CHAPTER III

THE UNIQUENESS OF THE POISSON INTEGRAL

We shall show that if the temperature is always positive for positive time, then it cannot be zero at time zero. That is, there are no positive solutions for the equation $\Delta u = \frac{\partial}{\partial t} u$ subject to $u(x,0) = 0$.

Define $E_{r,c} = \left\{ (x,t) / -r < x_i < r,\ i=1,2,\dots,n,\ 0 < t \leq c \right\}$

and $G_{r,c} = \bar{E}_{r,c} - E_{r,c}$.

Lemma 3.1.1. If δ is a positive constant, we get

$$z^n \exp(-\delta^2 z^2) \leq \frac{n^{n/2}}{\delta^n (2e)^{n/2}}, \text{ for } z \geq 0.$$

Proof: Let $h(z) = z^n \exp(-\delta^2 z^2)$, $z \geq 0$

$$h'(z) = z^{n-1} \exp(-\delta^2 z^2)(n - 2z^2\delta^2)$$

the critical point is that $z = \frac{\sqrt{n}}{\delta\sqrt{2}}$.

The critical point gives the absolute maximum value of $h(z)$. In fact,

$$h'(z) > 0 \quad \text{whenever} \quad 0 < z < \frac{\sqrt{n}}{\delta\sqrt{2}}$$

$$h'(z) < 0 \quad \text{whenever} \quad z > \frac{\sqrt{n}}{\delta\sqrt{2}} .$$

Therefore, $h(\frac{\sqrt{n}}{\delta\sqrt{2}})$ is the absolute maximum value of $h(z)$.

i.e.

$$z^n \exp(-\delta^2 z^2) \leq \frac{n^{n/2}}{\delta^n (2e)^{n/2}} \ ; \text{ for all } z \geq 0.$$

Theorem 3.1.2. If

1. $\phi(y) \in L(B(0,A))$,

2. $F_A(x,t) = \int_{B(0,A)} K(y-x,t)\phi(y)\,dy \ , \ t>0$,

3. $g(x) = \sup_{0<t\leq c} \left| F_A(x,t) \right|$, for some $c>0$,

then $\lim_{|x|\to\infty} g(x) = 0$.

Proof: When $|x|>A$, $|y|<A$,

$$(|x|-|y|)^2 < |y-x|^2 = \sum_{i=1}^{n}(y_i-x_i)^2.$$

But $0<|x|-A<|x|-|y|$, therefore

$$(|x|-A)^2 < \sum_{i=1}^{n}(y_i-x_i)^2.$$

Since $f(r) = \exp(-r)$ is a strictly decreasing function, for $r\geq 0$, we get

$$\exp\left(-\frac{(|x|-A)^2}{4t}\right) \geq \exp\left(-\frac{\sum_{i=1}^{n}(y_i-x_i)^2}{4t}\right)$$

$$(4\pi t)^{-n/2}\exp\left(-\frac{(|x|-A)^2}{4t}\right) \geq (4\pi t)^{-n/2}\exp\left(-\frac{\sum_{i=1}^{n}(y_i-x_i)^2}{4t}\right)$$

$$= K(y-x,t).$$

By Lemma 3.1.1., with $\delta = |x|-A$, $z^2 = \frac{1}{4t}$, we have

$$K(y-x,t) \leq \frac{n^{n/2}}{(2\pi e)^{n/2}} \cdot \frac{1}{(|x|-A)^n}, \ |y|<A.$$

Therefore,

$$\left| F_A(x,t) \right| \leq \frac{n^{n/2}}{(2\pi e)^{n/2}} \cdot \frac{1}{(|x|-A)^n} \int_{B(0,A)} |\phi(y)|\,dy,$$

whenever $|x| > A$.

By definition of $g(x)$,

$$(3.1) \quad 0 \leq g(x) \leq \frac{n^{n/2}}{(2\pi e)^{n/2}} \cdot \frac{1}{(|x|-A)^n} \int_{B(0,A)} |\phi(y)| \, dy.$$

Since $\phi(y) \in L(B(0,A))$, the right hand of (3.1) tends to 0 as $|x| \to \infty$.

$$\lim_{|x| \to \infty} g(x) = 0.$$

Theorem 3.1.3. If

1. $u(x,t) \in \mathcal{H}$ in $E_{r,c}$,

2. $\lim_{(x,t) \to (x_0,t_0)} u(x,t) \geq 0$ for all (x_0,t_0) on

$G_{r,c}$, then

$$u(x,t) \geq 0 \quad \text{in } E_{r,c} .$$

Proof: By hypothesis 2 of the Theorem,

$$\sup_{\delta > 0} \; \inf_{(x,t) \in B((x_0,t_0), \delta) \cap E_{r,c}} u(x,t) \geq 0 .$$

For $(x_0,t_0) \in G_{r,c}$, given $\varepsilon > 0$, $\exists\ \delta_0 > 0$ such that

$$u(x,t) > -\varepsilon, \quad (x,t) \in B((x_0,t_0), \delta_0) \cap E_{r,c} .$$

By the Heiene Borel Theorem, for an $\varepsilon > 0$, there are positive real numbers $\Delta_1, \Delta_2, \ldots, \Delta_m$, such that

$$u(x,t) > -\varepsilon ,$$

for all $(x,t) \in \bigcup_{i=1}^{m} B((x^i,t^i); \delta_i) \cap E_{r,c}$. Note that $G_{r,c}$ is contained in $\bigcup_{i=1}^{m} B((x^i,t^i); \delta_i)$, where (x^i,t^i) $G_{r,c}$. Hence for $\varepsilon > 0$, there is a $\delta > 0$ such that

$$u(x,t) > -\varepsilon \tag{3.2}$$

for all points of $E_{r,c}$ with in a distance δ of $G_{r,c}$.

Suppose that there exists $(x^1,t^1) \in E_{r,c}$ such that $u(x^1,t^1) = -\ell < 0$. Set

$$v(x,t) = u(x,t) + kt - kt^1, \text{ where } 0 < k < \frac{\ell}{t^1}.$$

Choose $\varepsilon > 0$ such that $kt^1 + \varepsilon < \ell$. Such an ε, by (3.2), there is a $\delta > 0$ such that $u(x,t) > -\varepsilon$, whenever $(x,t) \in E_{r,c}$ within a distance δ of $G_{r,c}$. Then for these points of $E_{r,c}$,

$$\begin{aligned} v(x,t) = u(x,t) + kt - kt^1 &> -\varepsilon + kt - kt^1 \\ &> -\ell + kt \quad (\; kt^1 + \varepsilon < \ell) \\ &> -\ell . \end{aligned}$$

Since $v(x^1,t^1) = u(x^1,t^1) = -\ell$, the minimum of $v(x,t)$ is at a point (x°,t°) of $E_{r,c}$

Since $\Delta v(x,t) = \Delta u(x,t)$, $\frac{\partial}{\partial t} v(x,t) = \frac{\partial}{\partial t} u(x,t) + k$, we have

$$\Delta v(x,t) = \frac{\partial}{\partial t} v(x,t) - k. \tag{3.3}$$

At the minimum point, $\Delta v(x^\circ,t^\circ) \geqslant 0$, $\frac{\partial}{\partial t} v(x^\circ,t^\circ) = 0$.

This contradicts (3.3). Hence the assumption is false.

Therefore $\quad u(x,t) \geqslant 0 \quad$ in $E_{r,c}$.

Theorem 3.1.4. If

1. $u(x,t) \in \mathcal{H}_e$, $0 < t < c$,
2. $u(x,t) \geq 0$, $0 < t < c$,

then the integral $\int_{\mathbb{R}^n} K(y-x,t)\ u(y,\delta)dy$ exists, $x \in \mathbb{R}^n$, $0 < t < c-\delta$, and

$$\int_{\mathbb{R}^n} K(y-x,t)\ u(y,\delta)dy \leq u(x,t+\delta).$$

Proof: Let $0 < \delta < c$, and let

$$\phi(y) = \begin{cases} u(y,\delta) & , |y| \leq A, \\ 0 & , |y| > A. \end{cases}$$

Since $u(x,t) \in \mathcal{H}_e$, $\phi(y)\exp(-a|y|^2) \in L(\mathbb{R}^n)$ for some $a > 0$.

$$\begin{aligned} \text{Define } F(x,t) &= \int_{\mathbb{R}^n} K(y-x,t)\ \phi(y)dy \\ &= \int_{B(0,A)} K(y-x,t)\ \phi(y)dy \\ &= F_A(x,t) \quad \text{(by definition of } F_A(x,t) \text{ in the previous theorem.)} \end{aligned}$$

$\phi(y)\exp(-a|y|^2) \in L(\mathbb{R}^n)$ for some $a > 0$, since

$\phi(y)$ is continuous on B(0,A). Therefore by the Theorem 2.1.9.,

if $|x_0|<A$, $\lim\limits_{(x,t)\to(x_0,0^+)} F(x,t) = \phi(x_0) = u(x_0,\delta)$, and

if $|x_0|>A$,

$$\overline{\lim_{(x,t)\to(x_0,0^+)}} F(x,t) = \overline{\lim_{(x,t)\to(x_0,0^+)}} |F(x,t)|$$

$$\leq \overline{\lim_{y\to x_0}} |\phi(y)|$$

$$= \lim_{y\to x_0} \phi(y)$$

$$= 0 \quad .$$

$$\overline{\lim_{(x,t)\to(x_0,0^+)}} F(x,t) = \underline{\lim}_{(x,t)\to(x_0,0^+)} F(x,t)$$

$$= 0$$

$$\lim_{(x,t)\to(x_0,0^+)} F(x,t) = 0 , \quad \text{if } |x_0|>A .$$

Hence

$$\lim_{(x,t)\to(x_0,0^+)} F_A(x,t) = \begin{cases} u(x_0,\delta) , & |x_0|<A \\ 0 , & |x_0|>A. \end{cases}$$

At the point x_0 such that $|x_0| = A$, we have by the Theorem 2.1.9. that

$$\overline{\lim_{(x,t)\to(x_0,0^+)}} F_A(x,t) = \overline{\lim_{(x,t)\to(x_0,0^+)}} F(x,t)$$

$$\leq \lim_{y\to x_0} \phi(y)$$

$$= u(x_0,\delta) \quad .$$

Consider the function

(3.4) $$v(x,t) = u(x,t+\delta) - F_A(x,t).$$

By the Theorem 2.1.7., $v(x,t) \in \mathcal{H}_e$, in the strip $H_{(0,c-\delta)}$.

$$\underline{\lim}_{(x,t)\to(x_0,0^+)} v(x,t) \geq \underline{\lim}_{(x,t)\to(x_0,0^+)} u(x,t+\delta) + \underline{\lim}_{(x,t)\ (x_0,0^+)} - F_A(x,t)$$

$$= u(x_0,\delta) - \overline{\lim}_{(x,t)\to(x_0,0^+)} F(x,t)$$

$$\geq 0$$

Claim that $v(x,t) \geq 0$ in the strip $H_{(0,c-\delta)}$.

Suppose $v(x,t)$ is negative at some points, say, at (x_0,t_0). That is

(*) $$v(x_0,t_0) = -\ell < 0$$

If $g(x) = \sup_{0<t<c} |F_A(x,t)|$, for some $c>0$, therefore, by the Theorem 3.1.2.

$$\lim_{|x|\to\infty} g(x) = 0.$$

Given $\varepsilon>0$, there is an $N>0$ such that, $i = 1,2,\ldots,n.$,

$$|F_A(x,t)| \leq \varepsilon \text{ whenever } |x_i| \geq N,\ 0 < t < c.$$

In particular, for $\varepsilon = \frac{\ell}{2}$, there is an $N>0$ such that

$$F_A(x,t) \leq \frac{\ell}{2},\quad |x_i| \geq N,\ 0<t<c,\quad i=1,2,\ldots,n.$$

For $|x_i| \geq N$, $0 < t + \delta < c$, $i = 1,2,\ldots,n.$, we have by the definition of $v(x,t)$ and $u(x,t) \geq 0$ that

$$(3.5) \qquad v(x,t) = u(x,t+\delta) - F_A(x,t) \geqslant - F_A(x,t) \geqslant -\frac{\ell}{2}.$$

Hence $v(x,t) + \frac{\ell}{2} \geqslant 0$ on $|x_i| = N$, $0 < t < c$, $i = 1,2,\ldots,n$.

Since $v(x,t) \in \mathcal{H}$, $0 < t < c-\delta$, we get

$$\lim_{(x,t)\to(x_1,t_1)} v(x,t) + \frac{\ell}{2} \geqslant 0, \quad x_1 = (x_{1_1},\ldots,x_{1_n}).$$

where $|x_{1_i}| = N$, $0 < t_1 < c-\delta$, $1_i = 1_1$ or 1_2 or ....or 1_n.

Let $t_0 < c' < c-\delta$. The function $v(x,t) + \frac{\ell}{2}$ satifies all the hypothesis of the Theorem 3.1.3., thus

$$(3.6) \qquad v(x,t) + \frac{\ell}{2} \geqslant 0 \text{ in } E_{N,c'} .$$

From (3.5) and (3.6), $v(x,t) \geqslant -\frac{\ell}{2}$ in $H_{(0,c)}$. It then contradicts (*).

Hence $v(x,t) \geqslant 0$ in the strip $H_{(0,c-\delta)}$.

$$F(x,t) = F_A(x,t) \leqslant u(x,t+\delta) \text{ in the strip } H_{(0,c-\delta)}.$$

$$\int_{B(0,A)} K(y-x,t)u(y,\delta)\,dy \leqslant u(x,t+\delta)$$

in the strip $H_{(0,c-\delta)}$, for all $A > 0$.

Since $u(y,\delta) \geqslant 0$ for all $y \in \mathbb{R}^n$, and

$$\int_{B(0,A)} K(y-x,t)u(y,\delta)\,dy$$

is bounded and increasing on $A > 0$, we get

$$\lim_{A\to\infty} \int_{B(0,A)} K(y-x,t)u(y,\delta)\,dy$$

exists, and

$$\int_{\mathbb{R}^n} K(y-x,t)u(y,\delta)\,dy = \lim_{A\to\infty} \int_{B(0,A)} K(y-x,t)u(y,\delta)\,dy$$
$$\leq u(x,t+\delta).$$

Theorem 3.1.5. If

1. $u(x,t) \in \mathcal{H}$, in the strip $H_{(0,c]}$,

2. $\lim_{(x,t)\to(x_0,0^+)} u(x,t) = 0$, for all x_0,

3. $f(x) = \max_{0<t\leq c} |u(x,t)|$,

4. $f(x) = o(\exp(a|x|^2))$, as $|x|\to\infty$, for some $a>0$,

then

$u(x,t) = 0$, in the strip $H_{(0,c]}$.

Proof: Let $r>0$. Let $(x,t)\in H_{(0,c]}$. Denote $r_n = (r,r,\ldots,r)$. **Form** a function $U_r(x,t)$ by

$$U_r(x,t) = f(-r_n)K(x+r_n,t) + f(r_n)K(x-r_n,t).$$

Then $U_r(x,t)\in \mathcal{H}$, since $K(x,t)\in\mathcal{H}$, and

$$U_r(r_n,t) = f(-r_n)K(2r_n,t) + f(r_n)K(0,t)$$
$$\geq f(r_n)K(0,t)$$
$$= f(r_n)(4\pi t)^{-n/2}, \text{ where } 0<t\leq c.$$

$$U_r(-r_n,t) = f(-r_n)K(0,t) + f(r_n)K(-2r_n,t)$$

$$\geqslant f(-r_n)K(0,t)$$

$$= f(-r_n)(4\pi t)^{-n/2}, \text{ where } 0 < t \leqslant c.$$

From the definition of $f(x)$ we have, for $0 < t \leqslant c$, that

$$\left| u(r_n,t) \right| \leqslant f(r_n) \leqslant f(r_n)\left(\frac{c}{t}\right)^{n/2}, \quad \frac{c}{t} \geqslant 1$$

$$\leqslant U_r(r_n,t)(4\pi c)^{n/2}$$

and also $|u(-r_n,t)| \leqslant U_r(-r_n,t)(4\pi c)^{n/2}$. In other word,

$(4\pi c)^{n/2}U_r(x,t) + u(x,t)$ and $(4\pi c)^{n/2}U_r(x,t) - u(x,t)$

both belong to $\mathcal{H}$ in $E_{r,c}$ and are nonnegative on $G_{r,c}$. If

$t = 0$, $U_r(x,0) = 0$, and $\lim_{(x,t)\to(x_0,0^+)} u(x,t) = 0$, then

they both vanish except at $x_i = r$ or $x_i = -r$, and, therefore,

$$\lim_{(x,t)\to(x_0,t_0)} (4\pi c)^{n/2}U_r(x,t) + u(x,t) \geqslant 0,$$

$$\lim_{(x,t)\to(x_0,t_0)} (4\pi c)^{n/2}U_r(x,t) - u(x,t) \geqslant 0,$$

for all $(x_0,t_0) \in G_{r,c}$. Now apply the Theorem 3.1.3. to

$(4\pi c)^{n/2}U_r(x,t) + u(x,t)$ and $(4\pi c)^{n/2}U_r(x,t) - u(x,t)$,

we have

$(4\pi c)^{n/2}U_r(x,t) + u(x,t) \geqslant 0$, and

$(4\pi c)^{n/2}U_r(x,t) - u(x,t) \geqslant 0$, on $E_{r,c}$.

That is, $(4\pi c)^{n/2}U_r(x,t) \geqslant - u(x,t)$, and

$$(4\pi c)^{n/2}U_r(x,t) \geq u(x,t) \quad \text{on } E_{r,c}.$$

Hence, $(4\pi c)^{n/2}U_r(x,t) \geq |u(x,t)|$ on $E_{r,c}$.

Fix (x,t) and let r tends to infinite.

If $0 < t < \frac{1}{4a}$, and by the definition of $U_r(x,t)$,

$U_r(x,t)$ tends to zero.

Hence if $c \leq 1/4a$, $u(x,t) = 0$ on $H_{(0,c]}$. Otherwise, if $c > 1/4a$, let $w(x,t) = u(x,t+\frac{1}{4a})$, repeat the above argument with $w(x,t)$, and so forth.

$u(x,t) = 0$ on the strip $H_{(0,c]}$.

Theorem 3.1.6. If

1. $u(x,t) \in \mathcal{H}_e$, $0 \leq t < c$,
2. $u(x,t) \geq 0$, $0 < t < c$,
3. $u(x,0) = 0$, $x \in \mathbb{R}^n$,

then

$u(x,t) = 0$, on the strip $H_{[0,c)}$.

Proof: Set $w(x,t) = \int_0^t u(x,\tau)\,d\tau$, $0 \leq t < c$.

$$\frac{\partial}{\partial t}w(x,t) = u(x,t) \geq 0. \tag{3.7}$$

Since $u(x,t) \in \mathcal{H}_e$, for $x \in \mathbb{R}^n$, $0 \leq t < c$,

$$\frac{\partial^2}{\partial x_i^2}w(x,t) = \int_0^t \frac{\partial^2}{\partial x_i^2}u(x,\tau)\,d\tau,$$

Then $$w(x,t) = \int_0^t u(x,\tau)\,d\tau$$

$$= \int_0^t \frac{\partial}{\partial \tau} u(x,\tau)\,d\tau = u(x,t) \quad 0.$$

(*) That is w(x,t) is subharmonic. *

Hence w(x,t) is an increasing function of variable t and w(x,t) is subharmonic for variable x. And

$$\Delta w(x,t) = \frac{\partial}{\partial t} w(x,t)$$

$$w(x,t) \in \mathcal{H}_e, \ 0 < t < c,$$

$$w(x,0) = 0, \ w(x,t) \geqslant 0, \ 0 < t < c.$$

Let δ be such that $0 < \delta < c$ and let $x = 0$, and $t = t_0 < c - \delta$. Then

$$\int_{\mathbb{R}^n} K(y-x,t)w(y,\delta)\,dy = \int_{\mathbb{R}^n} K(y,t_0)w(y,\delta)\,dy.$$ Hence

by the Theorem 3.1.4. we have that

$$\int_{\mathbb{R}^n} K(y,t_0)w(y,\delta)\,dy \leq w(0,t_0+\delta), \ 0 < \delta < c.$$

Since $w(x,t) \in \mathcal{H}_e$ on the strip $H_{o,c)}$, we have that

$$(3.8) \quad \int_{\mathbb{R}^n} \exp\left(\frac{-|y|^2}{4t_0}\right)w(y,\delta)\,dy \leq (4\pi t_0)^{n/2} w(0,t_0+\delta)$$

$$< \infty$$

Since $w(x,t)$ is nonnegative and nondecreasing in t, we set

$$f(x) = \max_{0 \leq t \leq \delta} w(x,t) = w(x,\delta).$$

We know from (*) that $f(x)$ is subharmonic, then apply the Theorem 1.4.2. to $f(x)$ we have that

$$f(x) \leq \frac{1}{\nu_n(1)|x|^n} \int_{B(x,|x|)} f(y)dy, \text{ for all } |x| \neq 0 .$$

$$\nu_n(1)|x|^n f(x)\exp\left(-\frac{|x|^2}{t_0}\right) \leq \int_{B(x,|x|)} f(y)\exp\left(-\frac{|x|^2}{t_0}\right)dy.$$

Since $h(z) = \exp(-z)$, $z \geq 0$, is strictly decreasing, and $y \in B(x,|x|)$, i.e. $|y| < 2|x|$,

$$\nu_n(1)|x|^n f(x)\exp\left(-\frac{|x|^2}{t_0}\right) \leq \int_{B(x,|x|)} f(y)\exp\left(-\frac{|y|^2}{4t_0}\right)dy$$

$$\leq \int_{\mathbb{R}^n} f(y)\exp\left(-\frac{|y|^2}{4t_0}\right)dy.$$

By (3.8), there is a positive number A such that

$$\nu_n(1)|x|^n f(x)\exp\left(-\frac{|x|^n}{t_0}\right) \leq A,$$

$$0 \leq \frac{f(x)}{\exp\left(\frac{|x|^2}{t_0}\right)} \leq \frac{A}{\nu_n(1)|x|^n},$$

$$\lim_{|x|\to\infty} \frac{f(x)}{\exp\left(\frac{|x|^2}{t_0}\right)} = 0$$

$$f(x) = o\left(\exp\left(\frac{|x|^2}{t_0}\right)\right) \quad \text{as } |x| \to \infty .$$

Applying the Theorem 3.1.5. to the function $w(x,t)$, we get $$w(x,t) = 0 \text{ in } H_{(0,\delta]}.$$

Since δ is arbitrary,

$$w(x,t) = 0 \text{ in } H_{(0,c)}$$

Since $$\frac{\partial}{\partial t} w(x,t) = u(x,t) \text{ on } H_{(0,c)},$$

$$u(x,t) = 0.$$

The proof is complete.

Corollary 3.1.7. The theorem holds if $H_{(0,c)}$ is replaced by the half-space.

Proof: Assume that $u(x,t) \neq 0$ on the half-space, that is, there is $(x_0,t_0) \in H$ such that

(**) $$u(x_0,t_0) \neq 0.$$

Since t_0 is a point such that $0 < t_0 < \infty$, there is a point c such that $0 < t_0 < c$.

If we restrict t be such that $0 < t < c$, all the hypothesis of the theorem hold on $H_{(0,c)}$. Thus, $u(x,t) = 0$ on $H_{(0,c)}$. But $(x_0,t_0) \in H_{(0,c)}$, it contradicts (**) .

Hence the assumption fails. i.e.

$$u(x,t) = 0 \text{ on } H.$$