CHAPTER II

THE FOURIER - BESSEL REPRESENTATION OF STRAIGHT WAVES

Analytical Treatment

In this chapter we shall show that a straight wave can be represented by a linear superposition of circular waves by the formula

$$u = \sum_{n=-\infty}^{\infty} i^n J_n (kr) e^{in\phi}$$

$$= e^{ikr \cos \phi}$$

$$= e^{ikx}$$

where $x = r \cos \phi$, $y = r \sin \phi$.

We begin with the general expression for the scalar wave equation

$$\nabla^2 U(x,y,t) = \frac{1}{c^2} \frac{\partial^2 U}{\partial t^2}(x, y, t) \quad . \quad \text{..........} \; 1)$$

Assume the time dependence of the wave function to be harmonic,

$$U(x,y,t) = u(x,y) e^{-i\omega t} \quad \text{...............} \; 2)$$

where ω = circular frequency

$k = \frac{\omega}{c}$, wave number.

So we get the time - independent wave equation in two dimensions

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) + k^2 u(x,y) = 0 \quad \text{........} \; 3)$$

We shall first calculate the straight wave solutions by separating this equation in Cartesian coordinates.

Let $u(x,y) = X(x)\,Y(y)$. .... 4)

Then 3) becomes

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + k^2 = 0. \quad \text{......... 5)}$$

Since the second and third terms are independent of x and the first term is independent of y, the first term must be a constant. Since in addition we want the solution to be periodic on the x and y axes we let

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -a^2, \text{ where a is real, ... 6)}$$

that is $X(x) = Ae^{\pm iax}$, where A is constant. .7)

Also

$$\frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -b^2, \text{ where b is real . ...8)}$$

that is

$Y(y) = e^{\pm iby}$, where the arbitrary factor may be put equal to unity.

$$\left.\begin{aligned} \text{Now} \quad a^2 + b^2 &= k^2 \\ \text{or} \quad a &= k\cos\alpha \\ \text{and} \quad b &= k\sin\alpha\,. \end{aligned}\right\} \quad \text{....9)}$$

The solution of 3) is therefore of the form

$$u(x,y) = A\,e^{i(ax + by)}. \quad \text{...........10)}$$

We choose a plus sign in the exponent to represent a single wave that progresses in the positive direction.

Introducing the plane polar coordinates, r, ϕ, with

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots 11)$$

and by using 9) 11) and 10) we obtain :

$$\begin{aligned} u(r,\phi) &= A\, e^{i\,(kr\cos\alpha\cos\phi + kr\sin\alpha\sin\phi)} \\ &= A\, e^{i\,k\,r\cos(\phi-\alpha)} \\ &= A\, e^{i\rho\cos(\phi-\alpha)}, \text{ where } \rho = kr. \quad \ldots\ldots\ldots 12) \end{aligned}$$

Equation 12) represents a plane wave which progresses in the direction $\phi = \alpha$.

Now we shall find the circular wave solution by separating equation 3) in polar coordinates. Equation 3) in terms of r, ϕ is

$$\frac{\partial^2 u}{\partial r^2}(r,\phi) + \frac{1}{r}\frac{\partial u}{\partial r}(r,\phi) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2}(r,\phi) + k^2 u(r,\phi) = 0. \quad ..13)$$

Assume $u(r,\phi) = R(r)\,\Phi(\phi)$ $\ldots\ldots\ldots\ldots\ldots\ldots 14)$

then 13) gives

$$\frac{d^2\Phi(\phi)}{d\phi^2} + n^2\Phi(\phi) = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots 15)$$

and

$$\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left(k^2 - \frac{n^2}{r^2}\right)R(r) = 0, \qquad \ldots\ldots\ldots\ldots\ldots 16)$$

where n is a constant. From 15) the eigenvalue must be negative and n must be an integer for single valuedsolution. Hence the solution of 15) is of the form

$$\Phi(\phi) = e^{in\phi}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots 17)$$

The general solution of 16) is of the form

$$R(r) = c_n J_n(kr) + d_n Y_n(kr) \qquad \ldots\ldots 18)$$

where J_n and Y_n are the Bessel and Neumann functions respectively. But we want the solution to be finite at $r = 0$ so we let $d_n = 0$. Hence the solution of 16) is of the form

$$R(r) = c_n J_n(kr).$$

Therefore the required solutions of 13) are of the form

$$u(r, \emptyset) = \sum_{n=-\infty}^{\infty} c_n J_n(kr)\, e^{in\emptyset}. \qquad \text{.............19)}$$

We want to find the coefficients c_n such that equation 19) represents a straight wave.

Without loss of generality we may let $\alpha = 0$ in 12). Then we see that $u = e^{i\rho \cos\emptyset}$ is the simplest solution of 3) which represents a straight wave progressing in the direction of the x-axis.

We shall develop $u = e^{i\rho \cos\emptyset}$ into a Fourier-Bessel series that is,

$$u = e^{i\rho\cos\emptyset} = \sum_{n=-\infty}^{\infty} c_n J_n(\rho)\, e^{in\emptyset} \qquad \text{..........20)}$$

and

$$c_n J_n(\rho) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\rho \cos\emptyset}\, e^{-in\emptyset}\, d\emptyset. \qquad \text{...21)}$$

Since the integral representation for $J_n(\rho)$ is

$$J_n(\rho)^{*} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\rho \cos\theta}\, e^{in(\theta-\pi/2)}\, d\theta. \qquad \text{..22)}$$

in which we may replace θ by $-\emptyset$, using 21) and 22) we get

$$c_n = e^{in\,\pi/2}$$

$$= \cos\frac{n\pi}{2} + i \sin\frac{n\pi}{2}.$$

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*Korn, Granino A. and Korn Theresa M. Mathematical Handbook for Scientists and Engineers. (McGraw-Hill Book Company 1968.) pp.860.

Now

$$
\begin{aligned}
c_o &= 1 \\
c_1 &= i \\
c_2 &= i^2 \\
&\cdots\cdots\cdots \\
&\cdots\cdots\cdots \\
c_n &= i^n ,
\end{aligned}
$$

and hence from 19) we have

$$
\begin{aligned}
u &= e^{i\rho \cos\phi} \\
&= \sum_{n=-\infty}^{\infty} i^n J_n(kr)\, e^{in\phi} \\
&= \ldots + i^{-n} J_{-n}(kr)\, e^{-in\phi} + \ldots + i^{-2} J_{-2}(kr)\, e^{-i2\phi} + \\
&\quad + i^{-1} J_{-1}(kr)\, e^{-i\phi} + J_o(kr) + i\, J_1(kr)\, e^{i\phi} + \\
&\quad + i^2 J_2(kr)\, e^{i2\phi} + \ldots + i^n J_n(kr)\, e^{in\phi} + \ldots\ldots\ldots \\
&= \ldots\, i^{-n}(-1)^n J_n(kr)\, e^{-in\phi} + \ldots + i^{-2}(-1)^2 J_2(kr)\, e^{-i2\phi} \\
&\quad + i^{-1}(-1)^1 J_1(kr)\, e^{-i\phi} + J_o(kr) + i\, J_1(kr)\, e^{i\phi} + \\
&\quad + i^2 J_2(kr)\, e^{i2\phi} + \ldots + i^n J_n(kr)\, e^{in\phi} + \ldots\ldots\ldots \\
&= \ldots + i^{n} J_n(kr)\, e^{-in\phi} + \ldots - J_2(kr)\, e^{-i2\phi} + i\, J_1(kr)\, e^{-i\phi} \\
&\quad + J_o(kr) + i\, J_1(kr)\, e^{i\phi} - J_2(kr)\, e^{i2\phi} + \ldots + i^n J_n(kr)\, e^{in\phi} + \ldots
\end{aligned}
$$

$$= J_o(kr) + iJ_1(kr)\left[e^{i\phi} + e^{-i\phi}\right] - J_2(kr)\left[e^{i2\phi} + e^{-i2\phi}\right]$$

$$+ \ldots + i^n J_n(kr)\left[e^{in\phi} + e^{-in\phi}\right] + \ldots\ldots\ldots\ldots$$

$$= J_o(kr) + 2i\, J_1(kr) \cos\phi - 2\, J_2(kr) \cos 2\phi + \ldots + \ldots\ldots$$

$$+ 2i^n\, J_n(kr) \cos n\phi + \ldots\ldots\ldots$$

$$= \sum_{m=0}^{\infty} \epsilon_m\, i^m\, J_m(kr) \cos m\,\phi\,, \qquad \ldots\ldots\ldots 23)$$

where $\epsilon_m = 1$ if $m = 0$

$= 2$ if $m > 0$.

Numerical Calculation and Graphical Representation

In one-dimension, let $f(x)$ be defined in the interval $(-\ell, \ell)$ and assumed that $f(x)$ has the period 2ℓ. The Fourier expansion corresponding to $f(x)$ is given by

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{\ell} + b_n \sin \frac{n\pi x}{\ell} \right),$$

where the Fourier coefficients a_n and b_n are

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \frac{n\pi x}{\ell}\, dx$$

$$, \quad n = 0,1,2,\ldots$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \frac{n\pi x}{\ell}\, dx\,.$$

To illustrate the calculation of a Fourier series, let $f(x) = x$, $-\pi < x < \pi$.

Calculating the coefficients a_n and b_n gives

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx \, dx = 0$$

and

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx = -\frac{2}{n}\cos n\pi .$$

Hence,

$$y = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right].$$

On the function

$$2 \sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x,$$

which contains only the first four terms of the series is an approximation to the function $f(x)$. The closeness of the approximation is indicated by the upper curves in Fig. 2.1, which show this partial sum of four terms together with the sums of six and ten terms. As the number of terms increases, the approximating curves approach $y = x$ for each fixed x on $-\pi < x < \pi$.

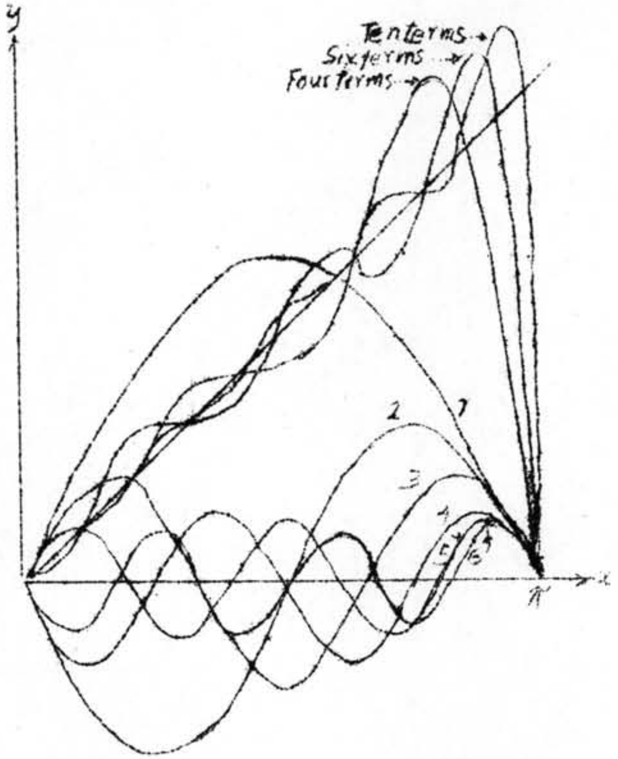

Figure 2.1 Shows the approximating curve of the partial sum of four terms, six terms and ten terms, and the first six terms are shown by the number 1,2,3,4,5,6 respectively.

The function $f(x) = x$, $-\pi < x < \pi$ is plotted in figure 2.2

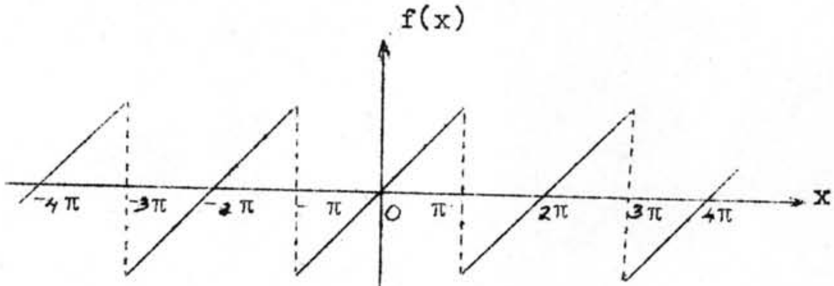

Figure 2.2

We shall use the same method to describe the two dimensional straight wave represented by the Fourier Bessel series. The straight wave in two dimensions is shown in figure 2.3. We shall approximate the straight wave by partial sums of the Fourier Bessel series.

Let us consider the equation obtained earlier :

$$u = \sum_{m=0}^{\infty} \epsilon_m i^m J_m(kr) \cos m\phi.$$

For simplicity let $k = 1$ and

$$u_n = \sum_{m=0}^{n} \epsilon_m i^m J_m(r) \cos m\phi$$

For $n = 0$

$$u_o = J_o(r),$$

$n = 1$

$$u_1 = J_o(r) + 2iJ_1(r) \cos \phi ,$$

$n = 2$

$$u_2 = J_o(r) - 2 J_2(r) \cos 2\phi + 2i J_1(r) \cos \phi ,$$

$n = 3$

$$u_3 = J_o(r) - 2 J_2(r) \cos 2\phi + 2i \left[J_1(r) \cos \phi - J_3(r) \cos 3\phi \right]$$

$n = 4$

$$u_4 = J_o(r) - 2 J_2(r) \cos 2\phi + 2J_4(r) \cos 4\phi + 2i \left[J_1(r) \cos \phi - J_3(r) \cos 3\phi \right].$$

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We shall consider only the real part of u_n. Let m and n be even so that we have

$$u_n = u_{n-2} - 2J_n(r)\cos n\phi\ , \quad n = 2, 6, 10, 14, \ldots$$

$$= u_{n-2} + 2\, J_n(r)\cos n\phi, \ n = 4, 8, 12, 16, \ldots$$

We shall find the value of u_n for $n = 0, 2, \ldots 8$;

We have

$$u_o = J_o(r)$$

$$u_2 = u_o - 2J_2(r)\cos 2\phi$$

$$u_4 = u_2 + 2\,J_4(r)\cos 4\,\phi$$

$$u_6 = u_4 - 2\,J_6(r)\cos 6\,\phi$$

$$u_8 = u_6 + 2\,J_8(r)\cos 8\,\phi\ ,$$

Where $r = \sqrt{x^2 + y^2}$, $x = 0, 1, 2, \ldots, 9$

$y = 0, 1, 2, \ldots, 9$

A discussion of the method of finding the values of the Bessel functions with orders greater than one is given in the appendix.

$u_o, u_2, \ldots, u_8$ are the partial sums of the Fourier Bessel series for one, two, three, four and five terms respectively. The numerical values for $u_o, u_2, \ldots, u_8$ are shown in Tables 2.1 - 2.5. From these tables we can obtain the approximations of the straight wave by circular waves. Table 2.1 contains values of the first term in the series. The function has circular symmetry and the nodal curves are circular as shown in figure 2.4 Table 2.3 gives values of u_4 which are shown graphically in figure 2.5. The nodal curves depart from the circular shape and approach the shape for straight waves near the origin. Table 2.5 gives values of u_8 which are shown graphically in figure 2.6. The nodal curves are almost straight lines.

Table 2.1, Values of $u_o = J_o(r)$ to four decimal places.

y \ x	0	1	2	3	4	5	6	7	8	9
0	1.000	0.7652	0.2240	-0.2601	-0.3971	-0.1776	0.1506	0.3001	0.1717	-0.0903
1	0.7652	0.5614	0.0937	-0.3094	-0.3865	-0.1477	0.1721	0.2997	0.1573	-0.1024
2	0.2240	0.0937	-0.1932	-0.3918	-0.3274	-0.0481	0.2279	0.2898	0.1118	-0.1389
3	-0.2601	-0.3094	-0.3918	-0.3707	-0.1776	0.1010	0.2851	0.2500	0.0310	-0.1907
4	-0.3971	-0.3865	-0.3274	-0.1776	0.0436	0.2433	0.2945	0.1573	-0.0754	-0.2358
5	-0.1776	-0.1477	-0.0481	0.1010	0.2433	0.2997	0.2134	0.0146	-0.1821	-0.2480
6	0.1506	0.1721	0.2279	0.2851	0.2945	0.2134	0.0474	-0.1389	-0.2459	-0.2018
7	0.3001	0.2997	0.2898	0.2500	0.1573	0.0146	-0.1389	-0.2396	-0.2245	-0.0880
8	0.1717	0.1573	0.1118	0.0310	-0.0754	-0.1821	-0.2459	-0.2245	-0.1099	0.0566
9	-0.0903	-0.1024	-0.1389	-0.1907	-0.2358	-0.2480	-0.2018	-0.0880	0.0566	0.1792

Table 2.2 Values of $u_2 = u_o - 2 J_2(r) \cos 2 \emptyset$ to four decimal places.

y \ x	0	1	2	3	4	5	6	7	8	9
0	1.000	0.5354	-0.4816	-1.2323	-1.1253	-0.2708	0.6364	0.9029	0.3977	-0.3799
1	0.9950	0.5614	-0.3874	-1.0852	-0.9740	-0.1765	0.6596	0.8663	0.3455	-0.4054
2	0.9296	0.5748	-0.1932	-0.7339	-0.6005	0.0683	0.6953	0.7461	0.1972	-0.4755
3	0.7121	0.4664	-0.0497	-0.3707	-0.2037	0.2950	0.6613	0.5364	-0.0187	-0.5517
4	0.3311	0.2010	-0.0543	-0.1515	0.0436	0.3750	0.5088	0.2559	-0.2336	-0.5745
5	-0.0844	-0.1189	-0.1645	-0.0930	0.1116	0.2997	0.2716	-0.0171	-0.3744	-0.5041
6	-0.3352	-0.3154	-0.2395	-0.0911	0.0802	0.1552	0.0474	-0.1957	-0.3885	-0.3365
7	-0.3027	-0.2669	-0.1665	-0.0364	0.0587	0.0463	-0.0821	-0.2396	-0.2787	-0.1121
8	-0.0543	-0.0309	0.0264	0.0807	0.0828	0.0102	-0.1033	-0.1703	-0.1099	0.0784
9	0.1993	0.2006	0.1977	0.1703	0.0849	0.0081	-0.0671	-0.0639	0.0348	0.1792

Table 2.3 Values of $u_4 = u_2 + 2J_4(r) \cos 4 \phi$ to four decimal places.

y \ x	0	1	2	3	4	5	6	7	8	9
0	1.000	0.5404	-0.4136	-0.9683	-0.5631	0.5116	1.3516	1.2185	0.1869	-0.9109
1	1.000	0.5428	-0.4154	-0.9990	-0.6411	0.3802	1.2078	1.1017	0.1355	-0.8903
2	0.9976	0.5468	-0.4114	-1.0435	-0.7936	0.1071	0.8685	0.8199	0.0199	-0.8267
3	0.9761	0.5526	-0.3593	-1.0017	-0.8634	-0.1240	0.5318	0.5370	-0.0766	-0.7029
4	0.8933	0.5339	-0.2474	-0.8112	-0.7360	-0.1572	0.3646	0.3717	-0.0873	-0.5254
5	0.6980	0.4378	-0.1257	-0.5120	-0.4206	0.0205	0.3806	0.3355	-0.0393	-0.3534
6	0.3800	0.2328	-0.0663	-0.2206	-0.0640	0.2642	0.4562	0.3263	-0.0182	-0.2531
7	0.0129	-0.0315	-0.0927	-0.0358	0.1745	0.3989	0.4399	0.2296	-0.0855	-0.2483
8	-0.2651	-0.2409	-0.1509	0.0228	0.2291	0.3453	0.2670	0.0229	-0.2215	-0.2867
9	-0.3317	-0.2843	-0.1535	0.0191	0.1340	0.1588	0.0163	-0.2001	-0.3303	-0.2796

Table 2.4 Values of $u_6 = u_4 - 2J_6(r)\cos 6\phi$ to four decimal places.

y \ x	0	1	2	3	4	5	6	7	8	9
0	1.000	0.5404	-0.4160	-0.9911	-0.6613	0.2496	0.8600	0.5401	-0.4883	-1.3195
1	1.000	0.5428	-0.4154	-0.9884	-0.6525	0.2743	0.9283	0.6497	-0.3540	-1.1968
2	0.9952	0.5427	-0.4114	-0.9892	-0.6405	0.3329	1.0667	0.8893	-0.0441	-0.9098
3	0.9989	0.5420	-0.4136	-1.0017	-0.6663	0.3255	1.1271	1.0720	0.2325	-0.6298
4	0.9915	0.5453	-0.4005	-1.0083	-0.7360	0.2002	1.0110	1.0380	0.3163	-0.4901
5	0.9600	0.5437	-0.3515	-0.9615	-0.7780	0.0205	0.7407	0.7914	0.1854	-0.5084
6	0.8716	0.5123	-0.2645	-0.8159	-0.7104	-0.0959	0.4562	0.4706	-0.0400	-0.5764
7	0.6913	0.4205	-0.1621	-0.5708	-0.4918	-0.0570	0.2956	0.2296	0.1978	-0.5748
8	0.4101	0.2486	-0.0869	-0.5954	-0.1745	0.1206	0.2888	0.1352	-0.2215	-0.4535
9	0.0769	0.0222	-0.0704	-0.0540	0.0987	0.3138	0.3396	0.1264	-0.1635	-0.2796

Table 2.5 Values of $u_8 = u_6 + 2\,J_8(r)\cos 8\phi$ to four decimal places

y \ x	0	1	2	3	4	5	6	7	8	9
0	1.000	0.5404	-0.4160	-0.9901	-0.6533	0.2864	0.9730	0.7961	-0.0413	-0.7093
1	1.000	0.5428	-0.4156	-0.9897	-0.6563	0.2740	0.9584	0.7629	-0.1046	-0.8071
2	0.9952	0.5425	-0.4108	-0.9892	-0.6552	0.2744	0.9390	0.7024	-0.2308	-1.0215
3	0.9999	0.5407	-0.4136	-0.9895	-0.6508	0.2892	0.9532	0.7036	-0.2929	-1.1745
4	0.9995	0.5415	-0.4152	-0.9928	-0.6566	0.3027	1.0086	0.7946	-0.1923	-1.1217
5	0.9968	0.5434	-0.4100	-0.9978	-0.6755	0.2887	1.0472	0.9285	0.0301	-0.8770
6	0.9846	0.5424	-0.3922	-0.9898	-0.7128	0.2106	0.9916	0.9859	0.2283	-0.5805
7	0.9473	0.5337	-0.3490	-0.9392	-0.7352	0.0801	0.8109	0.8722	0.2709	-0.4024
8	0.8571	0.4980	-0.2736	-1.1208	-0.6831	-0.0347	0.5571	0.6830	0.1301	-0.3873
9	0.6871	0.4119	-0.1821	-0.5987	-0.5329	-0.0548	0.3355	0.2988	-0.0973	-0.4686

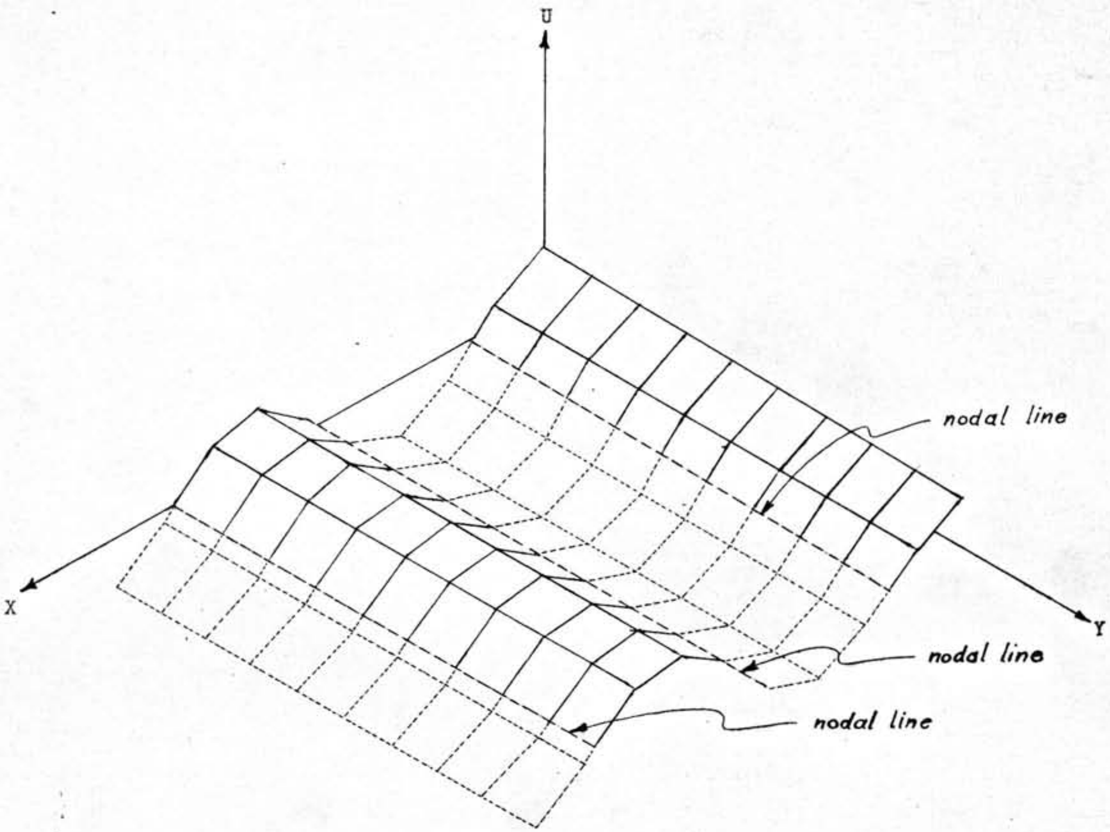

Figure 2.3 Shows the straight wave in two dimensions .

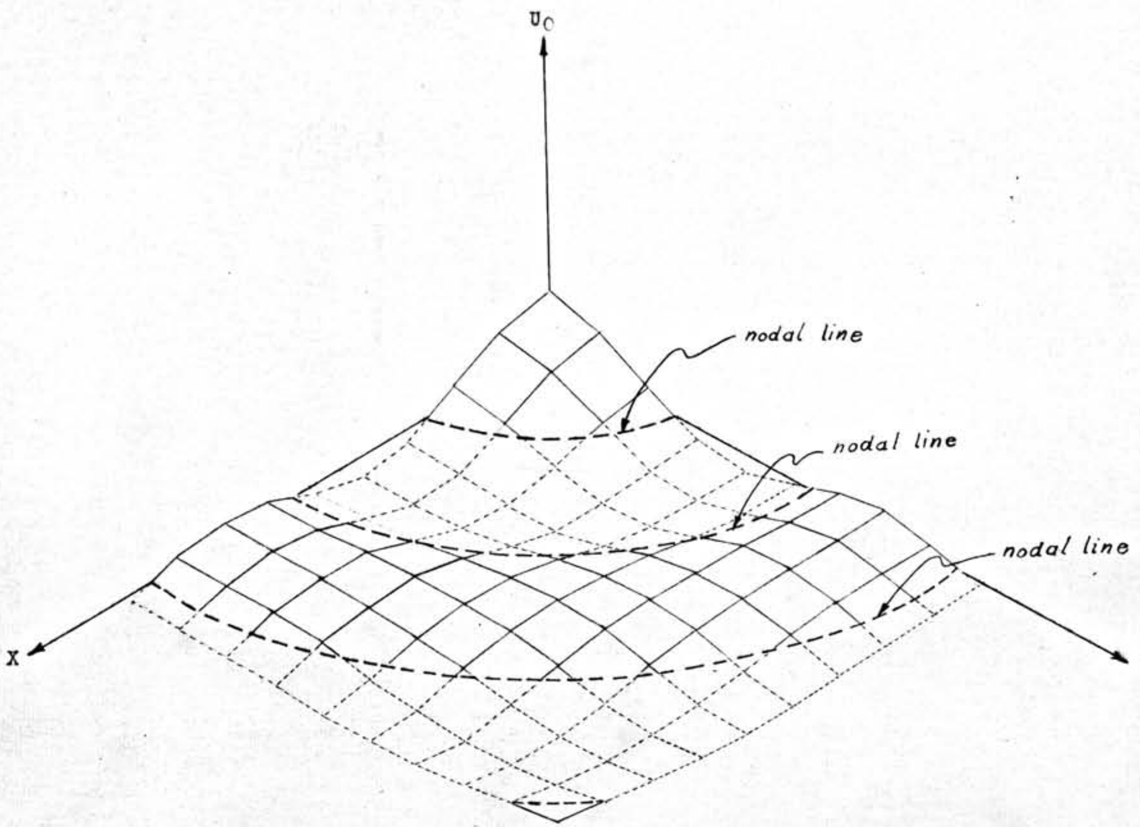

Figure 2.4 The graph representation of u_o, the nodal line is circular.

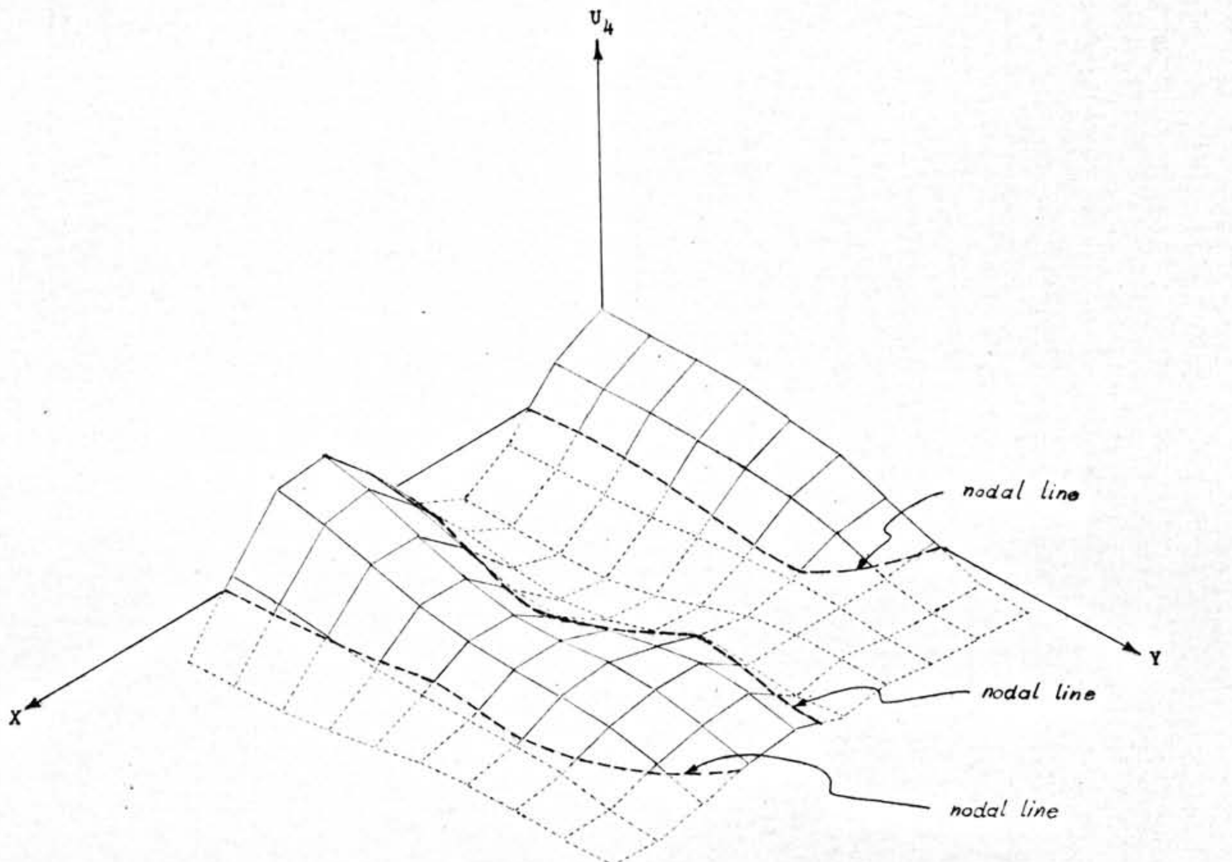

Figure 2.5 The graph representation of u_4, the nodal line departs from the circular shape.

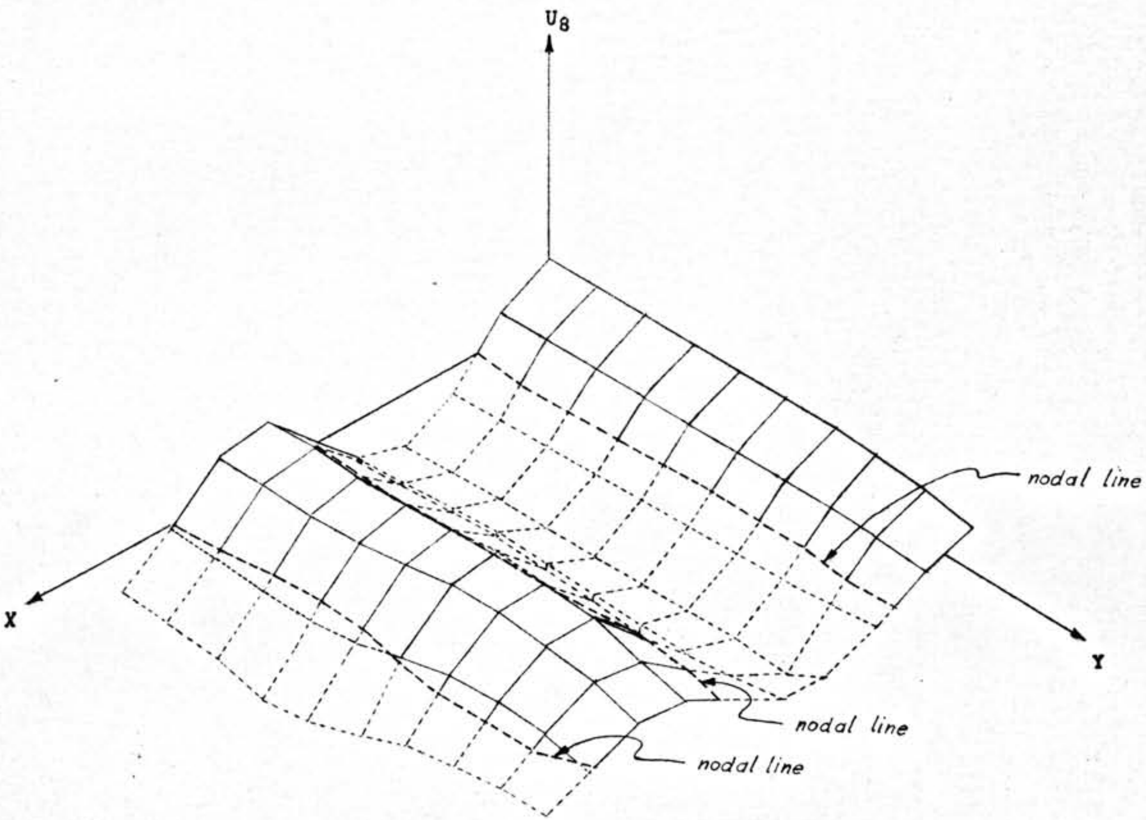

Figure 2.6 The graph representation of u_8, the nodal line is almost straight line.