Chapter 3

Electrostatic Boundary Value Problem of Nonlinear Dielectric Composites

In order to get more accurate solutions of the electric potentials of nonlinear dielectric composites, we have extended the general nonlinear partial differential equations of the electric potentials up to the third order (Eqs. (2.24)-(2.27)) as shown in the last chapter. In this chapter, we will apply these equations to solve for the electric potentials up to the third order of a weakly nonlinear dielectric composite consisting of dilute linear cylindrical inclusions randomly dispersed in a nonlinear host medium.

3.1 Physical modeling

We consider a case of nonlinear dielectric composite, consisting of dilute linear cylindrical inclusions randomly dispersed in a nonlinear host medium. In this case, the system is assumed to be a single inclusion model. We apply a uniform electric field ($\mathbf{E}_0$) perpendicular to the inclusion axis and assume that the cylindrical dielectric inclusion has such a tiny radius relative to its length. Then this problem is considered to be a two dimensional problem as shown in Figure 3.1, where

i denotes the quantities in the inclusion region;

m denotes the quantities in the host region (medium);

a denotes the radius of the cylindrical inclusion;

ε_i is linear coefficient or the dielectric constant of the inclusion;

ε_m is linear coefficient or the dielectric constant of the medium;

χ_m is the third-order nonlinear coefficient or nonlinear susceptibility of the medium.

We use the cylindrical coordinate (r, θ, z) help to solve this problem. Since the system has a symmetric property along z-axis, the three dimensional system can be reduced to a two dimensional system which is described by two variables, r and θ.

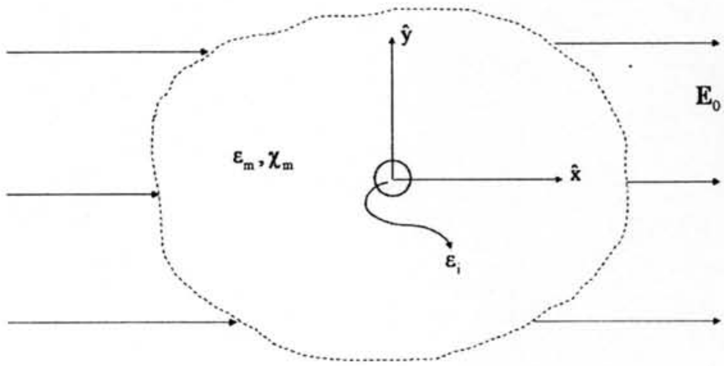

Figure 3.1: A single inclusion model of a nonlinear dilute dielectric composite in an external uniform electric field ($\mathbf{E}_0$).

3.2 Boundary conditions for the case of linear inclusions in a nonlinear medium

The first boundary condition

The first boundary condition is located at the remote distance from the cylindrical inclusion, which is

$$\phi^m(r \to \infty) = -E_0 r \cos\theta. \tag{3.1}$$

Comparing Eq. (3.1) with Eq. (2.8), we obtain

$$\phi_0^m + \lambda\phi_1^m + \lambda^2\phi_2^m + \lambda^3\phi_3^m = -E_0 r \cos\theta\ ; r \to \infty\ . \tag{3.2}$$

We will consider each term as follows:

The zeroth order:

$$\phi_0^m(r \to \infty) = -E_0 r \cos\theta. \tag{3.3}$$

The first order:

$$\phi_1^m(r \to \infty) = 0. \tag{3.4}$$

The second order:

$$\phi_2^m(r \to \infty) = 0. \tag{3.5}$$

The third order:

$$\phi_3^m(r \to \infty) = 0. \tag{3.6}$$

Eqs. (3.4), (3.5) and (3.6) show that the electric potentials at the remote distance from the inclusion which the orders are higher than zero do not exist.

The secondary boundary condition

The secondary boundary condition lies within the linear inclusion at $r = 0$. Every order of the electric potential must be finite at this point, let say

$$\phi_j^i(r = 0,\ \theta) = \text{finite},\ j = 0,\ 1,\ 2,\ 3\ . \tag{3.7}$$

The third boundary condition

The third boundary condition is situated at the connecting area, the inclusion surface ($r = a$). The tangential of the electric field (E_t) continues at the inclusion surface for every order, which are

$$E_{tj}^i = E_{tj}^m,\ j = 0,\ 1,\ 2,\ 3\ . \tag{3.8}$$

From $\mathbf{E} = -\boldsymbol{\nabla}\phi$ and for the tangential component of the electric field, Eq. (3.8) becomes

$$\left.\frac{\partial \phi_j^i}{\partial \theta}\right|_{r=a} = \left.\frac{\partial \phi_j^m}{\partial \theta}\right|_{r=a}, \tag{3.9}$$

or

$$\phi_j^i|_{r=a} = \phi_j^m|_{r=a},\ j = 0,\ 1,\ 2,\ 3\ . \tag{3.10}$$

Eq. (3.10) shows the continuity of electric potential at the inclusion surface for every order.

The fourth boundary condition

Because the normal component of the electric displacement is continuous on the inclusion surface ($r = a$), we have

$$\hat{n} \cdot \mathbf{D}_j^i = \hat{n} \cdot \mathbf{D}_j^m, \; j = 0,\ 1,\ 2,\ 3, \tag{3.11}$$

where $\hat{n}$ is a unit normal vector to the inclusion surface.

Substituting Eqs. (2.20)-(2.23) into Eq. (3.11) and consider each term in cylindrical coordinate as follows:

The zeroth order:

$$\varepsilon_i \frac{\partial \phi_0^i}{\partial r}\bigg|_a = \varepsilon_m \frac{\partial \phi_0^m}{\partial\, r}\bigg|_a. \tag{3.12}$$

The first order:

$$\varepsilon_i \frac{\partial \phi_1^i}{\partial r}\bigg|_a = \varepsilon_m \frac{\partial \phi_1^m}{\partial\, r}\bigg|_a + \beta_m G_0^m \frac{\partial \phi_0^m}{\partial r}\bigg|_a. \tag{3.13}$$

The second order:

$$\varepsilon_i \frac{\partial \phi_2^i}{\partial r}\bigg|_a = \varepsilon_m \frac{\partial \phi_2^m}{\partial\, r}\bigg|_a + \beta_m \Big(G_0^m \frac{\partial \phi_1^m}{\partial r}\bigg|_a + G_1^m \frac{\partial \phi_0^m}{\partial r}\bigg|_a \Big). \tag{3.14}$$

The third order:

$$\varepsilon_i \frac{\partial \phi_3^i}{\partial r}\bigg|_a = \varepsilon_m \frac{\partial \phi_3^m}{\partial\, r}\bigg|_a + \beta_m \Big(G_0^m \frac{\partial \phi_2^m}{\partial r}\bigg|_a + G_1^m \frac{\partial \phi_1^m}{\partial r}\bigg|_a + G_2^m \frac{\partial \phi_0^m}{\partial r}\bigg|_a \Big). \tag{3.15}$$

3.3 Derivation of the electric potentials of nonlinear composites

From Eqs. (2.24)-(2.27), we can write the equations describing the electric potential in the inclusion and host medium as follows:

For the inclusion region:

$$\nabla^2 \phi_0^i = 0, \tag{3.16}$$

$$\nabla^2 \phi_1^i = 0, \tag{3.17}$$

$$\nabla^2 \phi_2^i = 0, \tag{3.18}$$

$$\nabla^2 \phi_3^i = 0. \tag{3.19}$$

Note that Eqs. (3.16)-(3.19) are the Laplace equations.

For the host medium: For convenient, we let $\beta_m = 1$ or ε_m of the following equations is in unit of β_m.

$$\nabla^2\phi_0^m = 0, \tag{3.20}$$

$$\nabla^2\phi_1^m = -\frac{1}{\varepsilon_m}\Big[G_0^m\,\nabla^2\phi_0^m + \nabla G_0^m\cdot\nabla\phi_0^m\Big], \tag{3.21}$$

$$\begin{aligned}\nabla^2\phi_2^m &= -\frac{1}{\varepsilon_m}\Big[G_0^m\,\nabla^2\phi_1^m + \nabla\phi_1^m\cdot\nabla G_0^m + G_1^m\,\nabla^2\phi_0^m \\ &\quad + \nabla\phi_0^m\cdot\nabla G_1^m\Big],\end{aligned} \tag{3.22}$$

$$\begin{aligned}\nabla^2\phi_3^m &= -\frac{1}{\varepsilon_m}\Big[G_0^m\,\nabla^2\phi_2^m + \nabla\phi_2^m\cdot\nabla G_0^m + G_1^m\,\nabla^2\phi_1^m + \nabla\phi_1^m\cdot\nabla G_1^m \\ &\quad + G_2^m\,\nabla^2\phi_0^m + \nabla\phi_0^m\cdot\nabla G_2^m\Big].\end{aligned} \tag{3.23}$$

The zeroth-order electric potential

We first determine the zeroth-order electric potentials, $\phi_0^i(r,\theta)$ and $\phi_0^m(r,\theta)$, from Eqs. (3.16) and (3.20). The general solution of these equations, which have two variables r and θ, in cylindrical coordinate is [17],

$$\begin{aligned}\phi_o^\alpha(r,\theta) &= A_0^\alpha + B_0^\alpha\ln r + \sum_{n=1}^{\infty} r^n\Big[A_{0n}^\alpha\cos(n\theta) + B_{0n}^\alpha\sin(n\theta)\Big] \\ &\quad + \sum_{n=1}^{\infty} r^{-n}\Big[C_{0n}^\alpha\cos(n\theta) + D_{0n}^\alpha\sin(n\theta)\Big],\ \alpha = i,\ m.\end{aligned} \tag{3.24}$$

From the first boundary condition in Eq. (3.3), we obtain

$$\phi_0^m(r,\theta) = -E_0 r\cos\theta + \sum_{n=1}^{\infty} r^{-n}\Big[C_{0n}^m\cos(n\theta) + D_{0n}^m\sin(n\theta)\Big]. \tag{3.25}$$

From the second boundary condition in Eq. (3.7), the coefficients C_{on}^i and D_{on}^i are zero so

$$\phi_0^i(r,\theta) = \sum_{n=1}^{\infty} r^n\Big[A_{0n}^i\cos(n\theta) + B_{0n}^i\sin(n\theta)\Big]. \tag{3.26}$$

Using the third boundary condition in Eq. (3.10) and the orthogonal property between cosine and sine functions, we obtain

$$A_{o1}^i = \frac{C_{o1}^m}{a^2} - E_0. \tag{3.27}$$

From the fourth boundary condition in Eq. (3.12) and use the orthogonal property between cosine and sine functions again, we obtain

$$\varepsilon_i A^i_{o1} = -\varepsilon_m E_0 - \frac{\varepsilon_m C^m_{o1}}{a^2}. \tag{3.28}$$

From Eqs. (3.27) and (3.28), the coefficients A^i_{o1} and C^m_{o1} can be solved. The solutions are

$$A^i_{o1} = \frac{-2\varepsilon_m E_0}{(\varepsilon_i + \varepsilon_m)}, \tag{3.29}$$

and

$$C^m_{o1} = \frac{-E_0 a^2(\varepsilon_m - \varepsilon_i)}{(\varepsilon_i + \varepsilon_m)}. \tag{3.30}$$

Substituting A^i_{o1} and C^m_{o1} from Eqs. (3.29) and (3.30) into Eqs. (3.26) and (3.25) respectively, we obtain the zeroth-order electric potential in the inclusion and host medium as follows:

$$\phi^i_0(r,\theta) = -cE_0 r\cos\theta, \tag{3.31}$$

$$\phi^m_0(r,\theta) = -E_0(r + br^{-1})\cos\theta, \tag{3.32}$$

where $b = \frac{(\varepsilon_m - \varepsilon_i)a^2}{\varepsilon}$, $c = \frac{2\varepsilon_m}{\varepsilon}$ and $\varepsilon = \varepsilon_i + \varepsilon_m$. We note that $\mathbf{E}^i_0 = -\boldsymbol{\nabla}\phi^i_0 = c\mathbf{E}_0$ which is a uniform or constant field.

The first-order electric potential

We will determine the first-order electric potential, $\phi^i_1(r,\theta)$ and $\phi^m_1(r,\theta)$, from Eqs. (3.17) and (3.21). The electric potential equation in the inclusion, Eq. (3.17), is a homogeneous partial differential equation (Laplace equation) which the solution is known. In the host medium, Eq. (3.21) is a nonhomogeneous partial differential equation. Therefore, there are two parts of the solutions (Eq. (3.21)) which are the particular solution ($\phi^m_{1p}(r,\theta)$) and the complementary solution ($\phi^m_{1c}(r,\theta)$). The complementary solution is considered as a homogeneous equation (the right-hand side of Eq. (3.21) is considered to be zero) which this solution is known. For the particular solution, we need to compute the quantities of the right-hand side of Eq. (3.21). We omit the detail of calculation and give the result

$$\Big[G_0^m \nabla^2\phi_0^m + \nabla G_0^m \cdot \nabla\phi_0^m\Big] = \Big[(8b^2r^{-5} - 4b^3r^{-7})\cos\theta - 4br^{-3}\cos 3\theta\Big]E_0^3. \quad (3.33)$$

So, Eq. (3.21) becomes

$$\nabla^2\phi_1^m = -\frac{1}{\varepsilon_m}\Big[(8b^2r^{-5} - 4b^3r^{-7})\cos\theta - 4br^{-3}\cos 3\theta\Big]E_0^3. \quad (3.34)$$

The particular solution of $\phi_1^m(r,\theta)$ is

$$\phi_{1p}^m(r,\theta) = -\frac{1}{\varepsilon_m}\Big[(b^2r^{-3} - \frac{1}{6}\,b^3r^{-5})\cos\theta + \frac{1}{2}\,br^{-1}\cos 3\theta\Big]E_0^3. \quad (3.35)$$

The details of calculation of the particular solution, Eq. (3.35), are shown in Appendix A.

For the complementary solution, we consider

$$\nabla^2\phi_{1c}^m(r,\theta) = 0. \quad (3.36)$$

The general solution of $\phi_{1c}^m(r,\theta)$, Eq. (3.36), is equivalent to Eq. (3.24). We obtain

$$\begin{aligned}\phi_{1c}^m(r,\theta) &= A_1^m + B_1^m \ln r + \sum_{n=1}^{\infty} r^n\Big[A_{1n}^m\cos(n\theta) + B_{1n}^m\sin(n\theta)\Big] \\ &\quad + \sum_{n=1}^{\infty} r^{-n}\Big[C_{1n}^m\cos(n\theta) + D_{1n}^m\sin(n\theta)\Big]. \quad (3.37)\end{aligned}$$

Combining the particular solution, Eq. (3.35), and the complementary solution, Eq. (3.37), we obtain

$$\begin{aligned}\phi_1^m(r,\theta) &= A_1^m + B_1^m \ln r + \sum_{n=1}^{\infty} r^n\Big[A_{1n}^m\cos(n\theta) + B_{1n}^m\sin(n\theta)\Big] \\ &\quad + \sum_{n=1}^{\infty} r^{-n}\Big[C_{1n}^m\cos(n\theta) + D_{1n}^m\sin(n\theta)\Big] \\ &\quad - \frac{1}{\varepsilon_m}\Big[(b^2r^{-3} - \frac{1}{6}\,b^3r^{-5})\cos\theta + \frac{1}{2}\,br^{-1}\cos 3\theta\Big]E_0^3. \quad (3.38)\end{aligned}$$

For the inclusion region, the general solution of $\phi_1^i(r,\theta)$, from Eq. (3.17) is Laplace equation, can be written as

$$\begin{aligned}\phi_1^i(r,\theta) &= A_1^i + B_1^i \ln r + \sum_{n=1}^{\infty} r^n \Big[A_{1n}^i \cos(n\theta) + B_{1n}^i \sin(n\theta)\Big] \\ &\quad + \sum_{n=1}^{\infty} r^{-n} \Big[C_{1n}^i \cos(n\theta) + D_{1n}^i \sin(n\theta)\Big]. \qquad (3.39)\end{aligned}$$

From the first boundary condition in Eq. (3.4), $\phi_1^m(r \to \infty) = 0$, we find that the coefficients $A_1^m = 0$, $B_1^m = 0$, $A_{1n}^m = 0$ and $B_{1n}^m = 0$. Thus, Eq. (3.38) is rewritten as

$$\begin{aligned}\phi_1^m(r,\theta) &= \sum_{n=1}^{\infty} r^{-n} \Big[C_{1n}^m \cos(n\theta) + D_{1n}^m \sin(n\theta)\Big] \\ &\quad - \frac{1}{\varepsilon_m}\Big[(b^2 r^{-3} - \frac{1}{6}\, b^3 r^{-5}) \cos\theta + \frac{1}{2}\, b r^{-1} \cos 3\theta\Big] E_0^3. \qquad (3.40)\end{aligned}$$

From the secondary boundary condition in Eq. (3.7), $\phi_1^i(r = 0, \theta) =$ finite, the coefficients $A_1^i = 0$, $B_1^i = 0$, $C_{1n}^i = 0$ and $D_{1n}^i = 0$. Also, Eq. (3.39) can be expressed as

$$\phi_1^i(r,\theta) = \sum_{n=1}^{\infty} r^n \Big[A_{1n}^i \cos(n\theta) + B_{1n}^i \sin(n\theta)\Big]. \qquad (3.41)$$

Substituting Eqs. (3.40) and (3.41) into the third boundary condition, Eq. (3.10), we obtain

$$\begin{aligned}&\sum_{n=1}^{\infty} a^{-n} \Big[C_{1n}^m \cos(n\theta) + D_{1n}^m \sin(n\theta)\Big] - \frac{1}{\varepsilon_m}\Big[(b^2 a^{-3} - \frac{1}{6}\, b^3 a^{-5}) \cos\theta \\ &\quad + \frac{1}{2}\, b a^{-1} \cos 3\theta\Big] E_0^3 = \sum_{n=1}^{\infty} a^n \Big[A_{1n}^i \cos(n\theta) + B_{1n}^i \sin(n\theta)\Big]. \qquad (3.42)\end{aligned}$$

By using the orthogonal property between cosine and sine functions, we obtain

$$A_{11}^i = a^{-2} C_{11}^m - \frac{1}{\varepsilon_m}(b_0^2 - \frac{1}{6}\, b_0^3) E_0^3, \qquad (3.43)$$

and

$$A_{13}^i = a^{-6} C_{13}^m - \frac{1}{2\varepsilon_m}\, b a^{-4} E_0^3, \qquad (3.44)$$

where $b_0 = ba^{-2}$.

Substituting Eqs. (3.40) and (3.41) into the fourth boundary condition, Eq. (3.13), we obtain

$$\varepsilon_m\Bigg[\sum_{n=1}^{\infty}-na^{-n-1}\Big[C^m_{1n}\cos(n\theta)+D^m_{1n}\sin(n\theta)\Big]-\frac{1}{\varepsilon_m}\Big[(-3b_0^2+\frac{5}{6}\,b_0^3)\cos\theta$$
$$-\frac{1}{2}\,b_0\cos 3\theta\Big]E_0^3\Bigg]-\Big[(1-2b_0+2b_0^2-b_0^3)\cos\theta+(-b_0+b_0^2)\cos 3\theta\Big]E_0^3$$
$$=\varepsilon_i\Bigg[\sum_{n=1}^{\infty}na^{n-1}\Big[A^i_{1n}\cos(n\theta)+B^i_{1n}\sin(n\theta)\Big]\Bigg], \quad (3.45)$$

where ε_m and ε_i are in unit of β_m.

Again, using the orthogonal property between cosine and sine functions, we obtain

$$\varepsilon_m\Bigg[-a^{-2}C^m_{11}-\frac{1}{\varepsilon_m}\Big[(-3b_0^2+\frac{5}{6}\,b_0^3)E_0^3\Big]\Bigg]-(1-2b_0+2b_0^2-b_0^3)E_0^3=\varepsilon_iA^i_{11}, \quad (3.46)$$

and

$$\varepsilon_m\Big[-3a^{-4}C^m_{13}+\frac{b_0}{2\varepsilon_m}\,E_0^3\Big]-(-b_0+b_o^2)E_0^3=3\varepsilon_ia^2A^i_{13}. \quad (3.47)$$

From Eqs. (3.43), (3.44), (3.46) and (3.47), the coefficients A^i_{11}, C^m_{11}, A^i_{13} and C^m_{13} can be solved. The solutions are

$$A^i_{11} = -\frac{1}{\varepsilon}\Big[-(2b_0^2-\frac{2}{3}\,b_0^3)+(1-2b_0+2b_0^2-b_0^3)\Big]E_0^3, \quad (3.48)$$

$$C^m_{11} = -\frac{a^{-2}}{\varepsilon}\Big[-(3b_0^2-\frac{5}{6}\,b_0^3)+(1-2b_0+2b_0^2-b_0^3) -\frac{\varepsilon_i}{\varepsilon_m}(b_0^2-\frac{b_0^3}{6})\Big]E_0^3, \quad (3.49)$$

$$A^i_{13} = -\frac{a^{-2}}{\varepsilon}\Big[\frac{b_0^2}{3}\Big]E_0^3, \quad (3.50)$$

and

$$C^m_{13}=-\frac{a^{-4}}{\varepsilon}\Big[-\frac{b_0}{6}-\frac{\varepsilon_i}{\varepsilon_m}(\frac{b_0}{2})+(-\frac{b_0}{3}+\frac{b_0^2}{3})\Big]E_0^3. \quad (3.51)$$

Substituting C^m_{11} and C^m_{13} into Eq. (3.40), we obtain the first-order electric potential in the host medium as

$$\phi_1^m(r,\theta) = -\Big[\Big(b_1r^{-1}+\frac{1}{\varepsilon_m}(b^2r^{-3}-\frac{1}{6}\,b^3r^{-5})\Big)\cos\theta +\Big(b_2r^{-3}+\frac{1}{\varepsilon_m}(\frac{1}{2}\,br^{-1})\Big)\cos 3\theta\Big]E_0^3, \quad (3.52)$$

where

$$b_1 = \frac{a^2}{\varepsilon}\Big[1 - 2b_0 - b_0^2 - \frac{b_0^3}{6} - \frac{\varepsilon_i}{\varepsilon_m}(b_0^2 - \frac{b_0^3}{6})\Big], \quad (3.53)$$

$$b_2 = \frac{a^4}{\varepsilon}\Big[-\frac{b_0}{2} + \frac{b_0^2}{3} - \frac{\varepsilon_i}{\varepsilon_m}(\frac{b_0}{2})\Big]. \quad (3.54)$$

Substituting A^i_{11} and A^i_{13} into Eq. (3.41), we obtain the first-order electric potential in the inclusion as

$$\phi^i_1(r,\theta) = -\Big[b_3 r\cos\theta + b_4 r^3 \cos 3\theta\Big]E_0^3, \quad (3.55)$$

where

$$b_3 = \frac{1}{\varepsilon}\Big[1 - 2b_0 - \frac{b_0^3}{3}\Big], \quad (3.56)$$

$$b_4 = \frac{a^{-2}}{\varepsilon}(\frac{b_0^2}{3}), \quad (3.57)$$

where $\varepsilon = \varepsilon_i + \varepsilon_m$ is in unit of $\beta_m = \chi_m/\lambda$.

The second-order electric potential

We continue to determine the second-order electric potentials $\phi^i_2(r,\theta)$ from Eq. (3.18) and $\phi^m_2(r,\theta)$ from Eq. (3.22). In the inclusion region, the general solution of $\phi^i_2(r,\theta)$ is

$$\begin{aligned}\phi^i_2(r,\theta) &= A^i_2 + B^i_2 \ln r + \sum_{n=1}^{\infty} r^n\Big[A^i_{2n}\cos(n\theta) + B^i_{2n}\sin(n\theta)\Big] \\ &\quad + \sum_{n=1}^{\infty} r^{-n}\Big[C^i_{2n}\cos(n\theta) + D^i_{2n}\sin(n\theta)\Big]. \quad (3.58)\end{aligned}$$

In the host medium, we need to compute the quantities of the right-hand side of Eq. (3.22) by using mathematica program because the process is much time-consuming. We give the result as

$$\begin{aligned}\nabla^2\phi^m_2 &= -\frac{1}{\varepsilon_m}\Big[\Big(\frac{16bb_1}{r^5} - \frac{20b^2}{r^5} - \frac{12b^2b_1}{r^7} + \frac{24bb_2}{r^7} + \frac{68b^3}{\varepsilon_m r^7} - \frac{24b^2b_2}{r^9} - \frac{76b^4}{\varepsilon_m r^9} \\ &\quad + \frac{56b^5}{3\varepsilon_m r^{11}}\Big)\cos\theta + \Big(-\frac{4b_1}{r^3} + \frac{8b}{\varepsilon_m r^3} + \frac{48bb_2}{r^7} + \frac{32b^3}{\varepsilon_m r^7} - \frac{24b^2b_2}{r^9} \\ &\quad - \frac{44b^4}{3\varepsilon_m r^9} + \frac{2b^5}{\varepsilon_m r^{11}}\Big)\cos 3\theta + \Big(-\frac{6b}{\varepsilon_m r^3} - \frac{24b_2}{r^5} - \frac{12b^2}{\varepsilon_m r^5}\Big)\cos 5\theta\Big]E_0^5, \\ &\quad (3.59)\end{aligned}$$

where ε_m is in unit of $\beta_m = \chi_m/\lambda$.

The solutions of $\phi_2^m(r,\theta)$ are composed of the particular solution ($\phi_{2p}^m(r,\theta)$) and the complementary solution ($\phi_{2c}^m(r,\theta)$). The particular solution of $\phi_2^m(r,\theta)$ (see Appendix A) is

$$\begin{aligned}\phi_{2p}^m &= -\frac{1}{\varepsilon_m}\Big[(b_5 r^{-3} + b_6 r^{-5} + b_7 r^{-7} + b_8 r^{-9})\cos\theta \\ &+ (b_9 r^{-1} + b_{10} r^{-5} + b_{11} r^{-7} + b_{12} r^{-9})\cos 3\theta \\ &+ (b_{13} r^{-1} + b_{14} r^{-3})\cos 5\theta\Big] E_0^5, \end{aligned} \tag{3.60}$$

where;

$$b_5 = 2bb_1 - \frac{5b^2}{2\varepsilon_m}, \tag{3.61}$$

$$b_6 = -\frac{b^2 b_1}{2} + bb_2 + \frac{17b^3}{6\varepsilon_m}, \tag{3.62}$$

$$b_7 = -\frac{b^2 b_2}{2} - \frac{19b^4}{12\varepsilon_m}, \tag{3.63}$$

$$b_8 = \frac{7b^5}{30\varepsilon_m}, \tag{3.64}$$

$$b_9 = \frac{b_1}{2} - \frac{b}{\varepsilon_m}, \tag{3.65}$$

$$b_{10} = 3bb_2 + \frac{2b^3}{\varepsilon_m}, \tag{3.66}$$

$$b_{11} = -\frac{3b^2 b_2}{5} - \frac{11b^4}{30\varepsilon_m}, \tag{3.67}$$

$$b_{12} = \frac{b^5}{36\varepsilon_m}, \tag{3.68}$$

$$b_{13} = \frac{b}{4\varepsilon_m}, \tag{3.69}$$

$$b_{14} = \frac{3b_2}{2} + \frac{3b^2}{4\varepsilon_m}. \tag{3.70}$$

which ε_m is in unit of $\beta_m = \chi_m/\lambda$ and $\lambda^2\phi_{2p}^m$ is the second-order potential.

The calculations of the particular solution, $\phi_{2p}^m(r,\theta)$, are shown in detail (see Appendix A).

For the complementary solution, we consider

$$\nabla^2 \phi_{2c}^m(r,\theta) = 0. \tag{3.71}$$

The general solution of $\phi_{2c}^m(r,\theta)$, Eq. (3.71), is

$$\phi_{2c}^m(r,\theta) = A_2^m + B_2^m \ln r + \sum_{n=1}^{\infty} r^n \Big[A_{2n}^m \cos(n\theta) + B_{2n}^m \sin(n\theta)\Big] + \sum_{n=1}^{\infty} r^{-n} \Big[C_{2n}^m \cos(n\theta) + D_{2n}^m \sin(n\theta)\Big]. \qquad (3.72)$$

Combining the particular solution, Eq. (3.60), and the complementary solution, Eq. (3.72), we obtain

$$\begin{aligned} \phi_2^m(r,\theta) = {} & A_2^m + B_2^m \ln r + \sum_{n=1}^{\infty} r^n \Big[A_{2n}^m \cos(n\theta) + B_{2n}^m \sin(n\theta)\Big] \\ & + \sum_{n=1}^{\infty} r^{-n} \Big[C_{2n}^m \cos(n\theta) + D_{2n}^m \sin(n\theta)\Big] \\ & - \frac{1}{\varepsilon_m}\Big[(b_5 r^{-3} + b_6 r^{-5} + b_7 r^{-7} + b_8 r^{-9}) \cos\theta + (b_9 r^{-1} + b_{10} r^{-5} \\ & + b_{11} r^{-7} + b_{12} r^{-9}) \cos 3\theta + (b_{13} r^{-1} + b_{14} r^{-3}) \cos 5\theta\Big] E_0^5. \end{aligned} \qquad (3.73)$$

Applying the four boundary conditions from Eqs. (3.5), (3.7), (3.10) and (3.14) to the two solutions, Eqs. (3.58) and (3.73), mentioned above. The computation of $\phi_2^m(r,\theta)$ is far more tedious, but similar to that of $\phi_1^m(r,\theta)$. We finally obtain the second-order electric potentials in the host medium and in the inclusion as follows:

$$\begin{aligned} \phi_2^m(r,\theta) = {} & -\Big[\Big(c_1 r^{-1} + \frac{1}{\varepsilon_m}\,(b_5 r^{-3} + b_6 r^{-5} + b_7 r^{-7} + b_8 r^{-9})\Big) \cos\theta \\ & + \Big(c_2 r^{-3} + \frac{1}{\varepsilon_m}\,(b_9 r^{-1} + b_{10} r^{-5} + b_{11} r^{-7} + b_{12} r^{-9})\Big) \cos 3\theta \\ & + \Big(c_3 r^{-5} + \frac{1}{\varepsilon_m}\,(b_{13} r^{-1} + b_{14} r^{-3})\Big) \cos 5\theta\Big] E_0^5, \end{aligned} \qquad (3.74)$$

$$\phi_2^i(r,\theta) = -\Big[c_4 r \cos\theta + c_5 r^3 \cos 3\theta + c_6 r^5 \cos 5\theta\Big] E_0^5, \qquad (3.75)$$

where

$$
\begin{aligned}
c_1 &= \frac{a}{\varepsilon}\Big(C_1\varepsilon_i - D_1 a\Big), && (3.76)\\
c_2 &= \frac{a^3}{\varepsilon}\Big(C_3\varepsilon_i - \frac{D_3 a}{3}\Big), && (3.77)\\
c_3 &= \frac{a^5}{\varepsilon}\Big(C_5\varepsilon_i - \frac{D_5 a}{5}\Big), && (3.78)\\
c_4 &= -\frac{a^{-1}}{\varepsilon}\Big(C_1\varepsilon_m + D_1 a\Big), && (3.79)\\
c_5 &= -\frac{a^{-3}}{\varepsilon}\Big(C_3\varepsilon_m + \frac{D_3 a}{3}\Big), && (3.80)\\
c_6 &= -\frac{a^{-5}}{\varepsilon}\Big(C_5\varepsilon_m + \frac{D_5 a}{5}\Big), && (3.81)
\end{aligned}
$$

where

$$
\begin{aligned}
C_1 &= -\frac{1}{\varepsilon_m}\Big(b_5 a^{-3} + b_6 a^{-5} + b_7 a^{-7} + b_8 a^{-9}\Big), && (3.82)\\
C_3 &= -\frac{1}{\varepsilon_m}\Big(b_9 a^{-1} + b_{10} a^{-5} + b_{11} a^{-7} + b_{12} a^{-9}\Big), && (3.83)\\
C_5 &= -\frac{1}{\varepsilon_m}\Big(b_{13} a^{-1} + b_{14} a^{-3}\Big), && (3.84)\\
D_1 &= \Big(3b_5 a^{-4} + 5b_6 a^{-6} + 7b_7 a^{-8} + 9b_8 a^{-10}\Big) + \Big(2b_1 - \frac{b}{2\varepsilon_m}\Big)a^{-2}\\
&\quad + \Big(-4b_1 b + \frac{6b^2}{\varepsilon_m}\Big)a^{-4} + \Big(-6b_2 b + 3b_1 b^2 - \frac{13b^3}{\varepsilon_m}\Big)a^{-6}\\
&\quad + \Big(3b_2 b^2 + \frac{34b^4}{3\varepsilon_m}\Big)a^{-8} - \frac{13b^5}{6\varepsilon_m}a^{-10}, && (3.85)\\
D_3 &= \Big(b_9 a^{-2} + 5b_{10} a^{-6} + 7b_{11} a^{-8} + 9b_{12} a^{-10}\Big) + \Big(b_1 + \frac{b}{\varepsilon_m}\Big)a^{-2}\\
&\quad + \Big(6b_2 - 2b_1 b + \frac{b^2}{\varepsilon_m}\Big)a^{-4} + \Big(-6b_2 b - \frac{11b^3}{2\varepsilon_m}\Big)a^{-6}\\
&\quad + \Big(6b_2 b^2 + \frac{8b^4}{3\varepsilon_m}\Big)a^{-8} - \frac{b^5}{3\varepsilon_m}a^{-10}, && (3.86)\\
D_5 &= \Big(b_{13} + \frac{b}{\varepsilon_m}\Big)a^{-2} + \Big(3b_2 + 3b_{14} - \frac{b^2}{\varepsilon_m}\Big)a^{-4}\\
&\quad + \Big(-6b_2 b - \frac{b^3}{2\varepsilon_m}\Big)a^{-6}, && (3.87)
\end{aligned}
$$

which ε_m and ε_i are in unit of $\beta_m = \chi_m/\lambda$ and $\varepsilon = \varepsilon_i + \varepsilon_m$. We note that $\lambda^2\phi_2^m$ is the second-order potential which is independent of λ and depend on χ_m^2.

The results of our calculations show some miss prints of the reported work of Gu and Yu [6] as confirmed by showing that our solutions satisfying the differential equations and the boundary conditions.

The third-order electric potential

We have just obtained the zeroth, the first and the second-orders electric potentials of a linear dielectric inclusion embedded in nonlinear dielectric medium. Next we calculate the third-order electric potential of the system which is the new result of this work. We will proceed to find the $\phi_3^i(r,\theta)$ and $\phi_3^m(r,\theta)$ from Eqs. (3.19) and (3.23), respectively. In the inclusion region, the general solution of $\phi_3^i(r,\theta)$ is

$$\begin{aligned} \phi_3^i(r,\theta) &= A_3^i + B_3^i \ln r + \sum_{n=1}^{\infty} r^n \Big[A_{3n}^i \cos(n\theta) + B_{3n}^i \sin(n\theta)\Big] \\ &\quad + \sum_{n=1}^{\infty} r^{-n} \Big[C_{3n}^i \cos(n\theta) + D_{3n}^i \sin(n\theta)\Big]. \end{aligned} \tag{3.88}$$

In the host medium, the quantities of the right-hand side of Eq. (3.23) are computed by using mathematica program. We give the result as

$$\begin{aligned} \nabla^2 \phi_3^m(r,\theta) &= -\frac{1}{\varepsilon_m}\Big[(g_1 r^{-5} + g_2 r^{-7} + g_3 r^{-9} + g_4 r^{-11} + g_5 r^{-13} + g_6 r^{-15}) \cos\theta \\ &\quad + (g_7 r^{-3} + g_8 r^{-5} + g_9 r^{-7} + g_{10} r^{-9} + g_{11} r^{-11} + g_{12} r^{-13} \\ &\quad + g_{13} r^{-15}) \cos 3\theta + (g_{14} r^{-3} + g_{15} r^{-5} + g_{16} r^{-7} + g_{17} r^{-9} \\ &\quad + g_{18} r^{-11} + g_{19} r^{-13} + g_{20} r^{-15}) \cos 5\theta + (g_{21} r^{-3} + g_{22} r^{-5} \\ &\quad + g_{23} r^{-7}) \cos 7\theta\Big] E_0^7, \end{aligned} \tag{3.89}$$

where ε_m is in unit of $\beta_m = \chi_m/\lambda$ and

$$g_1 = 8b_1^2 + 16bc_1 + \frac{8b^2}{\varepsilon_m^2} - \frac{20b_5}{\varepsilon_m}, \tag{3.90}$$

$$\begin{aligned} g_2 &= -12bb_1^2 + 24b_1b_2 - 12b^2c_1 + 24bc_2 - \frac{4b^3}{\varepsilon_m^2} + \frac{52b^2b_1}{\varepsilon_m} \\ &+ \frac{60bb_2}{\varepsilon_m} + \frac{64bb_5}{\varepsilon_m} - \frac{60b_6}{\varepsilon_m} - \frac{12b^2b_9}{\varepsilon_m} - \frac{4b_{10}}{\varepsilon_m}, \end{aligned} \tag{3.91}$$

$$\begin{aligned} g_3 &= -48bb_1b_2 + 144b_2^2 - 24b^2c_2 + \frac{100b^4}{\varepsilon_m^2} - \frac{128b^3b_1}{\varepsilon_m} + \frac{24b^2b_2}{\varepsilon_m} \\ &- \frac{52b^2b_5}{\varepsilon_m} + \frac{144bb_6}{\varepsilon_m} - \frac{120b_7}{\varepsilon_m} + \frac{64bb_{10}}{\varepsilon_m} - \frac{12b_{11}}{\varepsilon_m}, \end{aligned} \tag{3.92}$$

$$\begin{aligned} g_4 &= -144bb_2^2 - \frac{832b^5}{3\varepsilon_m^2} + \frac{112b^4b_1}{3\varepsilon_m} - \frac{192b^3b_2}{\varepsilon_m} - \frac{112b^2b_6}{\varepsilon_m} + \frac{256bb_7}{\varepsilon_m} \\ &- \frac{200b_8}{\varepsilon_m} - \frac{40b^2b_{10}}{\varepsilon_m} + \frac{120bb_{11}}{\varepsilon_m} - \frac{24b_{12}}{\varepsilon_m}, \end{aligned} \tag{3.93}$$

$$g_5 = \frac{146b^6}{\varepsilon_m^2} + \frac{56b^4b_2}{\varepsilon_m} - \frac{192b^2b_7}{\varepsilon_m} + \frac{400bb_8}{\varepsilon_m} - \frac{60b^2b_{11}}{\varepsilon_m} + \frac{192bb_{12}}{\varepsilon_m}, \tag{3.94}$$

$$g_6 = -\frac{21b^7}{\varepsilon_m^2} - \frac{292b^2b_8}{\varepsilon_m} - \frac{84b^2b_{12}}{\varepsilon_m}, \tag{3.95}$$

$$g_7 = -4c_1 + \frac{16b_9}{\varepsilon_m} - \frac{4b_{13}}{\varepsilon_m}, \tag{3.96}$$

$$g_8 = \frac{12bb_1}{\varepsilon_m} - \frac{12b_5}{\varepsilon_m} + \frac{24bb_9}{\varepsilon_m} - \frac{24bb_{13}}{\varepsilon_m}, \tag{3.97}$$

$$\begin{aligned} g_9 &= 48b_1b_2 + 48bc_2 + \frac{20b^3}{\varepsilon_m^2} + \frac{32b^2b_1}{\varepsilon_m} + \frac{24bb_5}{\varepsilon_m} \\ &- \frac{24b_6}{\varepsilon_m} + \frac{8b^2b_9}{\varepsilon_m} - \frac{32b_{10}}{\varepsilon_m} - \frac{24b^2b_{13}}{\varepsilon_m}, \end{aligned} \tag{3.98}$$

$$\begin{aligned} g_{10} &= -48bb_1b_2 - 24b^2c_2 + 40bc_3 + \frac{124b^4}{3\varepsilon_m^2} - \frac{56b^3b_1}{3\varepsilon_m} + \frac{168b^2b_2}{\varepsilon_m} \\ &- \frac{4b^2b_5}{\varepsilon_m} + \frac{64bb_6}{\varepsilon_m} - \frac{40b_7}{\varepsilon_m} + \frac{88bb_{10}}{\varepsilon_m} - \frac{80b_{11}}{\varepsilon_m} - \frac{40b^2b_{14}}{\varepsilon_m}, \end{aligned} \tag{3.99}$$

$$\begin{aligned} g_{11} &= -60b^2c_3 - \frac{70b^5}{\varepsilon_m^2} + \frac{4b^4b_1}{\varepsilon_m} - \frac{276b^3b_2}{\varepsilon_m} - \frac{12b^2b_6}{\varepsilon_m} + \frac{120bb_7}{\varepsilon_m} \\ &- \frac{60b_8}{\varepsilon_m} - \frac{72b^2b_{10}}{\varepsilon_m} + \frac{144bb_{11}}{\varepsilon_m} - \frac{144b_{12}}{\varepsilon_m}, \end{aligned} \tag{3.100}$$

$$g_{12} = \frac{104b^6}{3\varepsilon_m^2} + \frac{76b^4b_2}{\varepsilon_m} - \frac{24b^2b_7}{\varepsilon_m} + \frac{192bb_8}{\varepsilon_m} - \frac{136b^2b_{11}}{\varepsilon_m} + \frac{216bb_{12}}{\varepsilon_m}, \quad (3.101)$$

$$g_{13} = -\frac{46b^7}{9\varepsilon_m^2} - \frac{40b^2b_8}{\varepsilon_m} - \frac{216b^2b_{12}}{\varepsilon_m}, \quad (3.102)$$

$$g_{14} = -\frac{12b_9}{\varepsilon_m} + \frac{48b_{13}}{\varepsilon_m}, \quad (3.103)$$

$$g_{15} = -24c_2 + \frac{4b^2}{\varepsilon_m^2} - \frac{12bb_1}{\varepsilon_m} - \frac{24bb_9}{\varepsilon_m} + \frac{56bb_{13}}{\varepsilon_m} + \frac{32b_{14}}{\varepsilon_m}, \quad (3.104)$$

$$g_{16} = \frac{10b^3}{\varepsilon_m^2} - \frac{40b_{10}}{\varepsilon_m} + \frac{40b^2b_{13}}{\varepsilon_m} + \frac{80bb_{14}}{\varepsilon_m}, \quad (3.105)$$

$$g_{17} = 120bc_3 + \frac{8b^4}{\varepsilon_m^2} + \frac{120b^2b_2}{\varepsilon_m} + \frac{40bb_{10}}{\varepsilon_m} - \frac{60b_{11}}{\varepsilon_m} + \frac{8b^2b_{14}}{\varepsilon_m}, \quad (3.106)$$

$$g_{18} = -36bb_2^2 - 40b^2c_3 - \frac{4b^5}{\varepsilon_m^2} - \frac{64b^3b_2}{\varepsilon_m} - \frac{4b^2b_{10}}{\varepsilon_m} + \frac{96bb_{11}}{\varepsilon_m} - \frac{84b_{12}}{\varepsilon_m}, \quad (3.107)$$

$$g_{19} = \frac{12b^4b_2}{\varepsilon_m} - \frac{12b^2b_{11}}{\varepsilon_m} + \frac{168bb_{12}}{\varepsilon_m}, \quad (3.108)$$

$$g_{20} = -\frac{24b^2b_{12}}{\varepsilon_m}, \quad (3.109)$$

$$g_{21} = -\frac{24b_{13}}{\varepsilon_m}, \quad (3.110)$$

$$g_{22} = -\frac{14b^2}{\varepsilon_m^2} - \frac{64bb_{13}}{\varepsilon_m} - \frac{40b_{14}}{\varepsilon_m}, \quad (3.111)$$

$$g_{23} = -60c_3 - \frac{5b^3}{\varepsilon_m^2} - \frac{48bb_2}{\varepsilon_m} - \frac{4b^2b_{13}}{\varepsilon_m} - \frac{40bb_{14}}{\varepsilon_m}. \quad (3.112)$$

The solutions of $\phi_3^m(r,\theta)$, Eq. (3.89), are composed of the particular solution ($\phi_{3p}^m(r,\theta)$) and the complementary solution ($\phi_{3c}^m(r,\theta)$). The particular solution of $\phi_3^m(r,\theta)$ is

$$\begin{aligned}\phi_{3p}^m(r,\theta) = & -\frac{1}{\varepsilon_m}\Big[(b_{15}r^{-3} + b_{16}r^{-5} + b_{17}r^{-7} + b_{18}r^{-9} + b_{19}r^{-11} \\ & + b_{20}r^{-13})\cos\theta + (b_{21}r^{-1} + b_{22}r^{-3}\ln r + b_{23}r^{-5} + b_{24}r^{-7} \\ & + b_{25}r^{-9} + b_{26}r^{-11} + b_{27}r^{-13})\cos 3\theta + (b_{28}r^{-1} + b_{29}r^{-3} \\ & + b_{30}r^{-5}\ln r + b_{31}r^{-7} + b_{32}r^{-9} + b_{33}r^{-11} + b_{34}r^{-13})\cos 5\theta \\ & + (b_{35}r^{-1} + b_{36}r^{-3} + b_{37}r^{-5})\cos 7\theta\Big]E_0^7, \quad (3.113)\end{aligned}$$

where $b_{15} = \frac{1}{8}\,g_1$, $b_{16} = \frac{1}{24}\,g_2$, $b_{17} = \frac{1}{48}\,g_3$, $b_{18} = \frac{1}{80}\,g_4$, $b_{19} = \frac{1}{120}\,g_5$, $b_{20} = \frac{1}{168}\,g_6$,

$b_{21} = -\frac{1}{8}\, g_7$, $b_{22} = -\frac{1}{6}\, g_8$, $b_{23} = \frac{1}{16}\, g_9$, $b_{24} = \frac{1}{40}\, g_{10}$, $b_{25} = \frac{1}{72}\, g_{11}$, $b_{26} = \frac{1}{112}\, g_{12}$, $b_{27} = \frac{1}{160}\, g_{13}$, $b_{28} = -\frac{1}{24}\, g_{14}$, $b_{29} = -\frac{1}{16}\, g_{15}$, $b_{30} = -\frac{1}{10}\, g_{16}$, $b_{31} = \frac{1}{24}\, g_{17}$, $b_{32} = \frac{1}{56}\, g_{18}$, $b_{33} = \frac{1}{96}\, g_{19}$, $b_{34} = \frac{1}{144}\, g_{20}$, $b_{35} = -\frac{1}{48}\, g_{21}$, $b_{36} = -\frac{1}{40}\, g_{22}$ and $b_{37} = -\frac{1}{24}\, g_{23}$. The details of calculation of the particular solution, $\phi^m_{3p}(r,\theta)$, are similar to those of $\phi^m_{2p}(r,\theta)$ as shown in Appendix A.

For the complementary solution, we consider

$$\nabla^2 \phi^m_{3c}(r,\theta) = 0. \tag{3.114}$$

The general solution of $\phi^m_{3c}(r,\theta)$, Eq. (3.114), is

$$\begin{aligned} \phi^m_{3c}(r,\theta) &= A^m_3 + B^m_3 \ln r + \sum_{n=1}^{\infty} r^n \Big[A^m_{3n}\cos(n\theta) + B^m_{3n}\sin(n\theta)\Big] \\ &\quad + \sum_{n=1}^{\infty} r^{-n}\Big[C^m_{3n}\cos(n\theta) + D^m_{3n}\sin(n\theta)\Big]. \end{aligned} \tag{3.115}$$

Combining the particular solution, Eq. (3.113), and the complementary solution, Eq. (3.115), we obtain

$$\begin{aligned} \phi^m_3(r,\theta) &= A^m_3 + B^m_3 \ln r + \sum_{n=1}^{\infty} r^n \Big[A^m_{3n}\cos(n\theta) + B^m_{3n}\sin(n\theta)\Big] \\ &\quad + \sum_{n=1}^{\infty} r^{-n}\Big[C^m_{3n}\cos(n\theta) + D^m_{3n}\sin(n\theta)\Big] \\ &\quad - \frac{1}{\varepsilon_m}\Big[(b_{15}r^{-3} + b_{16}r^{-5} + b_{17}r^{-7} + b_{18}r^{-9} + b_{19}r^{-11} + b_{20}r^{-13})\cos\theta \\ &\quad + (b_{21}r^{-1} + b_{22}r^{-3}\ln r + b_{23}r^{-5} + b_{24}r^{-7} + b_{25}r^{-9} + b_{26}r^{-11} \\ &\quad + b_{27}r^{-13})\cos 3\theta + (b_{28}r^{-1} + b_{29}r^{-3} + b_{30}r^{-5}\ln r + b_{31}r^{-7} \\ &\quad + b_{32}r^{-9} + b_{33}r^{-11} + b_{34}r^{-13})\cos 5\theta + (b_{35}r^{-1} + b_{36}r^{-3} \\ &\quad + b_{37}r^{-5})\cos 7\theta\Big] E_0^7. \end{aligned} \tag{3.116}$$

Applying the four boundary conditions for the third-order electric potential from Eqs. (3.6), (3.7), (3.10) and (3.15) to the two solutions, Eqs. (3.88) and (3.116), mentioned above. The computation of $\phi^m_3(r,\theta)$ is far more tedious, but similar to that of $\phi_1^m(r,\theta)$ and $\phi_2^m(r,\theta)$. We finally obtain the third-order electric potentials in the host medium and in the inclusion which are the new results of this work as follows:

$$\begin{aligned}\phi_3^m(r,\theta) &= -\Big[\Big(c_7 r^{-1} + \frac{1}{\varepsilon_m}\,(b_{15}r^{-3} + b_{16}r^{-5} + b_{17}r^{-7} + b_{18}r^{-9} + b_{19}r^{-11} \\ &\quad + b_{20}r^{-13})\Big)\cos\theta + \Big(c_8 r^{-3} + \frac{1}{\varepsilon_m}\,(b_{21}r^{-1} + (b_{22}r^{-3}\ln r + b_{23}r^{-5} \\ &\quad + b_{24}r^{-7} + b_{25}r^{-9} + b_{26}r^{-11} + b_{27}r^{-13})\Big)\cos 3\theta \\ &\quad + \Big(c_9 r^{-5} + \frac{1}{\varepsilon_m}\,(b_{28}r^{-1} + b_{29}r^{-3} + b_{30}r^{-5}\ln r + b_{31}r^{-7} \\ &\quad + b_{32}r^{-9} + b_{33}r^{-11} + b_{34}r^{-13})\Big)\cos 5\theta \\ &\quad + \Big(c_{10}r^{-7} + \frac{1}{\varepsilon_m}\,(b_{35}r^{-1} + b_{36}r^{-3} + b_{37}r^{-5})\Big)\cos 7\theta\Big]E_0^7, \qquad (3.117)\end{aligned}$$

$$\phi_3^i(r,\theta) = -\Big[c_{11}r\cos\theta + c_{12}r^3\cos 3\theta + c_{13}r^5\cos 5\theta + c_{14}r^7\cos 7\theta\Big]E_0^7, \qquad (3.118)$$

where

$$\begin{aligned} c_7 &= \frac{1}{\varepsilon}\,(E_1 a\varepsilon_i + F_1 a^2 - H_1 a^2), & (3.119)\\ c_8 &= \frac{1}{\varepsilon}\,(3E_3 a^3\varepsilon_i + F_3 a^4 - H_3 a^4), & (3.120)\\ c_9 &= \frac{1}{\varepsilon}\,(5E_5 a^5\varepsilon_i + F_5 a^6 - H_5 a^6), & (3.121)\\ c_{10} &= \frac{1}{\varepsilon}\,(7E_7 a^7\varepsilon_i + F_7 a^8 - H_7 a^8), & (3.122)\\ c_{11} &= \frac{a^{-1}}{\varepsilon}\,(-E_1\varepsilon_m + F_1 a - H_1 a), & (3.123)\\ c_{12} &= \frac{a^{-3}}{\varepsilon}\,(-3E_3\varepsilon_m + F_3 a - H_3 a), & (3.124)\\ c_{13} &= \frac{a^{-5}}{\varepsilon}\,(-5E_5\varepsilon_m + F_5 a - H_5 a), & (3.125)\\ c_{14} &= \frac{a^{-7}}{\varepsilon}\,(-7E_7\varepsilon_m + F_7 a - H_7 a), & (3.126)\end{aligned}$$

and

$$E_1 = -\frac{1}{\varepsilon_m}(b_{15}a^{-3} + b_{16}a^{-5} + b_{17}a^{-7} + b_{18}a^{-9} + b_{19}a^{-11} + b_{20}a^{-13}), \tag{3.127}$$

$$E_3 = -\frac{1}{\varepsilon_m}(b_{21}a^{-1} + (b_{22}a^{-3}\ln a + b_{23}a^{-5} + b_{24}a^{-7} + b_{25}a^{-9} + b_{26}a^{-11} + b_{27}a^{-13}), \tag{3.128}$$

$$E_5 = -\frac{1}{\varepsilon_m}(b_{28}a^{-1} + b_{29}a^{-3} + b_{30}a^{-5}\ln a + b_{31}r^{-7} + b_{32}a^{-9} + b_{33}a^{-11} + b_{34}a^{-13}), \tag{3.129}$$

$$E_7 = -\frac{1}{\varepsilon_m}(b_{35}a^{-1} + b_{36}a^{-3} + b_{37}a^{-5}), \tag{3.130}$$

$$F_1 = -\frac{3b_{15}}{a^4} - \frac{5b_{16}}{a^6} - \frac{7b_{17}}{a^8} - \frac{9b_{18}}{a^{10}} - \frac{11b_{19}}{a^{12}} - \frac{13b_{20}}{a^{14}}, \tag{3.131}$$

$$F_3 = -\frac{b_{21}}{a^2} + \frac{b_{22}}{a^4} - \frac{3b_{22}\ln a}{a^4} - \frac{5b_{23}}{a^6} - \frac{7b_{24}}{a^8} - \frac{9b_{25}}{a^{10}} - \frac{11b_{26}}{a^{12}} - \frac{13b_{27}}{a^{14}}, \tag{3.132}$$

$$F_5 = -\frac{b_{28}}{a^2} - \frac{3b_{29}}{a^4} + \frac{b_{30}}{a^6} - \frac{5b_{30}\ln a}{a^6} - \frac{7b_{31}}{a^8} - \frac{9b_{32}}{a^{10}} - \frac{11b_{33}}{a^{12}} - \frac{13b_{34}}{a^{14}}, \tag{3.133}$$

$$F_7 = -\frac{b_{35}}{a^2} - \frac{3b_{36}}{a^4} - \frac{5b_{37}}{a^6}, \tag{3.134}$$

$$\begin{aligned} H_1 = {} & \frac{1}{a^2}\Big(2c_1 - \frac{b_9}{\varepsilon_m}\Big) + \frac{1}{a^4}\Big(-2b_1^2 - 4bc_1 - \frac{3b^2}{2\varepsilon_m^2} - \frac{bb_1}{\varepsilon_m} + \frac{7b_5}{\varepsilon_m} - \frac{2bb_9}{\varepsilon_m}\Big) \\ & \frac{1}{a^6}\Big(3bb_1^2 - 6b_1b_2 + 3b^2c_1 - 6bc_2 + \frac{b^3}{2\varepsilon_m^2} - \frac{10b^2b_1}{\varepsilon_m} - \frac{9bb_2}{\varepsilon_m} - \frac{12bb_5}{\varepsilon_m} \\ & + \frac{12b_6}{\varepsilon_m} + \frac{2b^2b_9}{\varepsilon_m} + \frac{b_{10}}{\varepsilon_m}\Big) + \frac{1}{a^8}\Big(6bb_1b_2 - 18b_2^2 + 3b^2c_2 - \frac{83b^4}{6\varepsilon_m^2} \\ & + \frac{58b^3b_1}{3\varepsilon_m} - \frac{3b^2b_2}{\varepsilon_m} + \frac{8b^2b_5}{\varepsilon_m} - \frac{20bb_6}{\varepsilon_m} + \frac{17b_7}{\varepsilon_m} - \frac{10bb_{10}}{\varepsilon_m} + \frac{2b_{11}}{\varepsilon_m}\Big) \\ & + \frac{1}{a^{10}}\Big(18bb_2^2 + \frac{94b^5}{3\varepsilon_m^2} - \frac{13b^4b_1}{3\varepsilon_m} + \frac{21b^3b_2}{\varepsilon_m} + \frac{13b^2b_6}{\varepsilon_m} - \frac{28bb_7}{\varepsilon_m} + \frac{22b_8}{\varepsilon_m} \\ & + \frac{4b^2b_{10}}{\varepsilon_m} - \frac{14bb_{11}}{\varepsilon_m} + \frac{3b_{12}}{\varepsilon_m}\Big) + \frac{1}{a^{12}}\Big(-\frac{27b^6}{2\varepsilon_m^2} - \frac{5b^4b_2}{\varepsilon_m} + \frac{18b^2b_7}{\varepsilon_m} \\ & - \frac{36bb_8}{\varepsilon_m} + \frac{5b^2b_{11}}{\varepsilon_m} - \frac{18bb_{12}}{\varepsilon_m}\Big) + \frac{1}{a^{14}}\Big(\frac{59b^7}{36\varepsilon_m^2} + \frac{23b^2b_8}{\varepsilon_m} + \frac{6b^2b_{12}}{\varepsilon_m}\Big), \end{aligned} \tag{3.135}$$

$$
\begin{aligned}
H_3 &= \frac{1}{a^2}\Big(c_1+\frac{2b_9}{\varepsilon_m}-\frac{2b_{13}}{\varepsilon_m}\Big)+\frac{1}{a^4}\Big(-b_1^2-2bc_1+6c_2-\frac{bb_1}{\varepsilon_m}+\frac{2b_5}{\varepsilon_m} \\
&\quad -\frac{2bb_9}{\varepsilon_m}-\frac{2bb_{13}}{\varepsilon_m}-\frac{b_{14}}{\varepsilon_m}\Big)+\frac{1}{a^6}\Big(-6b_1b_2-6bc_2-\frac{4b^3}{\varepsilon_m^2}-\frac{4b^2b_1}{\varepsilon_m} \\
&\quad -\frac{6bb_5}{\varepsilon_m}+\frac{3b_6}{\varepsilon_m}+\frac{2b^2b_9}{\varepsilon_m}+\frac{10b_{10}}{\varepsilon_m}+\frac{3b^2b_{13}}{\varepsilon_m}-\frac{6bb_{14}}{\varepsilon_m}\Big)+\frac{1}{a^8}\Big(12bb_1b_2 \\
&\quad +6b^2c_2-10bc_3-\frac{11b^4}{6\varepsilon_m^2}+\frac{11b^3b_1}{3\varepsilon_m}-\frac{24b^2b_2}{\varepsilon_m}+\frac{b^2b_5}{\varepsilon_m}-\frac{10bb_6}{\varepsilon_m}+\frac{4b_7}{\varepsilon_m} \\
&\quad -\frac{10bb_{10}}{\varepsilon_m}+\frac{14b_{11}}{\varepsilon_m}+\frac{4b^2b_{14}}{\varepsilon_m}\Big)+\frac{1}{a^{10}}\Big(5b^2c_3+\frac{8b^5}{\varepsilon_m^2}-\frac{2b^4b_1}{3\varepsilon_m}+\frac{37b^3b_2}{\varepsilon_m} \\
&\quad +\frac{2b^2b_6}{\varepsilon_m}-\frac{14bb_7}{\varepsilon_m}+\frac{5b_8}{\varepsilon_m}+\frac{10b^2b_{10}}{\varepsilon_m}-\frac{14bb_{11}}{\varepsilon_m}+\frac{18b_{12}}{\varepsilon_m}\Big)+ \\
&\quad \frac{1}{a^{12}}\Big(-\frac{43b^6}{12\varepsilon_m^2}-\frac{8b^4b_2}{\varepsilon_m}+\frac{3b^2b_7}{\varepsilon_m}-\frac{18bb_8}{\varepsilon_m}+\frac{14b^2b_{11}}{\varepsilon_m}-\frac{18bb_{12}}{\varepsilon_m}\Big) \\
&\quad +\frac{1}{a^{14}}\Big(\frac{4b^7}{9\varepsilon_m^2}+\frac{4b^2b_8}{\varepsilon_m}+\frac{18b^2b_{12}}{\varepsilon_m}\Big), \qquad (3.136)
\end{aligned}
$$

$$
\begin{aligned}
H_5 &= \frac{1}{a^2}\Big(\frac{2b_9}{\varepsilon_m}+\frac{2b_{13}}{\varepsilon_m}\Big)+\frac{1}{a^4}\Big(3c_2+\frac{3b^2}{4\varepsilon_m^2}-\frac{bb_1}{\varepsilon_m}-\frac{2bb_9}{\varepsilon_m}-\frac{2bb_{13}}{\varepsilon_m}+\frac{6b_{14}}{\varepsilon_m}\Big) \\
&\quad +\frac{1}{a^6}\Big(-6b_1b_2-6bc_2+10c_3-\frac{4b^3}{\varepsilon_m^2}-\frac{b^2b_1}{\varepsilon_m}+\frac{3bb_2}{\varepsilon_m}-\frac{b^2b_9}{\varepsilon_m}+\frac{4b_{10}}{\varepsilon_m} \\
&\quad +\frac{2b^2b_{13}}{\varepsilon_m}-\frac{6bb_{14}}{\varepsilon_m}\Big)+\frac{1}{a^8}\Big(-10bc_3+\frac{b^4}{6\varepsilon_m^2}-\frac{15b^2b_2}{\varepsilon_m}-\frac{10bb_{10}}{\varepsilon_m}+\frac{5b_{11}}{\varepsilon_m} \\
&\quad +\frac{6b^2b_{14}}{\varepsilon_m}\Big)+\frac{1}{a^{10}}\Big(9bb_2^2+10b^2c_3+\frac{b^5}{6\varepsilon_m^2}+\frac{11b^3b_2}{\varepsilon_m}+\frac{b^2b_{10}}{\varepsilon_m}-\frac{14bb_{11}}{\varepsilon_m} \\
&\quad +\frac{6b_{12}}{\varepsilon_m}\Big)+\frac{1}{a^{12}}\Big(-\frac{2b^4b_2}{\varepsilon_m}+\frac{2b^2b_{11}}{\varepsilon_m}-\frac{18bb_{12}}{\varepsilon_m}\Big)+\frac{3b^2b_{12}}{a^{14}\varepsilon_m}, \qquad (3.137)
\end{aligned}
$$

$$
\begin{aligned}
H_7 &= \frac{3b_{13}}{a^2\varepsilon_m}+\frac{1}{a^4}\Big(-\frac{2bb_{13}}{\varepsilon_m}+\frac{4b_{14}}{\varepsilon_m}\Big)+\frac{1}{a^6}\Big(5c_3-\frac{3b^3}{4\varepsilon_m^2}-\frac{3bb_2}{\varepsilon_m}-\frac{2b^2b_{13}}{\varepsilon_m} \\
&\quad -\frac{6bb_{14}}{\varepsilon_m}\Big)+\frac{1}{a^8}\Big(-9b_2^2-10bc_3-\frac{3b^2b_2}{\varepsilon_m}-\frac{b^2b_{14}}{\varepsilon_m}\Big), \qquad (3.138)
\end{aligned}
$$

which ε_i and ε_m are in unit of $\beta_m = \chi_m/\lambda$ and $\varepsilon = \varepsilon_i + \varepsilon_m$. We note that $\lambda^3\phi_3^m$ depends on χ_m^3 and is independent of λ.

In order to confirm our solutions, $\phi_3^m(r,\theta)$ and $\phi_3^i(r,\theta)$, we first substitute $\phi_3^m(r,\theta)$ back to the third-order potential equation for the host medium in Eq. (3.23). We find that the quantities of both sides of Eq. (3.23) are the same and the boundary condition is also confirmed. For the inclusion region, we obtain the same result by substituting $\phi_3^i(r,\theta)$ into Eq. (3.19). Then our new results, $\phi_3^m(r,\theta)$ and $\phi_3^i(r,\theta)$, are reliable. We learn through the calculations that higher-order terms of the electric potential can be solved by the same procedure and analytic solutions for them are all available. There will be no problems if we use mathematica program help to solve higher-order terms of the electric potential. We will use these results of the electric potential to calculate the effective nonlinear dielectric coefficients of nonlinear composites and the details will be shown in chapter 5.