CHAPTER IV

SEMILATTICE DECOMPOSITIONS

In this chapter, semilattice decompositions on orthodox semigroups and on quasi-inverse semigroups are studied.

Let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a semilattice Y of semigroups S_α. Let $\alpha \in Y$ and $a \in S_\alpha$. Assume $x \in S$ and x is an inverse of a in S. Then $x \in S_\beta$ for some $\beta \in Y$ and a = axa, x = xax. Because a = axa, $a \in S_\alpha \cap S_{\alpha\beta\alpha} = S_\alpha \cap S_{\alpha\beta}$, so $\alpha = \alpha\beta$. From x = xax, it follows that $x \in S_\beta \cap S_{\beta\alpha\beta} = S_\beta \cap S_{\alpha\beta}$ and hence $\beta = \alpha\beta$. Therefore $\alpha = \beta$. This shows that for any $\alpha \in Y$ and $a \in S_\alpha$, $V(a) \subseteq S_\alpha$.

Let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a semilattice Y of semigroups S_α. If S_α is regular for each $\alpha \in Y$, then S is clearly regular. Assume S is regular. Let $\alpha \in Y$ and $a \in S_\alpha$. Because S is regular, there is $x \in S$ such that a = axa and x = xax. Because $a \in S_\alpha$, $V(a) \subseteq S_\alpha$ and hence $x \in S_\alpha$. Therefore S_α is regular for each $\alpha \in Y$.

Let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a semilattice Y of semigroups S_α. If S is orthodox, then S_α is a regular subsemigroup of S for all $\alpha \in Y$ and hence, by Proposition 1.2, S_α is orthodox for all $\alpha \in Y$. The converse is also true. A proof is given in [6].

4.1 <u>Theorem</u> [6, Corollary IV.3.2]. Let $S = \bigcup_{\alpha\in Y} S_\alpha$ be a semilattice Y of semigroups S_α. Then S is orthodox if and only if S_α is orthodox for each $\alpha \in Y$.

The following proposition shows that Theorem 4.1 is still true if we replace "orthodox" by "right-inverse" :

4.2 Proposition. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of semigroups S_α. Then S is right-inverse if and only if S_α is right-inverse for each $\alpha \in Y$.

Proof : Assume S is right-inverse. Then S is regular. Therefore S_α is regular for all $\alpha \in Y$, so for each $\alpha \in Y$, S_α is a regular subsemigroup of the right-inverse semigroup S. By Proposition 1.7(1), S_α is right-inverse for each $\alpha \in Y$.

Conversely, assume S_α is right-inverse for all $\alpha \in Y$. Then S_α is regular for all $\alpha \in Y$, and therefore S is regular. Let e, f $\in E(S)$. Then $e \in E(S_\alpha)$ and $f \in E(S_\beta)$ for some $\alpha, \beta \in Y$ which imply ef, fe $\in S_{\alpha\beta}$. Since S_γ is right-inverse for each $\gamma \in Y$, S_γ is orthodox for each $\gamma \in Y$. By Theorem 4.1, S is orthodox, so E(S) is a subsemigroup of S. Then ef, fe $\in E(S_{\alpha\beta})$ because ef, fe $\in E(S)$ and ef, fe $\in S_{\alpha\beta}$. Therefore

efe = effe = (fe)(ef)(fe) (since $S_{\alpha\beta}$ is right-inverse and ef, fe $\in E(S_{\alpha\beta})$)

= fe.

Hence S is right-inverse. #

Because a generalized inverse semigroup is regular and a regular subsemigroup of a generalized inverse semigroup is generalized inverse, the following proposition follows directly :

4.3 Proposition. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of semigroups S_α. If S is generalized inverse , then S_α is generalized inverse for all $\alpha \in Y$.

The converse of Proposition 4.3 is not true in general. A counter example is given as follows :

Example. Let $S = \{I, E_1, E_2, E_3\}$ where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $E_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_3 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$. Then, under the usual matrix multiplication, the table of multiplication of S is as follows :

$\cdot$	I	E_1	E_2	E_3
I	I	E_1	E_2	E_3
E_1	E_1	E_1	E_1	E_1
E_2	E_2	E_2	E_2	E_2
E_3	E_3	E_2	E_1	I

Then S is a semigroup with identity I. Let $Y = \{\alpha, \beta\}$ be a semilattice with identity α and zero β. Let $S_\alpha = \{I, E_3\}$ and $S_\beta = \{E_1, E_2\}$. Then $S = S_\alpha \cup S_\beta$ and from the table S_α and S_β are subsemigroups of S and $S_\alpha S_\beta \subseteq S_\beta = S_{\alpha\beta}$, $S_\beta S_\alpha \subseteq S_\beta = S_{\beta\alpha}$. Then S is a semilattice Y of semigroups S_α and S_β. Because S_α is a group, S_α is generalized inverse. From the table, $E(S) = \{I, E_1, E_2\}$. Because $E(S_\beta) = S_\beta$ and $E_1E_1E_2E_1 = E_1 = E_1E_2E_1E_1$, $E_1E_1E_2E_2 = E_1 = E_1E_2E_1E_2$, $E_2E_1E_2E_1 = E_2 = E_2E_2E_1E_1$ and $E_2E_1E_2E_2 = E_2 = E_2E_2E_1E_2$, it then follows that S_β is generalized inverse. But $IE_1E_2I = E_1 \neq E_2 = IE_2E_1I$. Then S is not generalized inverse. #

If T is a subsemigroup of a semigroup S and M is an inverse subsemigroup of T, then M is an inverse subsemigroup of S. Thus, from the definition of being quasi-inverse, it clearly follows that any semigroup which is a union of quasi-inverse subsemigroups is quasi-inverse. Hence, a semilattice of quasi-inverse semigroups is quasi-inverse.

It has been shown in Chapter I that a regular subsemigroup of a quasi-inverse semigroup is not necessarily quasi-inverse. However, we show in the next theorem that if $S = \bigcup_{\alpha \in Y} S_\alpha$ is a semilattice Y of semigroups S_α and S is quasi-inverse, then S_α is quasi-inverse for each $\alpha \in Y$.

4.4 <u>Theorem</u>. Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of semigroups S_α. Then S is quasi-inverse if and only if S_α is quasi-inverse for each $\alpha \in Y$.

<u>Proof</u> : Assume S is quasi-inverse. Let $\alpha \in Y$. To show S_α is quasi-inverse, let $a \in S_\alpha$. Because $a \in S$ which is quasi-inverse, there exists an inverse subsemigroup T of S containing a. Next, to show $T \cap S_\alpha$ is an inverse subsemigroup of S_α containing a. Because $T \cap S_\alpha \neq \phi$ and T and S_α are subsemigroups of S, $T \cap S_\alpha$ is a subsemigroup of S. But $T \cap S_\alpha \subseteq S_\alpha$. Then $T \cap S_\alpha$ is a subsemigroup of S_α. Let $x \in T \cap S_\alpha$. Then, there exists $x' \in T$ such that $x = xx'x$ and $x' = x'xx'$. Because $x \in S_\alpha$, $V(x) \subseteq S_\alpha$ and hence $x' \in S_\alpha$. Therefore $T \cap S_\alpha$ is a regular subsemigroup of T. Because a regular subsemigroup

of an inverse semigroup is an inverse semigroup, $T \cap S_\alpha$ is an inverse subsemigroup of T. Then $T \cap S_\alpha$ is an inverse subsemigroup of S_α containing a. Hence S_α is quasi-inverse. #

Let S be a semigroup and ρ be a semilattice congruence on S. Then ρ decomposes S to be a semilattice S/ρ of subsemigroups. Hence, by Theorem 4.1, Proposition 4.2, Proposition 4.3 and Theorem 4.4, the following proposition directly follows :

4.5 Proposition. Let S be a semigroup and ρ be a semilattice congruence on S. Then the following hold :

(1) If S is an orthodox semigroup, then each ρ-class forms an orthodox subsemigroup of S.

(2) If S is a right-inverse semigroup, then each ρ-class forms a right-inverse subsemigroup of S.

(3) If S is a generalized inverse semigroup, then each ρ-class forms a generalized inverse subsemigroup of S.

(4) If S is a quasi-inverse semigroup, then each ρ-class forms a quasi-inverse subsemigroup of S.

It has been shown in Theorem 3.9 that for any regular subsemigroup T of an orthodox semigroup S,

$$\delta(T) = \delta(S) \cap (T \times T)$$

where $\delta(S)$ and $\delta(T)$ are the minimum inverse congruences on S and on T; respectively.

Let $S = \bigcup_{\alpha \in Y} S_\alpha$ be a semilattice Y of orthodox semigroups S_α. Then S is orthodox and for each $\alpha \in Y$, S_α is a regular subsemigroup of S. Therefore, for each $\alpha \in Y$,

$$\delta(S_\alpha) = \delta(S) \cap (S_\alpha \times S_\alpha).$$

Let ρ be congruence on a semigroup S. Then ρ is called a <u>semilattice-of-inverse semigroups congruence</u> on S if S/ρ is a semilattice of inverse semigroups.

Every inverse congruence on a semigroup S is clearly a semilattice-of-inverse semigroups congruence on S.

Let S be a semigroup and ρ be a semilattice-of-inverse semigroups congruence on S. Then S/ρ is a semilattice of inverse semigroups. Because a semilattice of inverse semigroups is an inverse semigroup [2, Theorem 7.5], S/ρ is an inverse semigroup and hence ρ is an inverse congruence on S.

Hence, the following remark follows :

4.6 <u>Remark</u>. In any orthodox semigroup S, the relation

$$\{(a, b) \in S \times S \mid V(a) = V(b)\}$$

gives the greatest decomposition of S to a semilattice of inverse semigroups.