CHAPTER II

THEORETICAL ANALYSIS

2.1 The Principle of Energy[15]

The total potential energy of a deformed flexible plate, I, is defined as the sum of the potential energy of deformation U and the potential energy of loading W.

$$I = U + W \tag{2.1}$$

The quantity U can also be written as the sum of the energy corresponding to the bending V and the energy corresponding to the deformation of the middle surface T of the plate.

$$U = V + T \tag{2.2}$$

For the case of uniform in-plane pressure acting on the edge of circular plates, the potential energy of loading disappears.

$$W = 0 \tag{2.3}$$

Therefore, the total potential energy of such a plate is the potential energy of deformation.

$$I = U = V + T \tag{2.4}$$

2.2 Strain Energy Due to Bending[12]

As knowing that a half of the product of a stress component and the corresponding strain component represents work done by the stress. The sum of the work done by all stress components is a half of the summation of all the products of stresses and their corresponding strains.

$$V = \frac{1}{2}\int_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \sigma_{zz}\varepsilon_{zz} + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz})dv. \qquad (2.5)$$

The plane stress condition allows

$$\sigma_{zz} = \tau_{yz} = \tau_{xz} = 0 \qquad (2.6)$$

Thus, the sum of the work done is

$$V = \frac{1}{2}\int_V (\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy} + \tau_{xy}\gamma_{xy})dv \qquad (2.7)$$

By substituting the right-hand side of Eq.(2.7) with

$$\begin{aligned}
\sigma_{xx} &= -\frac{12z}{h^3}(D_x w_{,xx} + D_1 w_{,yy}) \\
\sigma_{yy} &= -\frac{12z}{h^3}(D_y w_{,yy} + D_1 w_{,xx}) \\
\tau_{xy} &= -\frac{12z}{h^3} D_{xy} w_{,xy} \\
\varepsilon_{xx} &= -z w_{,xx} \\
\varepsilon_{yy} &= -z w_{,yy} \\
\gamma_{xy} &= -2z w_{,xy}
\end{aligned} \qquad (2.8)$$

(see Appendix A.), Eq.(2.7) becomes

$$V = \frac{6}{h^3}\int_A \int_{-h/2}^{h/2} z^2 (D_x w_{,xx}^2 + D_y w_{,yy}^2 + 2D_1 w_{,xx} w_{,yy} + 2D_{xy} w_{,xy}^2) dz dA \quad (2.9)$$

or

$$V = \frac{1}{2}\int_A (D_x w_{,xx}^2 + D_y w_{,yy}^2 + 2D_1 w_{,xx} w_{,yy} + 2D_{xy} w_{,xy}^2) dA \qquad (2.10)$$

2.3 Potential Energy Due to Mid-plane Force During Bending[11]

Neglecting any stretching in the middle plane of a plane-stress plate, the potential energy due to the forces acting in the middle plane of the plate during bending can be represented as follows

$$T = \frac{h}{2} \int_A (\sigma^\circ_{xx} w^2_{,x} + \sigma^\circ_{yy} w^2_{,y} + 2\tau^\circ_{xy} w_{,x} w_{,y})dA \tag{2.11}$$

where superscript '°' indicates the membrane stresses due to the mid-plane forces.

2.4 Transformation from X-Y to r-θ Coordinates

By making the following substitutions

$$\begin{aligned} w_{,xx} &= w_{,rr} \\ w_{,yy} &= \frac{1}{r} w_{,r} + \frac{1}{r^2} w_{,\theta\theta} \\ w_{,xy} &= (\frac{1}{r} w_{,\theta})_{,r} \\ D_x &\longrightarrow D_r \\ D_y &\longrightarrow D_\theta \\ D_{xy} &\longrightarrow D_{r\theta} \end{aligned} \tag{2.12}$$

The strain energy and the potential energy due to mid-plane force in polar coordinates are

$$V = \frac{1}{2} \int_0^{2\pi} \int_a^b \Big\{ D_r w^2_{,rr} + 2D_1 w_{,rr} \cdot (\frac{1}{r} w_{,r} + \frac{1}{r^2} w_{,\theta\theta}) + D_\theta (\frac{1}{r} w_{,r} + \frac{1}{r^2} w_{,\theta\theta})^2 + 2D_{r\theta} \Big[(\frac{1}{r} w_{,\theta})_{,r}\Big]^2 \Big\} r dr d\theta \tag{2.13}$$

$$T = \frac{h}{2}\int_0^{2\pi}\int_a^b \left[\sigma_r^\circ w_{,r}^2 + \sigma_\theta^\circ\left(\frac{1}{r}w_{,\theta}\right)^2 + 2\tau_{r\theta}^\circ w_{,r}\left(\frac{1}{r}w_{,\theta}\right)\right] r\,dr\,d\theta \quad (2.14)$$

In this case, $\tau_{r\theta}^\circ = 0$, thus

$$T = \frac{h}{2}\int_0^{2\pi}\int_a^b \left[\sigma_r^\circ w_{,r}^2 + \sigma_\theta^\circ\left(\frac{1}{r}w_{,\theta}\right)^2\right] r\,dr\,d\theta \quad (2.15)$$

And the total potential energy is the sum of those two energies, by substituting Eq.(2.14) and Eq.(2.15) into Eq.(2.4), then

$$\begin{aligned} I = \frac{1}{2}\int_0^{2\pi}\int_a^b \Big\{ & D_r w_{,rr}^2 + 2D_1 w_{,rr}\left(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta}\right) + D_\theta\left(\frac{1}{r}w_{,r}\right. \\ & \left. + \frac{1}{r^2}w_{,\theta\theta}\right)^2 + 2D_{r\theta}\left[\left(\frac{1}{r}w_{,\theta}\right)_{,r}\right]^2 + h(P\cdot r^{k-1} + Q\cdot r^{-k-1})\, w_{,r}^2 \\ & + hk\cdot(P\cdot r^{k-1} - Q\cdot r^{-k-1})\cdot\left(\frac{1}{r}w_{,\theta}\right)^2 \Big\} r\,dr\,d\theta \end{aligned} \quad (2.16)$$

In our case, the membrane stress components[4] are

$$\begin{aligned} \sigma_r^\circ &= P\cdot r^{k-1} + Q\cdot r^{-k-1} \\ \sigma_\theta^\circ &= k\cdot(P\cdot r^{k-1} - Q\cdot r^{-k-1}) \end{aligned} \quad (2.17)$$

where $P = -\dfrac{P_o b^{k+1} - P_i a^{k+1}}{b^{2k} - a^{2k}}$

$$Q = \frac{(P_o a^{k-1} - P_i b^{k-1})\cdot(ab)^{k+1}}{b^{2k} - a^{2k}}$$

2.5 Governing Differential Equation and Boundary Conditions

The state of equilibrium of a deformed flexible plate can now be characterized as that for which the first variation of the total potential energy of the system is equal to zero.

$$\delta I = 0 \quad (2.18)$$

From Eq.(2.16), the total energy may be considered as the function of the following variables

$$I = f(w_{,r},\ w_{,\theta},\ w_{,rr},\ w_{,\theta\theta},\ w_{,r\theta}) \tag{2.19}$$

Therefore, the first variation of the total energy can be found as

$$\delta I = \frac{\partial I}{\partial w_{,r}}\delta w_{,r} + \frac{\partial I}{\partial w_{,\theta}}\delta w_{,\theta} + \frac{\partial I}{\partial w_{,rr}}\delta w_{,rr} + \frac{\partial I}{\partial w_{,\theta\theta}}\delta w_{,\theta\theta} + \frac{\partial I}{\partial w_{,r\theta}}\delta w_{,r\theta} \tag{2.20}$$

After substituting the total potential energy, I, from Eq.(2.16) into Eq.(2.20), and employed the technique of integration by parts, the Eq.(2.18) becomes

$$\begin{aligned}
&\int_0^{2\pi}\!\!\int_a^b \Big[rD_r w_{,rrrr} + 2D_r w_{,rrr} - \frac{D_\theta}{r} w_{,rr} + \frac{D_\theta}{r^2} w_{,r} - \frac{2}{r^2}(D_1 \\
&+ D_{r\theta}) w_{,r\theta\theta} + \frac{2}{r}(D_1 + D_{r\theta}) w_{,rr\theta\theta} + \frac{2}{r^3}(D_\theta + D_1 + D_{r\theta}) w_{,\theta\theta} \\
&+ \frac{D_\theta}{r^3} w_{,\theta\theta\theta\theta} - h(P\cdot r^k + Q\cdot \bar{r}^k) w_{,rr} - hk(P\cdot r^{k-1} - Q\cdot \bar{r}^{k-1}) \\
&\cdot (w_{,r} + \frac{1}{r} w_{,\theta\theta}) \Big] \delta w\, dr\, d\theta + \int_0^{2\pi} \Big[rD_r w_{,rr} + D_1 (w_{,r} + \\
&\frac{1}{r} w_{,\theta\theta}) \Big] \delta w_{,r}\, d\theta \Big|_{r=a}^{r=b} + \int_0^{2\pi} \Big[- rD_r w_{,rrr} - D_r w_{,rr} + \frac{D_\theta}{r} w_{,r} \\
&- \frac{1}{r}(D_1 + 2D_{r\theta}) w_{,r\theta\theta} + \frac{1}{r^2}(D_\theta + D_1 + 2D_{r\theta}) w_{,\theta\theta} + rh\sigma_r w_{,r} \Big] \\
&\cdot \delta w\, d\theta \Big|_{r=a}^{r=b} + \int_a^b \Big[-\frac{1}{r}(D_1 + 2D_{r\theta}) w_{,rr\theta} - \frac{1}{r^2}(D_\theta - 2D_{r\theta}) \\
&\cdot w_{,r\theta} - \frac{2D_{r\theta}}{r^3} w_{,\theta} - \frac{D_\theta}{r^3} w_{,\theta\theta\theta} + hk(P\cdot r^{k-2} - Q\cdot \bar{r}^{k-2}) w_{,\theta} \Big] \\
&\cdot \delta w\, dr \Big|_{\theta=0}^{\theta=2\pi} + \int_a^b \Big[\frac{D_1}{r} w_{,rr} + \frac{D_\theta}{r^2} w_{,r} + \frac{D_\theta}{r^3} w_{,\theta\theta} \Big] \delta w_{,\theta}\, dr \Big|_{\theta=0}^{\theta=2\pi} \\
&+ 2D_{r\theta} \Big[\frac{1}{r} w_{,r\theta} - \frac{1}{r^2} w_{,\theta} \Big] \delta w \Big|_{r=a}^{r=b} \Big|_{\theta=0}^{\theta=2\pi} = 0
\end{aligned} \tag{2.21}$$

The last three terms of the left-hand side vanish because of the definite integrals from $\theta = 0$ to $\theta = 2\pi$. The buckling equation is obtained by setting the first integrand of the left-hand side to zero, from which one obtains

$$D_r w_{,rrrr} + \frac{2D_r}{r} w_{,rrr} - \frac{D_\theta}{r^2} w_{,rr} + \frac{D_\theta}{r^3} w_{,r} - \frac{2}{r^3}(D_1 + D_{r\theta}) w_{,r\theta\theta} + \frac{2}{r^2}(D_1 + D_{r\theta}) w_{,rr\theta\theta} + \frac{2}{r^4}(D_\theta + D_1 + D_{r\theta}) \cdot w_{,\theta\theta} + \frac{D_\theta}{r^4} w_{,\theta\theta\theta\theta} - h\cdot(P\cdot r^{k-1} + Q\cdot\bar{r}^{k-1}) w_{,rr} - hk \cdot (P\cdot r^{k-1} - Q\cdot\bar{r}^{k-1})\cdot\left(\frac{1}{r} w_{,r} + \frac{1}{r^2} w_{,\theta\theta}\right) = 0 \qquad (2.22)$$

or

$$L(w) = 0 \qquad (2.23)$$

where $L(\ )$ is the linear differential operator as given by Eq.(2.22).

The boundary conditions along the edges obtained from Eq.(2.21) are

either $$D_r w_{,rr} + D_1\left(\frac{1}{r} w_{,r} + \frac{1}{r^2} w_{,\theta\theta}\right) = 0$$

or $$\delta w_{,r} = 0$$

either $$D_r w_{,rrr} + \frac{D_r}{r} w_{,rr} - \frac{D_\theta}{r^2} w_{,r} + \frac{1}{r^2}(D_1 + 2D_{r\theta}) w_{,r\theta\theta} - \frac{1}{r^3}(D_\theta + D_1 + 2D_{r\theta}) w_{,\theta\theta} - h\,(P\cdot r^{k-1} + Q\cdot\bar{r}^{k-1}) w_{,r} = 0 \qquad (2.24)$$

or $$\delta w = 0$$

The first boundary condition is either the bending moment or the slope must be zero, and the second, either the effective transverse shear force or the deflection must be zero.

2.6 Application of Galerkin's Method

For general buckling mode, the deflection function is assumed to be in the form

$$w = F(r)\cos n\theta \quad ; \qquad n = 0, 1, 2, \ldots\ldots \tag{2.25}$$

where n is the number of half-waves on the circumference. Note that $n = 0$ corresponds to the axisymmetrical buckling mode. By applying Galerkin's method(see Appendix B.) to the buckling equation (2.23),one has

$$\int_0^{2\pi}\int_a^b L(w)\cdot\eta_i(r,\theta)\cdot dr\,d\theta = 0 \tag{2.26}$$

After substituting Eq.(2.25) into Eq.(2.26), it is clearly seen that $\cos n\theta$ is a common factor, therefore, Eq.(2.26) becomes

$$\int_a^b L_1\left[F(r)\right]\cdot F(r)\cdot dr = 0 \tag{2.27}$$

where
$$\begin{aligned} L_1(\ ) = {} & D_r\frac{d^4(\ )}{dr^4} + \frac{2D_r}{r}\frac{d^3(\ )}{dr^3} - \frac{D_\theta}{r^2}\frac{d^2(\ )}{dr^2} + \frac{D_\theta}{r^3}\frac{d(\ )}{dr} + \frac{2n^2}{r^3}(D_1 + \\ & D_{r\theta})\frac{d(\ )}{dr} - \frac{2n^2}{r^2}(D_1 + D_{r\theta})\frac{d^2(\ )}{dr^2} - \frac{2n^2}{r^4}(D_\theta + D_1 + D_{r\theta}) \\ & \cdot(\ ) + \frac{D_\theta n^4}{r^4}(\ ) - h\cdot(P\cdot r^{k-1} + Q\cdot r^{-k-1})\frac{d^2(\ )}{dr^2} - hk \\ & \cdot(P\cdot r^{k-1} - Q\cdot r^{-k-1})\left(\frac{1}{r}\frac{d(\ )}{dr} - \frac{n^2}{r^2}(\ )\right) \\ = {} & 0 \end{aligned}$$

The Eq.(2.26) and Eq.(2.27) are applicable only when the boundary conditions are satisfied by the deflection function. Therefore, what needed to do next is to find the proper function F(r) which satisfies the boundary conditions. Four cases of different edge conditions are considered in the present research. They are

case	inner edge	outer edge
1	clamped	clamped
2	simply supported	clamped
3	clamped	simply supported
4	simply supported	simply supported

The selected deflection function is

$$F(r) = C_0 + C_1 r^2 + C_2 r^4 + C_3 r^6 + C_4 r^8 \tag{2.28}$$

where C_0, C_1, C_2, C_3, C_4 are arbitary constants.

For the function F(r) to satisfy the boundary conditions of various cases, the constants C_1, C_2, C_3, and C_4 must have the following values.

case 1 :

$$C_1 = -\frac{2C_0(\alpha^8 - 2\alpha^6 + 2\alpha^2 - 1)}{\alpha^8 - 3\alpha^6 + 3\alpha^4 - \alpha^2}$$

$$C_2 = -\frac{C_0(3\alpha^8 - 4\alpha^6 + 1)}{\alpha^8 - 2\alpha^6 + \alpha^4} - \frac{C_1(2\alpha^6 - 3\alpha^4 + 1)}{\alpha^6 - 2\alpha^4 + \alpha^2}$$

$$C_3 = -\frac{C_0(\alpha^8 - 1)}{\alpha^8 - \alpha^6} - \frac{C_1(\alpha^6 - 1)}{\alpha^6 - \alpha^4} - \frac{C_2(\alpha^4 - 1)}{\alpha^4 - \alpha^2}$$

$$C_4 = -C_0 - C_1 - C_2 - C_3$$

case 2 :

$$C_1 = \frac{-2C_0\begin{bmatrix}(13+\nu_0)\alpha^{10} - (33+3\nu_0)\alpha^8 + (18+2\nu_0)\alpha^6 \\ + (14+2\nu_0)\alpha^4 - (15+3\nu_0)\alpha^2 + (3+\nu_0)\end{bmatrix}}{\begin{bmatrix}(13+\nu_0)\alpha^{10} - (44+4\nu_0)\alpha^8 + (54+6\nu_0)\alpha^6 \\ - (28+4\nu_0)\alpha^4 + (5+\nu_0)\alpha^2\end{bmatrix}}$$

$$C_2 = -\frac{C_0(3\alpha^8 - 4\alpha^6 + 1)}{\alpha^8 - 2\alpha^6 + \alpha^4} - \frac{C_1(2\alpha^6 - 3\alpha^4 + 1)}{\alpha^6 - 2\alpha^4 + \alpha^2}$$

$$C_3 = -\frac{C_0(\alpha^8 - 1)}{\alpha^8 - \alpha^6} - \frac{C_1(\alpha^6 - 1)}{\alpha^6 - \alpha^4} - \frac{C_2(\alpha^4 - 1)}{\alpha^4 - \alpha^2}$$

$$C_4 = -C_0 - C_1 - C_2 - C_3$$

case 3 :

$$C_1 = \frac{-2C_0\begin{bmatrix}(3+\nu_0)\alpha^{10} - (15+3\nu_0)\alpha^8 + (14+2\nu_0)\alpha^6 \\ + (18+2\nu_0)\alpha^4 - (33+3\nu_0)\alpha^2 + (13+\nu_0)\end{bmatrix}}{\begin{bmatrix}(5+\nu_0)\alpha^{10} - (28+4\nu_0)\alpha^8 + (54+6\nu_0)\alpha^6 \\ - (44+4\nu_0)\alpha^4 + (13+\nu_0)\alpha^2\end{bmatrix}}$$

$$C_2 = -\frac{C_0(\alpha^8 - 4\alpha^2 + 3)}{\alpha^8 - 2\alpha^6 + \alpha^4} - \frac{C_1(\alpha^6 - 3\alpha^2 + 2)}{\alpha^6 - 2\alpha^4 + \alpha^2}$$

$$C_3 = -\frac{C_0(\alpha^8 - 1)}{\alpha^8 - \alpha^6} - \frac{C_1(\alpha^6 - 1)}{\alpha^6 - \alpha^4} - \frac{C_2(\alpha^4 - 1)}{\alpha^4 - \alpha^2}$$

$$C_4 = -C_0 - C_1 - C_2 - C_3$$

case 4 :

$$C_1 = \frac{-2C_0\begin{bmatrix}(39 - 16\nu_\theta + \nu_\theta^2)\alpha^{10} - (165 + 48\nu_\theta + 3\nu_\theta^2)\alpha^8 \\ + (126 + 32\nu_\theta + 2\nu_\theta^2)\alpha^6 + (126 + 32\nu_\theta + 2\nu_\theta^2)\alpha^4 \\ - (165 + 48\nu_\theta + 3\nu_\theta^2)\alpha^2 + (39 + 16\nu_\theta + \nu_\theta^2)\end{bmatrix}}{\begin{bmatrix}(65 + 18\nu_\theta + \nu_\theta^2)\alpha^{10} - (308 + 72\nu_\theta + 4\nu_\theta^2)\alpha^8 \\ + (486 + 108\nu_\theta + 6\nu_\theta^2)\alpha^6 - (308 + 72\nu_\theta + 4\nu_\theta^2)\alpha^4 \\ + (65 + 18\nu_\theta + \nu_\theta^2)\alpha^2\end{bmatrix}}$$

$$C_2 = -\frac{C_0\left[(15 + 3\nu_\theta)\alpha^8 - (28 + 4\nu_\theta)\alpha^6 + (13 + \nu_\theta)\right]}{(9 + \nu_\theta)\alpha^8 - (22 + 2\nu_\theta)\alpha^6 + (13 + \nu_\theta)\alpha^4}$$

$$- \frac{C_1\left[(14 + 2\nu_\theta)\alpha^6 - (27 + 3\nu_\theta)\alpha^4 + (13 + \nu_\theta)\right]}{(9 + \nu_\theta)\alpha^6 - (22 + 2\nu_\theta)\alpha^4 + (13 + \nu_\theta)\alpha^2}$$

$$C_3 = -\frac{C_0(\alpha^8 - 1)}{\alpha^8 - \alpha^6} - \frac{C_1(\alpha^6 - 1)}{\alpha^6 - \alpha^4} - \frac{C_2(\alpha^4 - 1)}{\alpha^4 - \alpha^2}$$

$$C_4 = -C_0 - C_1 - C_2 - C_3$$

where ν_θ = Poisson's ratio (ratio of radial strain to tangential strain)

Employing Eq.(2.28), $L_1[F(r)]$ can be written as

$$L_1[F(r)] = S_0 r^{-4} + S_1 r^{-2} + S_2 + S_3 r^2 + S_4 r^4 + \lambda[S_5 r^{k-3} + S_6 r^{-k-3} + S_7 r^{k-1} + S_8 r^{-k-1} + S_9 r^{k+1} + S_{10} r^{-k+1} + S_{11} r^{k+3} + S_{12} r^{-k+3} + S_{13} r^{k+5} + S_{14} r^{-k+5}] \qquad (2.29)$$

where λ(the dimensionless critical load parameter) = $\dfrac{P_0 hb^2}{D_r}$

$D_\theta = k^2 D_r$

$D_1 = \nu_\theta D_r$

$D_{r\theta} = (1 - \nu_\theta) D_r$

$$S_0 = -C_0(2n^2 + 2n^2 \cdot k^2 - n^4 \cdot k^2)$$

$$S_1 = -C_1(2n^2 + 2n^2 \cdot k - n^4 \cdot k^2)$$

$$S_2 = C_2(72 - 8k^2 - 18n^2 - 2n^2 \cdot k^2 + n^4 \cdot k^2)$$

$$S_3 = C_3(600 - 24k^2 - 50n^2 - 2n^2 \cdot k^2 + n^4 \cdot k^2)$$

$$S_4 = C_4(2352 - 48k^2 - 98n^2 - 2n^2 \cdot k^2 + n^4 \cdot k^2)$$

$$S_5 = C_0 n^2 \cdot kP$$

$$S_6 = -C_0 n^2 \cdot kQ$$

$$S_7 = C_1(n^2 \cdot k - 2k - 2)P$$

$$S_8 = -C_1(n^2 \cdot k - 2k + 2)Q$$

$$S_9 = C_2(n^2 \cdot k - 4k - 12)P$$

$$S_{10} = -C_2(n^2 \cdot k - 4k + 12)Q$$

$$S_{11} = C_3(n^2 \cdot k - 6k - 30)P$$

$$S_{12} = -C_3(n^2 \cdot k - 6k + 30)Q$$

$$S_{13} = C_4(n^2 \cdot k - 8k - 56)P$$

$$S_{14} = -C_4(n^2 \cdot k - 8k + 56)Q$$

where P and Q are now transformed into dimensionless terms as

$$P = -\frac{1 - \beta\alpha^{k+1}}{1 - \alpha^{2k}}$$

$$Q = \frac{\alpha^{k+1}(\alpha^{k-1} - \beta)}{1 - \alpha^{2k}}$$

By substituting Eq.(2.28) together with Eq.(2.29) into Eq.(2.27), the dimensionless critical load parameter can be determined as

$$\lambda = \frac{\begin{bmatrix} S_0 C_0 X(-3) + (S_1 C_0 + S_0 C_1) X(-1) + (S_2 C_0 + S_1 C_1 \\ + S_0 C_2) X(1) + (S_3 C_0 + S_2 C_1 + S_1 C_2 + S_0 C_3) \\ X(3) + (S_4 C_0 + S_3 C_1 + S_2 C_2 + S_1 C_3 + S_0 C_4) X(5) \\ + (S_4 C_1 + S_3 C_2 + S_2 C_3 + S_1 C_4) X(7) + (S_4 C_2 + \\ S_3 C_3 + S_2 C_4) X(9) + (S_4 C_3 + S_3 C_4) X(11) + \\ S_4 C_4 X(13) \end{bmatrix}}{\begin{bmatrix} S_5 C_0 X(k-2) + S_6 C_0 X(-k-2) + (S_7 C_0 + S_5 C_1) \\ X(k) + (S_8 C_0 + S_6 C_1) X(-k) + (S_9 C_0 + S_7 C_1 + \\ S_5 C_2) X(k+2) + (S_{10} C_0 + S_8 C_1 + S_6 C_2) X(-k+2) + \\ (S_{11} C_0 + S_9 C_1 + S_7 C_2 + S_5 C_3) X(k+4) + (S_{12} C_0 + \\ S_{10} C_1 + S_8 C_2 + S_6 C_3) X(-k+4) + (S_{13} C_0 + S_{11} C_1 + \\ S_9 C_2 + S_7 C_3 + S_5 C_4) X(k+6) + (S_{14} C_0 + S_{12} C_1 + S_{10} C_2 \\ + S_8 C_3 + S_6 C_4) X(-k+6) + (S_{13} C_1 + S_{11} C_2 + S_9 C_3 + \\ S_7 C_4) X(k+8) + (S_{14} C_1 + S_{12} C_2 + S_{10} C_3 + S_8 C_4) \\ X(-k+8) + (S_{13} C_2 + S_{11} C_3 + S_9 C_4) X(k+10) + (S_{14} C_2 \\ + S_{12} C_3 + S_{10} C_4) X(-k+10) + (S_{13} C_3 + S_{11} C_4) \\ X(k+12) + (S_{14} C_3 + S_{12} C_4) X(-k+12) + S_{13} C_4 \\ X(k+14) + S_{14} C_4 X(-k+14) \end{bmatrix}} \quad (2.30)$$

where $X(m) = \frac{b^m - a^m}{m} = \int_a^b x^{m-1} dx$

The minimization of Eq.(2.30) with respect to the number of half-waves n yields the buckling load of orthotropic annular plates.