CHAPTER V

LOCAL CYCLICITY AND THE ANTICENTERS OF GROUPS

The materials of this chapter are drawn from references [2] , [7] , [10]

In this final chapter , we only work with <u>additive abelian groups</u> . The main purposes of the chapter to state Hill's theorem in terms of local cyclicity and obtain a theorem of Lim as a consequence .

<u>Convention.</u> If G is a group , $\bar{G}$ will denote the intersection of all maximal locally cyclic subgroups of G .

<u>Lemma 5.1.</u> If G is a non-zero group , then

(a) $\bar{G} = G$ if and only if G is locally cyclic .

(b) if , further , G is torsion-free , then $\bar{G} = 0$ if and only if G is not locally cyclic .

<u>Proof</u>. (a) Assume $\bar{G} = G$. Since $\bar{G}$ is a subgroup of every maximal locally cyclic subgroup of G , $\bar{G}$ is also locally cyclic , and hence G is locally cyclic .

Conversely , assume G is locally cyclic . The only maximal locally cyclic subgroup of G is G , and hence $\bar{G} = G$.

(b) Assume $\bar{G} = 0$. Since G is non-zero , $G \neq \bar{G}$, and hence G is not locally cyclic by part (a) .

Conversely , assume G is not locally cyclic . Then $\bar{G} = 0$, by Corollary 3.12 , since G is torsion-free .

Hence the lemma is proved .

<u>Lemma 5.2.</u> If G is a group , then $\bar{G}$ is either torsion or else torsion-free .

Proof. If $\overline{G} = 0$, then the conclusion is obvious .

If $\overline{G} \neq 0$, then there exists a non-zero g in $\overline{G}$. For any x in G , x is in [x] . Let M be a maximal locally cyclic subgroup containing [x] . Since g is in $\overline{G}$, g is in M , and hence g and x generate a cyclic subgroup [c] of G .

If the order of g is finite , then [c] is the finite cyclic subgroup of G , by Lemma 3.7 and , therefore , the order of x is finite , by Lemma 3.8 .

If the order of g is infinite , then [c] is the infinite cyclic subgroup of G , by Lemma 3.8 and , therefore , the order of x is infinite , by Lemma 3.7 .

Thus G is either torsion or torsion-free .

Hence the lemma is proved .

Theorem 5.3. Let G be a non-zero group .

(a) If G is torsion-free , then

$$\overline{G} = \begin{cases} G & \text{if and only if } G \text{ is locally cyclic .} \\ 0 & \text{if and only if } G \text{ is not locally cyclic .} \end{cases}$$

(b) If G is mixed , then $\overline{G} = 0$.

(c) If G is torsion , then $\overline{G}$ is the direct sum of all the indecomposable primary components of G .

Proof. (a) and (b) follow from Lemma 5.1 and 5.2 , respectively .

(c) If G is torsion , then $G = \sum_{p \in \mathbb{P}} G_p$ its p-primary decomposition , by Theorem 3.2 . If G_p is indecomposable for some prime p , then G_p is a p-primary component of every maximal locally cyclic subgroup of G , by Theorem 3.6 , and hence G_p is the p-primary component of $\overline{G}$.

On the other hand , if G_q is decomposable for some prime q , let $G_q = A \oplus B$ be a decomposition of G_q into non-zero summands . For any non-zero x in G_q ,

$$x = a + b$$

for some a in A , b in B . Suppose neither a nor b is 0 . For each p in $\mathbb{P}^* = \mathbb{P} \setminus \{q\}$, choose a non-zero g_p in G_p and let

$$M = [a] \oplus \sum_{p \in \mathbb{P}^*} [g_p] .$$

Since the p-primary components of M are cyclic groups whose orders are powers of prime numbers , they are indecomposable, by Lemma 4.8 and , therefore , M is locally cyclic , by Lemma 3.5 . Let M^* be a maximal locally cyclic subgroup of G containing M . Since a and b cannot belong to the same cyclic subgroup of G_q , by Lemma 4.8 and since a is in M^* , b can not be in M^* . Then x is not in M^* , and hence x is not in $\bar{G}$. Suppose one of the a and b , say b , is 0 . Pick a non-zero b' in B and , as before , there is a maximal locally cyclic subgroup N^* of G such that b' is in N^* , but a is not in N^* . Then x = a is not in N^* and , therefore , x is not in $\bar{G}$.

Hence , in any case , x is not in $\bar{G}$. Thus , if G_p is decomposable , then the p-primary component of $\bar{G}$ is 0 , and hence (c) is proved .

The theorem is now completely proved .

Our next objective is to prove Hill's theorem in terms of local cyclicity . This we will obtain as the final step in a series of lemmas .

<u>Lemma 5.4</u>. If G is a torsion-free group , then its anticenter AC(G) is isomorphic to a subgroup of the additive abelian group $\mathbb{Q}$ of rationals .

Proof. Without loss of generality, we may assume that $AC(G) \neq 0$. Then there exists a non-zero g in AC(G). For any non-zero x in G, x and g generate a cyclic subgroup [c] of G, and hence

$$g = mc,$$
$$x = nc$$

for some non-zero integers m, n, so that

$$mx = ng.$$

Define $\varphi(x) = n/m$ and $\varphi(0) = 0$. One can show that φ is an isomorphism of G onto a subgroup of $\mathbb{Q}$ as in the Proof of Lemma 3.10.

Lemma 5.5. Let G be a non-zero group. Then

(a) AC(G) = G if and only if G is locally cyclic.

(b) if, further, G is torsion-free, then AC(G) = 0 if and only if G is not locally cyclic.

Proof. (a) Assume AC(G) = G. Then every two elements of G generate a cyclic subgroup of G, and hence G is locally cyclic.

Conversely, assume G is locally cyclic. Let x be a non-zero element in G. For any y in G, x and y generate a cyclic subgroup of G, then x is in AC(G), and hence AC(G) = G.

(b) Assume AC(G) = 0. Since G is non-zero, $G \neq AC(G)$, and hence G is not locally cyclic by part (a).

If $AC(G) \neq 0$, then G is isomorphic to a subgroup of the additive rationals $\mathbb{Q}$ by Lemma 5.4, and hence G is locally cyclic. Thus, if G is not locally cyclic, then AC(G) = 0.

Hence the lemma is proved.

Theorem 5.6. Let G be a non-zero group.

(a) If G is torsion-free, then

$$AC(G) = \begin{cases} G \text{ if and only if } G \text{ is locally cyclic }. \\ 0 \text{ if and only if } G \text{ is not locally cyclic }. \end{cases}$$

(b) If G is mixed , then $AC(G) = 0$.

(c) If G is torsion , then $AC(G)$ is the direct sum of all the indecomposable primary components of G .

<u>Proof</u>. Parts (a) and (b) follow from Lemma 5.5 and 4.9 respectively .

(c) If G is torsion , then $G = \sum_{p \in \mathbb{P}} G_p$ its p-primary decomposition , by Theorem 3.2. By Lemma 4.6 , $AC(G) = \sum_{p \in \mathbb{P}} AC(G_p)$.

If G_p is indecomposable for some prime p , then G_p is a p-primary component of $\overline{G}$, by Theorem 5.3 . Since G_p is a subgroup of $\overline{G}$ which is locally cyclic , G_p is also locally cyclic , and hence $AC(G_p) = G_p$, by Corollary 4.4 .

If G_q is decomposable for some prime q , then let $G_q = A \oplus B$ be the decomposition of G_q into non-zero summands . If $AC(G_q) \neq 0$, then there exists a non-zero g in $AC(G_q)$ such that

$$g = a + b$$

for some a in A , b in B . Suppose neither a nor b is 0 . Then g and a generate a cyclic subgroup $[c_q]$ of G_q , and hence $g-a = b$ is in $[c_q]$, contradicting Lemma 4.8 . Suppose one of the a and b , say b , is 0 . Then the subgroup generated by $g = a$ and any non-zero element in B is cyclic , a contradiction as before .

Hence , in any case , we have a contradiction and , therefore , $AC(G_q) = 0$. Thus , if G_q is decomposable , then $AC(G_q) = 0$, and hence (c) is proved .

Now the theorem is completely proved .

As an application of Theorem 5.3 and 5.6 , we obtain the following theorem of Lim and , thus , we give a different proof than Lim's original one .

<u>Theorem 5.7</u>. If G is a group , then $AC(G) = \bar{G}$.