	0.2377	-0.2064	-0.5663
7	0	-0.8377	-0.9297	-0.3005	0.4732	0.9323	0.9680	0.7029	0.3148	-0.0776
8	0	-0.7645	-0.9895	-0.5888	0.1052	0.6963	0.9785	0.9494	0.7058	0.3646
9	0	-0.6993	-0.9992	-0.7759	-0.2097	0.3988	0.8288	0.9960	0.9307	0.7043

Table 30 : Values of cos 7 Ø four decimal places

y \ x	0	-1	-2	-3	-4	-5	-6	-7	-8	-9
0	0.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000	-1.0000
1	0.0000	-0.6923	0.9978	0.6192	0.1246	-0.2492	-0.4188	-0.5480	-0.6541	-0.7377
2	0.0000	-0.1130	-0.6923	0.5746	0.9952	0.8806	0.6192	0.3690	0.1246	-0.0698
3	0.0000	0.7759	-0.8311	-0.7043	0.2064	0.8022	0.9962	0.9495	0.8035	0.6192
4	0.0000	0.9895	-0.1052	-0.9785	-0.7015	-0.0071	0.5622	0.8822	0.9952	0.9742
5	0.0000	0.9826	0.4580	-0.5983	-0.9999	-0.7065	-0.1504	0.3641	0.7216	0.9204
6	0.0000	0.9152	0.7759	-0.1075	-0.8274	-0.9886	-0.7043	-0.2377	0.2064	0.5663
7	0.0000	0.8377	0.9297	0.3005	-0.4732	-0.9323	-0.9680	-0.7029	-0.3148	0.0776
8	0.0000	0.7645	0.9895	0.5888	-0.1052	-0.6963	-0.9785	-0.9494	-0.7058	-0.3646
9	0.0000	0.6993	0.9992	0.7759	0.2097	-0.3988	-0.8288	-0.9960	-0.9307	-0.7043

Table 31 .: Values of cos 8 ϕ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1	1.0000	1.0000	-0.8432	-0.8432	-0.3790	-0.0082	0.2464	0.4220	0.5438	0.6330
2	1.0000	-0.8432	1.0000	-0.0082	-0.8432	-0.9952	-0.8432	-0.6092	-0.3790	-0.1772
3	1.0000	-0.8432	-0.0082	1.0000	0.4220	-0.3796	-0.8432	-0.9952	-0.9634	-0.8432
4	1.0000	-0.3790	-0.8432	0.4220	1.0000	0.6330	-0.0082	-0.5308	-0.8432	-0.9792
5	1.0000	-0.0082	-0.9952	-0.3796	0.6330	1.0000	0.7484	0.2470	-0.2410	-0.6096
6	1.0000	0.2464	-0.8432	-0.8432	-0.0082	0.7484	1.0000	0.8172	0.4220	-0.0082
7	1.0000	0.4220	-0.6092	-0.9952	-0.5308	0.2470	0.8172	1.0000	0.8616	0.5446
8	1.0000	0.5438	-0.3790	-0.9634	-0.8432	-0.2410	0.4220	0.8616	1.0000	0.8920
9	1.0000	0.6330	-0.1772	-0.8432	-0.9792	-0.6096	-0.0082	0.5446	0.8920	1.0000

Table 32 : Values of cos x to four decimal places

x	0	± 1	± 2	± 3	± 4	± 5	± 6	± 7	± 8	± 9
cos x	1.0000	0.5403	-0.4176	-0.9902	-0.6524	0.2867	0.9611	0.7518	-0.1502	-0.9131