CHAPTER I

PRELIMINARY RESULTS

In this chapter, we will introduce some properties of K(x,t) and recall some facts from real analysis.

1.1 Some Properties of K(x,t)

It is easy to see that K(x,t) satisfies the equation (1), by directly partial differentiation.

Proposition 1.1.1. $\int_{\mathbb{R}^n} K(x,t)dx = 1$, for each $t > 0$.

Proof: Fix $t > 0$.

$$(1.1) \quad \int_{\mathbb{R}^n} K(x,t)dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (4\pi t)^{-n/2} \exp(-(\sum_{i=1}^{n} x_i^2)/4t)dx_n..dx_1$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \sqrt{4t}\ \sqrt{\pi}\ (4\pi t)^{-n/2} \exp(-(\sum_{i=1}^{n} x_i^2)/4t)dx_{n-1}..dx_1$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (4\pi t)^{-(n-1)/2} \exp(-(\sum_{i=1}^{n} x_i^2)/4t)dx_{n-1}..dx_2\ dx_1$$

Repeat the process (1.1) n-1 times, we get

$$\int_{\mathbb{R}^n} K(x,t)dx = 1.$$

Proposition 1.1.2. K(x,t) has a maximum value at $x = 0$, for fixed $t > 0$.

Proof: For fixed $t > 0$. Observe that the function

(1.2) $$h(s) = \exp(-s), \ s \geqslant 0$$

is strictly decreasing. Its maximum value occurs at $s = 0$.

Hence, since $|x| \geqslant 0$,

(1.3) $$K(0,t) \geqslant K(x,t), \quad x \in \mathbb{R}^n.$$

Proposition 1.1.3. $\lim_{t \to 0^+} K(x,t) = 0$, for $x \neq 0$.

Proof: $\lim_{t \to 0^+} K(x,t) = \lim_{u \to \infty} ((\pi)^{-n/2} u)/\exp(|x|^2 u^{2/n})$,

by L'Hospital rule, $x \neq 0$, the limit is zero.

Obviously, $K(0,t) = (4\pi t)^{-n/2}$ tends to infinite as t tends to 0^+.

1.2 Some Facts From Real Analysis

Theorem 1.2.1. (Lebesgue dominated convergence theorem) Let $\{f_n\}$ be a sequence of Lebesgue-integrable functions on a Lesbesgue measurable subset Y of $\mathbb{R}^n$. Assume that

(a) $\{f_n\}$ converges almost everywhere on Y to a function f, and

(b) there is a nonnegative function g in L(Y), such that, for all $n \geqslant 1$,

$$|f_n(y)| \leqslant g(y) \text{ , almost everywhere on Y.}$$

Then the limit function $f \in L(Y)$, the sequence $\left\{\int_Y f_n(y)dy\right\}$ converges, and

(1.4) $$\int_Y f(y)dy = \lim_{n \to \infty} \int_Y f_n(y)dy.$$

The proof of the theorem will be omitted (for the completed proof see [1], p.270).

Theorem 1.2.2. Let X be an open subset of $\mathbb{R}^m$ and Y be a Lebesgue measurable subset of $\mathbb{R}^n$, and let f be a function on $Y \times X$ and satisfy the following conditions:

(a) For each fixed $x \in X$, the function $f_x(y)$ defined on Y by

$$f_x(y) = f(y,x)$$

is measurable on Y, and $f_a(y) \in L(Y)$ for some $a \in X$.

(b) The partial derivative $D_{x_i} f(y,x)$ exists for each interior point (y,x) of $Y \times X$.

(c) There is a nonnegative function $G \in L(Y)$ such that

$$\left| D_{x_i} f(y,x) \right| \leq G(y)$$

for all interior points of $Y \times X$.

Then the Lebesgue integral $\int_Y f(y,x)dy$ exists for every $x \in X$, and the function F defined by

$$F(x) = \int_Y f(y,x)dy$$

is differentiable with respect to x_i at each interior point of X. Moreover, its derivative with respect to x_i is given by

$$D_{x_i} F(x) = \int_Y D_{x_i} f(y,x)dy.$$

Proof: Claim that $\left|f_x(y)\right| \leq \left|f_a(y)\right| + \left|x - a\right| G(y)$ for all interior points (y,x) of $Y \times X$.

The Mean - Value Theorem gives us

$$f(y,x) - f(y,a) = (x_i - a_i)\, D_{x_i} f(y,c)$$

where $a = (x_1, x_2, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_m)$ and $c = (x_1, x_2, \ldots, x_{i-1}, c_i, x_{i+1}, \ldots, x_m)$ and c_i lies between a_i and x_i. It follows that

$$\left|f(y,x)\right| \leq \left|f(y,a)\right| + \left|x_i - a_i\right| \left|D_{x_i} f(y,c)\right|$$

$$\leq \left|f(y,a)\right| + \left|x - a\right| G(y).$$

i.e., $$\left|f_x(y)\right| \leq \left|f_a(y)\right| + \left|x - a\right| G(y).$$

Since f_x is measurable on Y and dominated almost everywhere on Y by a nonnegative function in $L(Y)$, $f_x \in L(Y)$.

That is the integral $\int_Y f(y,x)\,dy$ exists for each $x \in X$.

Choose any sequence $\{x^k\}$ of points in X such that $x^k = (x_1, x_2, \ldots, x_{i-1}, x_i^k, x_{i+1}, \ldots x_m) \neq x$ but $\lim_{k\to\infty} x^k = x$ or $\lim_{k\to\infty} x_i^k = x_i$.

Define a sequence of functions $\{q_k\}$ on Y by

(1.5) $$q_k(y) = \frac{f(y,x^k) - f(y,x)}{x_i^k - x_i}.$$

Then $q_k \in L(Y)$ and $q_k(y) \longrightarrow D_{x_i} f(y,x)$ at each interior point of Y. By the Mean - Value Theorem and (1.5), we have

$$q_k(y) = D_{x_i} f(y, c^k)$$

where $c^k = (x_1, x_2, \ldots, x_{i-1}, c_i^k, x_{i+1}, \ldots, x_m)$ and c_i^k lies between x_i^k and x_i. Therefore, by hypothesis (c),

$$\left| q_k(y) \right| \leq G(y) \text{ almost everywhere on } Y.$$

Lebesgue's Dominated Convergence Theorem shows that $\left\{ \int_Y q_k(y)\,dy \right\}$ converges, the integral $\int_Y D_{x_i} f(y,x)\,dy$ exists,

and
$$\lim_{k \to \infty} \int_Y q_k(y)\,dy = \int_Y \lim_{k \to \infty} q_k(y)\,dy$$
$$= \int_Y D_{x_i} f(y,x)\,dy.$$

But
$$\int_Y q_k(y)\,dy = \int_Y \frac{f(y,x^k) - f(y,x)}{x_i^k - x_i}\,dy$$
$$= \frac{F(x^k) - F(x)}{x_i^k - x_i}.$$

Since the last quotient tends to a limit for all sequences $\left\{ x^k \right\}$, it follows that $\frac{\partial}{\partial x_i} F(x)$ exists and that

$$\frac{\partial}{\partial x_i} F(x) = \int_Y \frac{\partial}{\partial x_i} f(y,x)\,dy.$$

1.3 The Spherical Coordinate System.

The materials of this section are drawn from references [4], p 3 - 4.

If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are in $\mathbb{R}^n$, the length of a vector x is denoted by $|x|$, and

$$|x| = \sum_{i=1}^{n} x_i^2 .$$

The distance between two vectors x and y is defined to be $|x-y|$. The angle between two nonzero vectors x and y is defined to be the angle θ such that $0 \leq \theta \leq \pi$ and

$$\cos \theta = \frac{\sum_{i=1}^{n} x_i y_i}{|x||y|} .$$

Many of the functions that we shall deal with are functions only of the distance from the origin $0 \in \mathbb{R}^n$. For each functions it is more convenient to use a spherical coordinate system rather than a rectangular coordinate system. The spherical coordinates of $x \neq 0$ are defined as follows : if $r = |x|$, then

$$\theta = \left(\frac{x_1}{r}, \dots, \frac{x_n}{r} \right)$$

is a point of $\partial B(0,1)$, the unit sphere with centre at 0. The pair (θ, r) uniquely determines x and are called the spherical coordinates of x. The spherical coordinates of 0 is the pair $(0,0)$. This transformation from rectangular coordinates to spherical coordinates is essentially the mapping $(x_1, \dots, x_n) \mapsto (\theta_1, \dots, \theta_{n-1}, r)$ where

$$\theta_1 = \frac{x_1}{r}$$

$$\theta_2 = \frac{x_2}{r}$$
$$\cdots\cdots\cdots\cdots$$
$$\theta_{n-1} = \frac{x_{n-1}}{r}$$
$$r = (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}} .$$

We shall also let $\theta_n = \frac{x_n}{r}$, θ_n is the cosine of the angle between x and the vector $(0,0,\ldots.,0,1)$; that is, the angle between x and the "x_n- axis". The Jacobian of the mapping is easily calculated and absolute value is given by

$$\left| \frac{\partial (x_1,\ldots,x_n)}{\partial(\theta_1,\ldots,\theta_{n-1},r)} \right| = \frac{r^{n-1}}{(1-\theta_1^2 - \ldots - \theta_{n-1}^2)^{\frac{1}{2}}}$$

$$= \frac{r^{n-1}}{|\theta_n|} .$$

If $y = (y_1,\ldots.,y_n) \in R^n$ and $\rho > 0$, then $\partial B(y,\rho)$ is the surface defined by the equation

$$(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2 = \rho^2 .$$

Consider a Borel set $M \subset \partial B(y,\rho) \cap \{(x_1,\ldots,x_n)/x_n - y_n \geq 0\}$. Let M_n denote the projection of M onto the subspace $\{(x_1,\ldots.,x_n)/x_n = 0\}$; that is, $M_n = \{(x_1,\ldots,x_{n-1},0)/ (x_1,\ldots,x_n) \in M\}$. For each $x \in \partial B(y,\rho)$ let $\gamma = \gamma(x)$ be the angle between the "x_n - axis" and the outer normal to $\partial B(y,\rho)$ at x. Then

$$\sec\gamma = \frac{1}{\cos\gamma} = \frac{\rho}{x_n - y_n}$$

and

$$\sigma(M) = \int\int\cdots\int_{M_n} \sec \gamma \, dx_1 \cdots dx_{n-1}$$

represents the surface area of M. If $M \subset \partial B(y,\rho) \cap \{(x_1,\ldots, x_n)/ x_n - y_n \leq 0\}$, the surface area of M is given by the same integral with $\sec \gamma = -\rho/(x_n - y_n)$. The integral of a Borel function f defined on $\partial B(y,\rho)$ relative to the surface area σ is denoted by

$$\int_{\partial B(y,\rho)} f(x) d\sigma(x) .$$

Consider an extended real-valued function f with domain in $\mathbb{R}^n$. We shall take the following liberty with the functional notation. When f is considered as a function of the spherical coordinates (θ,r) of x, we shall denote the value of the composite function at (θ,r) by $f(\theta,r)$. Suppose f is integrable on $\bar{B}(0,\rho)$. Then the integral of f over $\bar{B}(0,\rho)$ can be evaluated using spherical coordinates as follows :

$$\int_{\bar{B}(0,\rho)} f(x)dx = \int_0^{\rho}\int\int\int_{|\theta|=1}\cdots\int f(\theta,r)\frac{r^{n-1}}{|\theta_n|} \, d\theta_1 \cdots d\theta_{n-1} dr$$

$$= \int_0^{\rho} r^{n-1}\left(\int\int\int_{|\theta|=1}\cdots\int f(\theta,r)d\sigma(\theta)\right)dr.$$

1.4 Subharmonic Functions

A function u having continuous second partial derivatives on an open subset Ω of $\mathbb{R}^n$ is said to be subharmonic on Ω if

$$\Delta u \geqslant 0 \qquad \text{on } \Omega .$$

Theorem 1.4.1 If u is subharmonic on the open set $\Omega \subseteq \mathbb{R}^n$, then

$$u(x) \leqslant \frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B} u\,d\sigma$$

whenever $\bar{B}(x,\delta) \subset \Omega$, σ_n denote the surface area of a sphere with radius 1.

Proof: Suppose first that

(*) $\Delta u > 0$ on $\bar{B}(x,\delta) \subset \Omega$.

Let

$$h(y) = \begin{cases} u(y) & ; \ y \in \partial B , \\ \frac{1}{\sigma_n \delta} \int_{\partial B} \frac{\delta^2 - |x-y|^2}{|z-y|^n} u(z)\,d\sigma(z) & ; \ y \in B \end{cases}$$

h is continuous on $\bar{B}(x,\delta)$ and harmonic on B .

Claim that $u \leqslant h$ on B .

Let $w = h-u$.

$\Delta w = \Delta h - \Delta u = -\Delta u < 0$ on B ,

$w = 0$ on ∂B , and

w is continuous on $\bar{B}$.

Suppose w attains its minimum on $\bar{B}$ at x_o.

If $x_o \in B$, then

$$\left.\frac{\partial^2}{\partial x_i^2} w\right|_{x_o} \geq 0 \ , \quad i = 1,2,3,\ldots\ldots,n.$$

and

$$\Delta w(x_o) \geq 0 \ , \text{ a contradiction .}$$

Therefore , $w = h-u \geq 0$ on $\bar{B}$,

$$\text{i.e.} \quad u(x) \leq h(x) = \frac{1}{\sigma_n \delta} \int_{\partial B} \frac{\delta^2}{|z-x|^n} u(z)d\sigma(z)$$

$$= \frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B} u d\sigma$$

Now $\Delta u \geq 0$ on B .

Letting $q(x) = -|x|^2$

$\Delta q = -2n$.

For $\varepsilon > 0$, $\Delta(u-\varepsilon q) > 0$ on Ω .

By the first part of the proof ,

$$u-\varepsilon q \leq \frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B} (u-\varepsilon q)d\sigma$$

$$= \frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B} u d\sigma - \frac{\varepsilon}{\sigma_n \delta^{n-1}} \int_{\partial B} q d\sigma$$

Letting $\varepsilon \to 0$, we obtain

$$u(x) \leq \frac{1}{\sigma_n \delta^{n-1}} \int_{\partial B} u\, d\sigma .$$

Theorem 1.4.2. If u is subharmonic on the open set $\Omega \subseteq R^n$ and $x \in \Omega$, then

$$u(x) \leq \frac{1}{\nu_n \delta^n} \int_{B(x,\delta)} u\, dz , \tag{1.8}$$

whenever $B(x,\delta) \subset \Omega$, $\nu_n = \frac{\sigma_n}{n}$.

Proof: Suppose that u is subharmonic on Ω, $x \in \Omega$, $B(x,\delta) \subset \Omega$. If $u(x) = -\infty$, (1.8) is trivially true. Assume that $u(x) > -\infty$. Since , by Theorem 1.4.1 ,

$$u(x) \leq \frac{1}{\sigma_n \rho^{n-1}} \int_{\partial B} u\, d\sigma , \quad 0 < \rho < \delta,$$

$$\sigma_n \rho^{n-1} u(x) \leq \int_{\partial B(x,\rho)} u\, d\sigma , \quad 0 < \rho < \delta.$$

Integrating over $(0, \delta)$

$$\sigma_n \frac{\delta^n}{n} u(x) \leq \int_0^{\delta} \int_{\partial B(x,\rho)} u\, d\sigma\, d\rho$$

$$= \int_{B(x,\delta)} u\, dz . \quad \text{(by section 1.3)}$$

Therefore, the theorem is proved.