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ROBUST CONSTRAINED MODEL PREDICTIVE CONTROL

WITH APPLICATIONS TO CHEMICAL PROCESSES

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A Dissertation Submitted in Partial Fulfillment of the Requirements

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Department of Chemical Engineering

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วิทยานิพนธ์เล่มนี้เสนอสามวิธีสำหรับการสังเคราะห์การควบคุมเชิงทำนายแบบจำลองที่มีเงื่อนไขบังคับคงทนเชิงออฟไลน์ ในวิธีแรกได้เสนอการสังเคราะห์เชิงออฟไลน์ของการควบคุมเชิงทำนายแบบจำลองที่มีเงื่อนไขบังคับคงทนโดยการใช้เซตยืนยงทรงโพลีฮีดรอล ภาระการคำนวณเชิงออนไลน์จะลดลงโดยการคำนวณเชิงออฟไลน์ลำดับของอัตราขยายป้อนกลับสถานะซึ่งสอดคล้องกับลำดับของเซตยืนยงทรงโพลีฮีดรอล ในแต่ละเวลาการสุ่มจะหาเซตยืนยงทรงโพลีฮีดรอลที่เล็กที่สุดที่สามารถบรรจุสถานะปัจจุบันที่วัดได้ และอัตราขยายป้อนกลับสถานะซึ่งสอดคล้องกันกับเซตยืนยงทรงโพลีฮีดรอลนี้จะถูกนำไปใช้กับกระบวนการ เมื่อเปรียบเทียบกับการสังเคราะห์เชิงออฟไลน์ของการควบคุมเชิงทำนายแบบจำลองที่มีเงื่อนไขบังคับคงทนโดยการใช้เซตยืนยงทรงรี วิธีการที่เสนอจะให้สมรรถนะการควบคุมที่ดีกว่า นอกจากนั้นยังให้บริเวณที่สามารถทำให้ระบบมีเสถียรภาพได้ที่ใหญ่กว่าอย่างมีนัยสำคัญ ในวิธีที่สองจะเสนอวิธีการควบคุมเชิงทำนายแบบจำลองที่มีพื้นฐานจากการประมาณค่าในช่วง ภาระการคำนวณเชิงออนไลน์จะลดลงโดยการคำนวณเชิงออฟไลน์ลำดับของอัตราขยายป้อนกลับสถานะซึ่งสอดคล้องกับลำดับของเซตยืนยงทรงรี อัตราขยายป้อนกลับสถานะเวลาจริงจะคำนวณโดยการประมาณค่าในช่วงเชิงเส้นระหว่างอัตราขยายป้อนกลับสถานะที่ถูกคำนวณไว้ก่อนหน้า เมื่อเปรียบเทียบกับวิธีการควบคุมเชิงทำนายแบบจำลองเชิงออฟไลน์ที่ไม่มีการประมาณค่าในช่วงระหว่างอัตราขยายป้อนกลับสถานะ วิธีการที่เสนอให้สมรรถนะการควบคุมที่ดีกว่า นอกจากนั้น วิธีการที่เสนอให้สมรรถนะการควบคุมเหมือนกับการควบคุมเชิงทำนายแบบจำลองเชิงออนไลน์ในขณะที่เวลาการคำนวณเชิงออนไลน์จะลดลงอย่างมีนัยสำคัญ ในวิธีสุดท้ายจะเสนอวิธีการลดความอนุรักษ์ที่มีพื้นฐานจากการทำนายสถานะหนึ่งขั้น ความอนุรักษ์ที่เกิดจากการกำหนดอัตราขยายป้อนกลับสถานะเพียงอย่างเดียวบนสัญญาณขาเข้าจะลดลงโดยการเพิ่มสัญญาณขาเข้าอิสระหนึ่งตัวซึ่งในแต่ละเวลาการสุ่ม จะมีเพียงการแก้ปัญหาออพติไมเซชันซึ่งต้องการเวลาคำนวณต่ำเท่านั้นที่จำเป็นต้องหาคำตอบออนไลน์

ภาควิชา วิศวกรรมเคมี ลายมือชื่อนิสิต ....

สาขาวิชา วิศวกรรมเคมี ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก ....

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KEYWORDS : OFF-LINE MODEL PREDICTIVE CONTROL / POLYHEDRAL INVARIANT SETS / INTERPOLATION / A ONE-STEP STATE PREDICTION STRATEGY

PORNCHAI BUMROONGSRI : ROBUST CONSTRAINED MODEL PREDICTIVE CONTROL WITH APPLICATIONS TO CHEMICAL PROCESSES. ADVISOR : ASST. PROF. SOORATHEP KHEAWHOM, Ph.D., 96 pp.

This research proposes three strategies for off-line robust model predictive control synthesis. In the first strategy, an off-line formulation of robust MPC using polyhedral invariant sets is proposed. The on-line computational burdens are reduced by computing off-line a sequence of state feedback gains corresponding to a sequence of polyhedral invariant sets. At each sampling time, the smallest polyhedral invariant set containing the current state measured is determined, and the corresponding state feedback gain is then implemented to the process. As compared with an off-line formulation of robust model predictive control using ellipsoidal invariant sets, the proposed strategy can achieve better control performance. Moreover, a significantly larger stabilizable region is obtained. In the second strategy, an interpolation-based model predictive control strategy is proposed. The on-line computational burdens are reduced by computing off-line the sequences of state feedback gains corresponding to the sequences of ellipsoidal invariant sets. The real-time state feedback gain is calculated by linear interpolation between the precomputed state feedback gains. As compared with an off-line model predictive control strategy with no interpolation between state feedback gains, the proposed strategy can achieve better control performance. Moreover, the proposed strategy gives the same control performance as compared with on-line model predictive control while the on-line computational time is significantly reduced. In the last strategy, an approach to reduce the conservativeness based on a one-step state prediction is proposed. The conservativeness arising from imposing only a state feedback gain on an input is reduced by adding an element of free input. At each sampling time, only a computationally low-demanding optimization problem is needed to be solved on-line.

Department : Chemical Engineering Student's Signature

Field of Study : Chemical Engineering Advisor's Signature

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Notation	Description
A^T	Transpose of A
A^{-1}	Inverse of A
$A(i,:)$	The row vector consisting of all elements in row i of A
$A > B$	A and B are square symmetric and $A - B$ is positive definite.
I	The identity matrix
$x(k/k)$	The state measured at real time k
$x(k+i/k)$	The state at prediction time $k+i$ predicted at real time k
$*$	The corresponding transpose of the lower block part of symmetric matrices
$V(i,k)$	The Lyapunov function $V(i,k) = x(k+i/k)^T P(i,k) x(k+i/k)$ where $\forall k, \forall i \geq 0,\ P(i,k) > 0$
Ω	The polytope
Co	The convex hull

CHAPTER I

INTRODUCTION

1.1 Backgrounds of This Research

Model predictive control (MPC) is an advance multivariable control algorithm that is popularly used in many chemical processes. At each sampling time, MPC explicitly uses a process model to predict the future plant behavior. Both input and output constraints are explicitly incorporated in formulating the optimization problem. Although an optimal input sequence is calculted, only the first element of input sequence is implemented to the process. At the next sampling time, this procedure is repeated by using a new measurement obtained from the process. The idea of MPC is shown in Fig. 1.1.

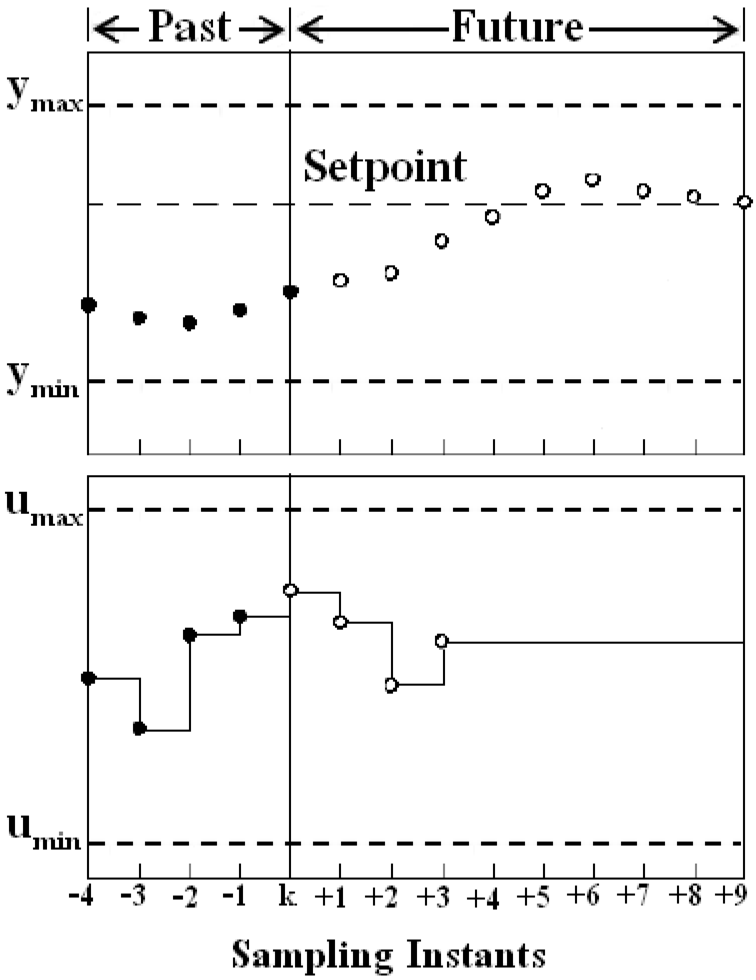


Figure 1.1 The idea of model predictive control.

Since MPC is based on an open-loop optimization problem, in the presence of plant uncertainty, robust stability of the closed-loop system cannot be guaranteed. Moreover, the control performance of MPC will deteriorate as the discrepency between the real plant and the process model used in the prediction increases. For this reason, robust MPC has been widely studied by many researchers.

One of the main approaches to guarantee robust stability in the presence of plant uncertainty is to impose the state feedback control law on the control input. The size of the stabilizable region is important because it provides a set of states that can be robustly stabilized. However, most of the current researches in the area of robust MPC (Kothare et al., 1996; Wan and Kothare, 2003; Ding et al., 2007) still calculate the stabilizable region based on an ellipsoidal approximation of a true stabilizable region. This leads to the conservative result because the stabilizable region obtained is significantly smaller than the true stabilizable region. The idea of using an ellipsoidal approximation of a true stabilizable region is illustrated in Fig. 1.2.

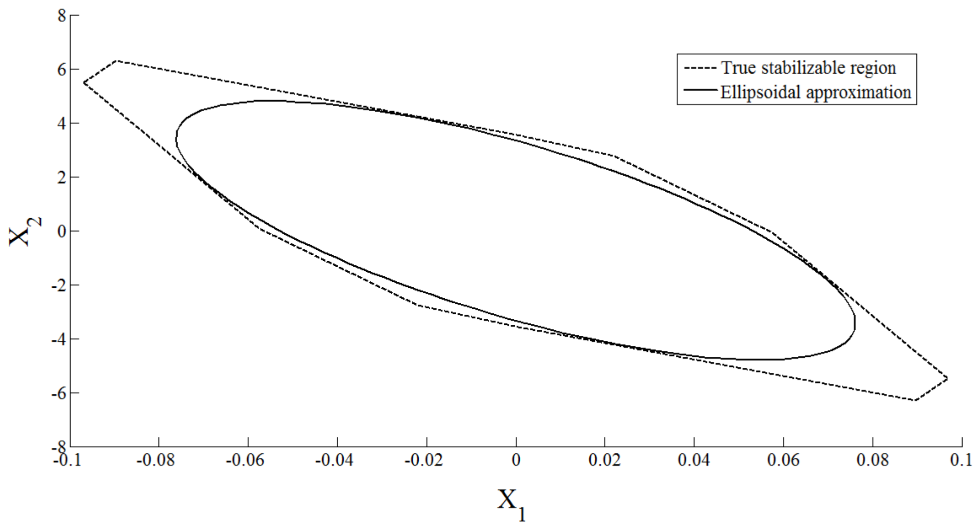


Figure 1.2 The idea of using an ellipsoidal approximation of a true stabilizable region.

Another important problem in an implementation of robust MPC is an on-line computational complexity. Although the significant advances of modern computers over the past few years have alleviated the computational problem of robust MPC, the application of robust MPC is rather restricted due to its on-line computational requirements. Moreover, the size of the optimization problem grows respectively with the number of independent uncertain process parameters (Kothare, 1996; Lu and Arkun, 2000; Wada, 2006).

The conservativeness is also an important problem of robust MPC. In order to guarantee robust stability, the state feedback control law has to be imposed on the control input. However, by doing so, the conservativeness is obtained because the control input only depends on the evolution of state. This problem is especially severe in the presence of tight constraints because the saturation at one point in the horizon as shown in Fig. 1.3 will require a small or zero gain for all steps in the horizon (Li and Marlin, 2011).

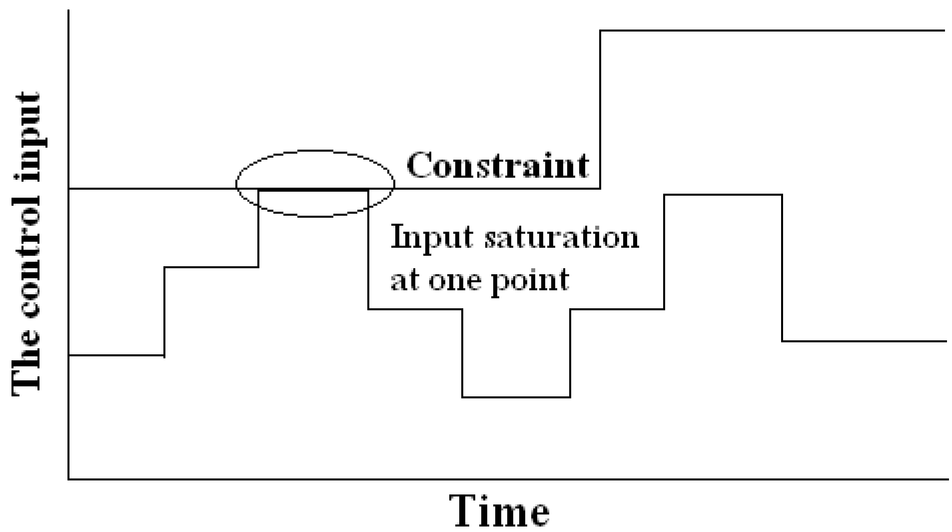


Figure 1.3 The input saturation at one point in the horizon.

As mentioned in the previous section, there are three important issues concerned in robust MPC synthesis including the size of stabilizable region, the on-line computational complexity and the conservativeness. In this research, the strategies to tackle these issues will be proposed.

In chapter 3, an off-line formulation of robust MPC using polyhedral invariant sets is proposed in order to solve the problem of the size of stabilizable region. A sequence of state feedback gains corresponding to a sequence of polyhedral invariant sets is precomputed off-line. At each sampling time, the smallest polyhedral invariant set containing the measured state is determined and the corresponding state feedback gain is then implemented to the process. As compared with an off-line formulation of robust MPC using ellipsoidal invariant sets of Wan and

Kothare (2003), the proposed algorithm gives a significantly larger stabilizable region because the true polyhedral invariant set is calculated. Moreover, the proposed algorithm can achieve better control performance. The proposed algorithm also solves the problem of on-line computational complexity because all of the optimization problems are solved off-line and no optimization problem is needed to be solved on-line.

In chapter 4, an interpolation-based MPC strategy for LPV systems is proposed to alleviate the problem of on-line computational complexity. The on-line computational burdens are reduced by precomputing off-line the sequences of state feedback gains corresponding to the sequences of nested ellipsoids. At each sampling instant, the real-time state feedback gain is calculated by linear interpolation between the state feedback gains of the smallest ellipsoid containing the measured state in each sequence. As compared with an on-line MPC algorithm for LPV systems of Lu and Arkun (2000), the proposed algorithm gives the same control performance with a significantly smaller on-line computational time.

In chapter 5, a strategy to reduce the conservativeness based on a one-step state prediction is presented. The conservativeness arising from imposing only a state feedback gain on the control input in chapters 3 and 4 is reduced by an addition of an element of free control input. At each sampling instant, only a computationally low-demanding optimization problem is needed to be solved on-line so the on-line computation is tractable. By using the proposed strategy, the control performance is improved because the number of degrees of freedom in adjusting the plant is increased.

All of the numerical simulations have been performed in Intel Core i-5 (2.4GHz), 2 GB RAM, using SeDuMi (Sturm, 1999) and YALMIP (Löfberg, 2004) within Matlab R2008a environment.

1.2 Objectives of This Research

The objectives of this research are as follows

1. To develop a robust MPC synthesis approach which gives a significantly larger stabilizable rigion as compared with a robust MPC synthesis approach based on an ellipsoidal invariant set.
2. To develop a robust MPC synthesis approach which can reduce high on-line computational requirement while still ensuring the same level of control performance.
3. To develop a robust MPC synthesis approach which can reduce the conservativeness arising from imposing only the state feedback control law on the control input.

CHAPTER II

BASIC KNOWLEDGES

Model predictive control (MPC) has originated in a chemical industry as an on-line computer control algorithm to solve multivariable problem. At each sampling time, an open-loop constrained optimization problem is solved and only the first computed input is implemented to the process. Although MPC is successfully applied to many chemical processes, there is no guarantee for robust stability of the control system. Moreover, the performace of MPC drastically deteriorates in the presence of plant uncertainty.

Due to the aforementioned problem, the attentions for MPC have been shifted towards robust MPC where the open-loop optimization is replaced by the closed-loop optimization (the deterministic control input is replaced by the state feedback control law). Then robust stability of the closed-loop system can be guaranteed by imposing the Lyapunov stability constraint.

In this chapter, some of the important basic knowledges in the design of robust MPC and several examples are presented. A problem description is presented in 2.1. The Lyapunov theorem and several illustrative examples are presented in 2.2. Then the classical robust MPC algorithm and its applications are presented in 2.3.

2.1 Problem Description

The model considered here is the following linear discrete-time systems with polytopic uncertainty

$$\begin{aligned} x(k+1) &= A(p(k))x(k) + B(p(k))u(k) \\ y(k) &= Cx(k) \end{aligned} \tag{2.1}$$

where $x(k)$ is the state of the plant, $u(k)$ is the control input and $y(k)$ is the plant output. We assume that

$$[A(p(k)), B(p(k))] \in \Omega, \Omega = Co\{[A_1, B_1], [A_2, B_2], \ldots, [A_L, B_L]\} \tag{2.2}$$

where Ω is the polytope, Co denotes convex hull, $[A_j, B_j]$ are vertices of the convex hull.

Any $[A(p(k)), B(p(k))]$ within the polytope Ω is a linear combination of the vertices such that

$$[A(p(k)), B(p(k))] = \sum_{j=1}^{L} p_j(k)[A_j, B_j], \sum_{j=1}^{L} p_j(k) = 1, 0 \le p_j(k) \le 1 \quad (2.3)$$

The aim is to find a state feedback control law

$$u(k+i/k) = g(x(k+i/k)) \quad (2.4)$$

that stabilizes (2.1) and achieves the following performance cost

$$\min_{u(k+i/k)} \max_{[A(p(k+i)),B(p(k+i))]\in\Omega} J_\infty(k)$$
$$J_\infty(k) = \sum_{i=0}^{\infty} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}^T \begin{bmatrix} \Theta & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix} \quad (2.5)$$

where $\Theta > 0$ and $R > 0$ are symmetric weighting matrices

, subject to input and output constraints

$$|u_h(k+i/k)| \le u_{h,\max}, h = 1,2,3,....,n_u \quad (2.6)$$

$$|y_r(k+i/k)| \le y_{r,\max}, r = 1,2,3,....,n_y \quad (2.7)$$

2.2 Lyapunov Theorem

Lyapunov theorem is an important basic theorem that gives the sufficient conditions to determine the stability of the considered linear discrete-time systems. Additionally, it is used in the formulation of the Lyapunov stability constraint to guarantee robust stability of the closed-loop system.

Consider the linear discrete-time system $x(k+1) = Ax(k)$, this system is said to be asymptotically stable if there exists a positive definite function $V(i,k) = x(k+i/k)^T Px(k+i/k) > 0$, $i \ge 0$ such that $V(i+1,k)\text{-}V(i,k) < 0$ for all non-zero $x(k+i/k)$ and $V(i,k) = 0$ at $x(k+i/k) = 0$.

Note that by substitution of $x(k+i+1/k) = Ax(k+i/k)$, an inequality $x(k+i+1/k)^T Px(k+i+1/k) - x(k+i/k)^T Px(k+i/k) < 0$ is equivalent to $x(k+i/k)^T A^T PAx(k+i/k)$ $-x(k+i/k)^T Px(k+i/k) < 0$. By pre-multiplying by $x(k+i/k)^{-T}$ and post-multiplying by $x(k+i/k)^{-1}$, an inequality $x(k+i/k)^T A^T PAx(k+i/k)\text{-}x(k+i/k)^T Px(k+i/k) < 0$ can be written as $A^T PA - P < 0$.

Example 2.1 Consider the system $x(k+1) = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix} x(k)$. Determine whether this system is asymptotically stable or not ?

Solution

By choosing $P = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, we can see that $A^T PA - P = \begin{bmatrix} -1.94 & 1.03 \\ 1.03 & -1.94 \end{bmatrix}$. The eigen values of $\begin{bmatrix} -1.94 & 1.03 \\ 1.03 & -1.94 \end{bmatrix}$ are -2.97 and -0.91 so $\begin{bmatrix} -1.94 & 1.03 \\ 1.03 & -1.94 \end{bmatrix}$ is negative definite. Then we can conclude that $A^T PA - P < 0$ and the system $x(k+1) = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix} x(k)$ is asymptotically stable by the existence of the Lyapunov function $V(i,k) = x(k+i/k)^T Px(k+i/k)$ where $P = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. Figure 2.1 shows the responses of this system. It can be observed that $\lim_{i\to\infty} x(k+i/k) = 0$. Therefore, the system is asymptotically stable.

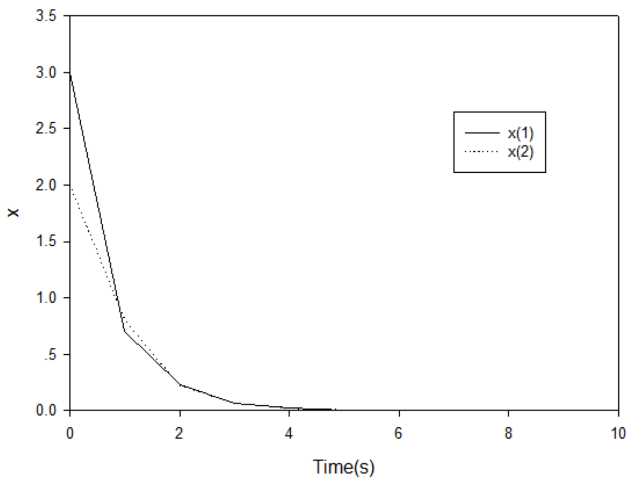


Figure 2.1 The responses of example 2.1.

From an example 2.1, it is seen that Lyapunov theorem can only be applied to the system $x(k+1) = Ax(k)$. In order to make it useful for the system with control input $x(k+1) = Ax(k) + Bu(k)$, we have to apply it with a little modification. By imposing the state feedback control law $u(k) = Kx(k)$, the system $x(k+1) = Ax(k) + Bu(k)$ can be written as $x(k+1) = (A+BK)x(k)$. Thus, it is seen that the system $x(k+1) = Ax(k) + Bu(k)$ is asymptotically stabilized by the control law $u(k) = Kx(k)$ if there exists a positive definite matrix P and a state feedback gain K such that $[A+BK]^T P\,[A+BK] - P < 0.$ The Lyapunov stability constraint $[A+BK]^T P\,[A+BK] - P < 0$ will be used in robust MPC synthesis in the next section.

Example 2.2 Consider the system $x(k+1) = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$. Determine whether this system is asymptotically stabilized by the feedback control law $u(k) = Kx(k)$ where $K = [-1.08 \quad -0.71]$ or not ?

Solution

For $K = [-1.08 \quad -0.71]$, the system $x(k+1) = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$ can be written as $x(k+1) = (A+BK)x(k) = \begin{bmatrix} 1 & 0.1 \\ -1.08 & -0.21 \end{bmatrix} x(k).$ By choosing $P = \begin{bmatrix} 2.575 & 0.1685 \\ 0.1685 & 0.248 \end{bmatrix}$, we can see that $[A+BK]^T P\,[A+BK] - P = \begin{bmatrix} -0.0747 & 0.0917 \\ 0.0917 & -0.2184 \end{bmatrix}$. Its eigen values are -0.26 and -0.03 so $\begin{bmatrix} -0.0747 & 0.0917 \\ 0.0917 & -0.2184 \end{bmatrix}$ is negative definite. Then we conclude that $[A+BK]^T P\,[A+BK] - P < 0.$ and the system $x(k+1) = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$ is asymptotically stabilized by the state feedback control law $u(k) = Kx(k)$ where $K = [-1.08 \quad -0.71]$. Figure 2.2 shows the closed-loop responses of this system. It can be observed that $\lim_{i \to \infty} x(k+i/k) = 0$. Thus, the system is asymptotically stabilized.

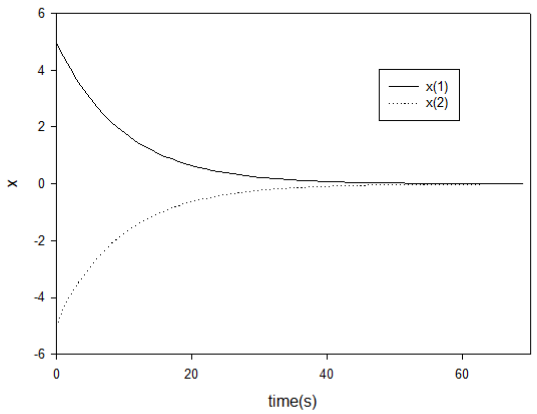


Figure 2.2 The responses of example 2.2.

2.3 The Classical Robust MPC Algorithm (Kothare et al., 1996)

In the preceding sections, the Lyapunov stability constraint is developed. In this section, it will be used in the design of robust MPC. Before proceeding to robust MPC synthesis, an important technique to formulate the efficiently solvable constraints will be presented.

Although we can guarantee robust stability of the closed-loop system by using the Lyapunov stability constraint $[A+BK]^T P [A+BK] - P < 0$, it is seen that the Lypunov stability constraint $[A+BK]^T P [A+BK] - P < 0$ is nonlinear. In the following section, we will introduce the technique to transform nonlinear inequality constraint to linear matrix inequality (LMI). LMI constraint is convex. Thus, it is computationally tractable. For more details, the reader is referred to Boyd et al. (1994).

2.3.1 Schur Complement

The Hermitian matrix $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ is positive definite if and only if $R > 0$ and $Q\text{-}SR^{-1}S^T > 0$.

Proof.

If $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ is positive definite then $\begin{bmatrix} I & -SR^{-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}\begin{bmatrix} I & 0 \\ -R^{-T}S^T & I \end{bmatrix} = \begin{bmatrix} Q-SR^{-1}S^T & 0 \\ 0 & R \end{bmatrix} > 0.$ $\begin{bmatrix} Q-SR^{-1}S^T & 0 \\ 0 & R \end{bmatrix} > 0$ if $R > 0$ and $Q\text{-}SR^{-1}S^T > 0.$

Example 2.3 Transform the Lyapunov stability constraint $[A+BK]^T P\,[A+BK]-P<0$ to LMI.

Solution

By following the Schur Complement, the Lyapunov stability constraint $[A+BK]^T P\,[A+BK]-P<0$ can be transformed to LMI as $\begin{bmatrix} P & [A+BK]^T P \\ P[A+BK] & P \end{bmatrix} > 0.$

2.3.2 Robust MPC Synthesis

Robust MPC synthesis that allows an explicit incorporation of model uncertainty in the problem formulation was first proposed by Kothare et al. (1996). The goal is to design the state feedback control law that minimizes the worst-case performance cost. The optimization problem is formulated as the convex optimization problem involving linear matrix inequalities. At each sampling instant, the state feedback control law that minimizes an upper bound γ on the performance cost $J_\infty(k) = \sum_{i=0}^{\infty} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}^T \begin{bmatrix} \Theta & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}$ and asymptotically stabilizes the system $x(k+1) = A(p(k))x(k) + B(p(k))u(k)$ is given by $u(k+i/k) = Kx(k+i/k),\ K = YQ^{-1}$ where Y and Q are obtained by solving the following optimization problem.

$$\min_{\gamma,Y,Q}\ \gamma \tag{2.8}$$

$$\text{s.t.} \begin{bmatrix} 1 & x(k/k)^T \\ x(k/k) & Q \end{bmatrix} \geq 0 \tag{2.9}$$

$$\begin{bmatrix} Q & QA_j^T + Y^T B_j^T & Q\Theta^{\frac{1}{2}} & Y^T R^{\frac{1}{2}} \\ A_j Q + B_j Y & Q & 0 & 0 \\ \Theta^{\frac{1}{2}} Q & 0 & \gamma I & 0 \\ R^{\frac{1}{2}} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0,\ j = 1,2,..,L \tag{2.10}$$

Proof. The proof will be divided in two steps. In step a), we will prove that γ is the upper bound on the performance cost $J_\infty(k) = \sum_{i=0}^{\infty} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}^T \begin{bmatrix} \Theta & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}$. In step b), we will prove that the state feedback gain $K = YQ^{-1}$ guarantees robust stability to the closed-loop system.

Step a) Suppose a quadratic function $V(i,k) = x(k+i/k)^T Px(k+i/k)$ satisfies the following inequality $V(i+1,k) - V(i,k) \le -(x(k+i/k)^T \Theta x(k+i/k) + u(k+i/k)^T Ru(k+i/k))$. By summing them from $i = 0$ to $i = \infty$, we get

$$\begin{aligned} &\text{-}V(0,k) + V(1,k)\text{-}V(1,k) + V(2,k)\text{-}V(2,k)\ldots \le \\ &-(x(k/k)^T \Theta x(k/k) + u(k/k)^T Ru(k/k)) - (x(k+1/k)^T \Theta x(k+1/k) + u(k+1/k)^T Ru(k+1/k))\ldots \end{aligned} \tag{2.11}$$

Thus, it is easy to see that $\text{-}V(0,k) \le -J_\infty(k)$ or equivalently $\max\ J_\infty(k)\ \le V(0,k)$. Thus, the minimization of cost function $J_\infty(k) = \sum_{i=0}^{\infty} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}^T \begin{bmatrix} \Theta & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}$ can be written as

$$\min_P \quad V(0,k) = x(k/k)^T Px(k/k) \tag{2.12}$$

By introducing the slack variable γ, $\min_P \quad x(k/k)^T Px(k/k)$ is equivalent to

$$\begin{aligned} &\min_{\gamma,P} \quad \gamma \\ &\text{s.t. } x(k/k)^T Px(k/k) \ \le\ \gamma \end{aligned} \tag{2.13}$$

By defining $P = \gamma Q^{-1}$, (2.13) can be written in the form of LMI as

$$\begin{aligned} &\min_{\gamma,P} \quad \gamma \\ &\text{s.t. } \begin{bmatrix} 1 & x(k/k)^T \\ x(k/k) & Q \end{bmatrix} \ge 0 \end{aligned} \tag{2.14}$$

From the proof in step a), we conclude that $\max\ J_\infty(k)\ \le V(0,k) \le \gamma$. Thus, γ is the upper bound on the performance cost $J_\infty(k)$.

Step b) From $V(i+1,k)-V(i,k)\le -(x(k+i/k)^T\Theta x(k+i/k)+u(k+i/k)^T Ru(k+i/k))$, by substituting $V(i+1,k)=x(k+i+1/k)^T Px(k+i+1/k)$, $V(i,k)=x(k+i/k)^T Px(k+i/k)$ and $u(k+i/k)=Kx(k+i/k)$, we get

$$(A(p(k+i))+B(p(k+i))K)^T P(A(p(k+i))+B(p(k+i))K)-P\le -(\Theta+K^T RK) \qquad (2.15)$$

By substituting $P=\gamma Q^{-1}$, pre-multiplying by Q^T, post-multiplying by Q, substituting $Y=KQ$ and applying Schur complement to the resulting inequality, we obtain

$$\begin{bmatrix} Q & QA(p(k+i))^T+Y^T B(p(k+i))^T & Q\Theta^{\frac{1}{2}} & Y^T R^{\frac{1}{2}} \\ A(p(k+i))Q+B(p(k+i))Y & Q & 0 & 0 \\ \Theta^{\frac{1}{2}}Q & 0 & \gamma I & 0 \\ R^{\frac{1}{2}}Y & 0 & 0 & \gamma I \end{bmatrix} \ge 0 \qquad (2.16)$$

This inequality is affine in $[A(p(k)),B(p(k))]=\sum_{j=1}^{L} p_j(k)[A_j,B_j]$. Thus, it is satisfied for all

$$\begin{bmatrix} Q & QA_j^T+Y^T B_j^T & Q\Theta^{\frac{1}{2}} & Y^T R^{\frac{1}{2}} \\ A_jQ+B_jY & Q & 0 & 0 \\ \Theta^{\frac{1}{2}}Q & 0 & \gamma I & 0 \\ R^{\frac{1}{2}}Y & 0 & 0 & \gamma I \end{bmatrix} \ge 0,\ j=1,2,..,L \qquad (2.17)$$

It is seen that $V(i+1,k)-V(i,k)\le -(x(k+i/k)^T\Theta x(k+i/k)+u(k+i/k)^T Ru(k+i/k))$ is equivalent to (2.17). Thus, $V(i,k)=x(k+i/k)^T Px(k+i/k)$ is a strictly decreasing Lyapunov function and robust stability is quaranteed.

Example 2.4

Consider the system $x(k+1)=\begin{bmatrix} 1 & 0.1 \\ 0 & 1-0.1\alpha \end{bmatrix}x(k)+\begin{bmatrix} 0 \\ 0.0787 \end{bmatrix}u(k)$. At any time *k*, α varies between $0.1\le\alpha\le 10$. Find the state feedback control law $u(k)=Kx(k)$ which robustly stabilizes this system.

Solution

It is seen that $A(k) \in Co\{A_1, A_2\}$ where $A_1 = \begin{bmatrix} 1 & 0.1 \\ 0 & 0.99 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0.1 \\ 0 & 0 \end{bmatrix}$. The state feedback control law $u(k) = Kx(k) = YQ^{-1}x(k) = [-86.22 \ \ -14.91]x(k)$ can be obtained by solving the following optimization problem

$$\min_{Y,Q} \ \gamma$$
s.t.
$$\begin{bmatrix} 1 & x(k/k)^T \\ x(k/k) & Q \end{bmatrix} \geq 0$$
$$\begin{bmatrix} Q & QA_1^T + Y^T B^T & Q\Theta^{\frac{1}{2}} & Y^T R^{\frac{1}{2}} \\ A_1 Q + BY & Q & 0 & 0 \\ \Theta^{\frac{1}{2}} Q & 0 & \gamma I & 0 \\ R^{\frac{1}{2}} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0$$
$$\begin{bmatrix} Q & QA_2^T + Y^T B^T & Q\Theta^{\frac{1}{2}} & Y^T R^{\frac{1}{2}} \\ A_2 Q + BY & Q & 0 & 0 \\ \Theta^{\frac{1}{2}} Q & 0 & \gamma I & 0 \\ R^{\frac{1}{2}} Y & 0 & 0 & \gamma I \end{bmatrix} \geq 0$$

Figure 2.3 shows the closed-loop responses of the system. It is seen that the state feedback control law $u(k) = Kx(k) = [-86.22 \ \ -14.91]x(k)$ robustly stabilize this system.

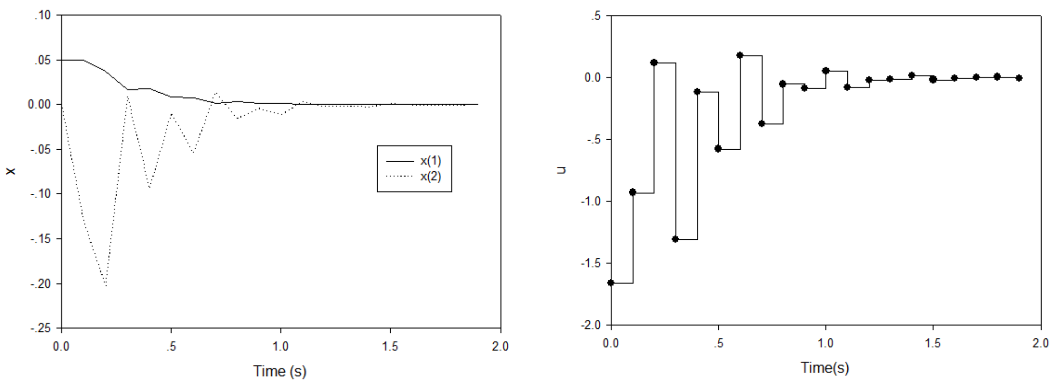


Figure 2.3 The responses of example 2.4(left), The control input(right).

2.3.3 Input and Output Constraints

One of the advantages of MPC is its ability to deal with constraints. In the presence of uncertainty, however, input and output constraints are not guaranteed to be satisfied. In order to guarantee robust constraint satisfaction, input and output constraints have to be explicitly incorporated into robust MPC synthesis. The sufficient conditions to guarantee input and output constraints satisfaction are presented as follows.

Input constraint; The state feedback control law $u(k+i/k) = Kx(k+i/k) = YQ^{-1}x(k+i/k)$ is guaranteed to satisfy $|u_h(k+i/k)| \le u_{h,\max}$ if there exists a symmetric matrix X such that $\begin{bmatrix} X & Y \\ Y^T & Q \end{bmatrix} \ge 0$ with $X_{hh} \le u_{h,\max}^2$, $h = 1,2,\ldots,n_u$.

Output constraints; The output constraint $|y_r(k+i/k)| \le y_{r,\max}$ is guaranteed to be satisfied if there exists a symmetric matrix T such that $\begin{bmatrix} T & C(A_jQ + B_jY) \\ (A_jQ + B_jY)^T C^T & Q \end{bmatrix} \ge 0$ with $T_{rr} \le y_{r,\max}^2$, $r = 1,2,\ldots,n_y$.

Example 2.5 Consider the system in the example 2.4. Find the state feedback control law $u(k) = Kx(k)$ which robustly stabilizes the system $x(k+1) = \begin{bmatrix} 1 & 0.1 \\ 0 & 1-0.1\alpha \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} u(k)$ and satisfies $|u(k+i/k)| \le 1$.

Solution

The state feedback control law $u(k+i/k) = Kx(k+i/k)$ that guarantees input constraint satisfaction can be obtained by solving the optimization problem in Example 2.4 with an incorporation of LMI constraint $\begin{bmatrix} u_{\max}^2 & Y \\ Y^T & Q \end{bmatrix} \ge 0$.

Figure 2.4 shows the closed-loop responses of the system. It is seen that the state feedback control law $u(k) = Kx(k) = [-86.33 \;\; -14.92]x(k)$ asymptotically stabilizes this system. Moreover, the control input is restricted in the range of $|u(k+i/k)| \le 1$.

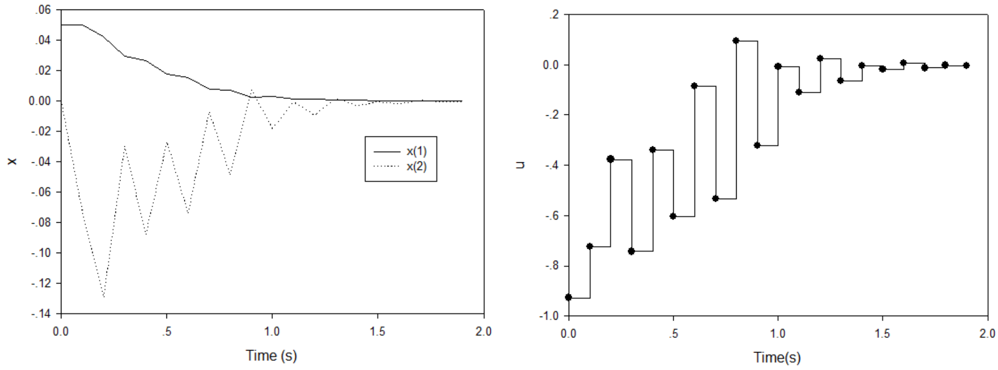


Figure 2.4 The responses of example 2.5(left), The control input(right).

2.4 Conclusions

In this chapter, we have presented some of the important basic knowledges in design of robust MPC. The Lyapunov theorem, which is an important theorem used to guarantee robust stability of the closed-loop systems, is described. Then the classical robust MPC algorithm and its applications are presented.

CHAPTER III

AN OFF-LINE FORMULATION OF ROBUST MPC USING POLYHEDRAL INVARIANT SETS

One of the main approaches to guarantee robust stability in the presence of plant uncertainty is to impose the state feedback gain on the control input as described in chapter 2. The size of stabilizable region of state feedback gain imposed is important because it provides a set of states that can be robustly stabilized. However, most of the current researches in the area of robust MPC (Kothare et al., 1996; Mao, 2003; Wan and Kothare, 2003; Ding et al., 2007) still calculate the stabilizable region based on an ellipsoidal approximation of the true polyhedral invariant set. This leads to the conservative result because a stabilizable region obtained is significantly smaller than the polyhedral counterpart.

This chapter presents an off-line synthesis approach to robust MPC using polyhedral invariant sets. The true polyhedral invariant set is computed so a significantly larger stabilizable region is obtained. Although the construction of polyhedral invariant set is computationally demanding, it is carried out off-line so an on-line computation is tractable. Most of the on-line computational time is reduced by computing off-line a sequence of state feedback control laws corresponding to a sequence of polyhedral invariant sets. At each sampling time, the smallest polyhedral invariant set that the currently measured state can be embedded is determined. Then the corresponding state feedback control law is implemented to the process.

3.1 Introduction

The main technique of on-line robust MPC (Kothare et al., 1996; Schuumans and Rossiter, 2000; Mao, 2003) to guarantee robust stability is to construct at each sampling instant, an ellipsoidal invariant set containing the currently measured state. Then the state feedback control law is imposed on the control input in order to drive the state towards the origin. However, by doing so, the conservative result is obtained because an ellipsoidal invariant set constructed is only an approximation of the true polyhedral invariant set. Moreover, the algorithm requires high on-line computational time bacause the optimization problem is really solved on-line at each sampling instant.

Since the application of on-line robust MPC is limited to only a slow dynamic process, many researchers have studied off-line robust MPC. Wan and Kothare (2003) proposed an off-line formulation of robust MPC using linear matrix inequalities (LMIs). A sequence of explicit control laws corresponding to a sequence of invariant ellipsoids is computed off-line. At each sampling time, the smallest ellipsoid containing the currently measured state is determined and the corresponding control law is implemented to the process. Although the algorithm substantially reduces on-line computational time, the conservative result is obtained due to the fact that the invariant ellipsoids constructed are only the approximations of the true polyhedral invariant sets. Ding et al. (2007) proposed an off-line robust MPC algorithm based on the nominal performance cost. The algorithm directly extends the algorithm of Wan and Kothare (2003) by choosing the nominal performance cost to substitute the worst-case performance cost in order to handle a wider class of systems. However, the algorithm is still designed by using an ellipsoidal approximation of an exact polyhedral invariant set. Thus, a significantly smaller stabilizable region is obtained.

From the preceding review, we can see that on-line robust MPC usually requires high on-line computational time. Thus, its ability is limited to a relatively slow dynamic process. For off-line robust MPC, the ellipsoidal approximations of the exact polyhedral invariant sets are usually used. Thus, a significantly smaller stabilizable region is obtained. In this chapter, an off-line synthesis approach to robust MPC using polyhedral invariant sets is presented. The true polyhedral invariant set is computed so a significantly larger stabilizable region is obtained. Moreover, all of the computational burdens are moved off-line so the on-line computation is tractable.

3.2 An Off-line Formulation of Robust MPC Using Polyhedral Invariant Sets

In this section, an off-line synthesis approach to robust MPC using polyhedral invariant sets is presented. Most of the computational burdens are moved off-line by precomputing a sequence of state feedback control laws corresponding to a sequence of polyhedral invariant sets. The approach to construct the polyhedral invariant set proposed by Pluymers et al. (2005) is adopted here to construct a sequence of polyhedral invariant sets. At each sampling time, the smallest polyhedral invariant set that the currently measured state can be embedded is determined. The corresponding state feedback control law is then implemented to the process. The definition of polyhedral invariant set is given as follows

Definition 3.1

The set $S = \{x / Mx \le d\}$ is said to be the polyhedral invariant set if it has the property that whenever $x(k) \in S$, then $x(k+i) \in S$, $\forall i = 1,2,\ldots,\infty$.

We can now formulate an off-line robust MPC algorithm using polyhedral invariant sets

Algorithm 3.1

Off-line step 1: Choose a sequence of states $x_i, i \in \{1,2,\ldots,N\}$ and solve the following problem to obtain the corresponding state feedback gains $K_i = Y_i Q_i^{-1}$. The states x_i should be chosen such that the distance between x_{i+1} and the origin is less than the distance between x_i and the origin.

$$\min_{\gamma_i, Y_i, Q_i} \gamma_i$$

$$\text{s.t.} \begin{bmatrix} 1 & * \\ x_i & Q_i \end{bmatrix} \ge 0 \tag{3.1}$$

$$\begin{bmatrix} Q_i & * & * & * \\ A_j Q_i + B_j Y_i & Q_i & * & * \\ \Theta^{\frac{1}{2}} Q_i & 0 & \gamma_i I & * \\ R^{\frac{1}{2}} Y_i & 0 & 0 & \gamma_i I \end{bmatrix} \ge 0, \forall j = 1,2,..,L \tag{3.2}$$

$$\begin{bmatrix} X & * \\ Y_i^T & Q_i \end{bmatrix} \ge 0,\ X_{hh} \le u_{h,\max}^2,\ h = 1,2,.,n_u \tag{3.3}$$

$$\begin{bmatrix} S & * \\ (A_j Q_i + B_j Y_i)^T C^T & Q_i \end{bmatrix} \ge 0, \tag{3.4}$$

$$S_{rr} \le y_{r,\max}^2, r = 1,2,..,n_y \forall j = 1,2,..,L$$

Off-line step 2: Given the state feedback gains $K_i = Y_i Q_i^{-1}$, $i \in \{1,2,\ldots,N\}$ from step 1. For each K_i, the corresponding polyhedral invariant sets $S_i = \{x / M_i x \le d_i\}$ is constructed by following these steps:

2.1) Set $M_i = [C^T, -C^T, K_i^T, -K_i^T]^T$, $d_i = [y_{\max}^T, y_{\min}^T, u_{\max}^T, u_{\min}^T]^T$ and $m = 1$.

2.2) Select row m from (M_i, d_i) and check $\forall j$ whether $M_{i,m}(A_j + B_j K_i)x \le d_{i,m}$ is redundant with respect to the constraints defined by (M_i, d_i) by solving the following problem:

$$\begin{aligned}
&\max_x \; W_{i,m,j} \\
&\text{s.t.} W_{i,m,j} = M_{i,m}(A_j + B_j K_i)x - d_{i,m} \\
&\quad M_i x \le d_i
\end{aligned} \tag{3.5}$$

If $W_{i,m,j} > 0$, the constraint $M_{i,m}(A_j + B_j K_i)x \le d_{i,m}$ is non-redundant with respect to (M_i, d_i). Then, add non-redundant constraints to (M_i, d_i) by assigning $M_i = [M_i^T, (M_{i,m}(A_j + B_j K_i))^T]^T$ and $d_i = [d_i^T, d_{i,m}^T]^T$.

2.3) Let $m = m+1$ and return to step 2.2. If m is strictly larger than the number of rows in (M_i, d_i) then terminate.

On-line: At each sampling time, determine the smallest polyhedral invariant set $S_i = \{x / M_i x \le d_i\}$ containing the measured state and implement the corresponding state feedback control law $u(k/k) = K_i x(k/k)$ to the process.

Remarks

1. The sequence of states $x_i, i \in \{1,2,\ldots,N\}$ should be chosen such that the distance between x_{i+1} and the origin is less than the distance between x_i and the origin. This is to ensure that the polyhedral invariant sets constructed are nested $(S_{i+1} \subset S_i)$. Thus, the state is guaranteed to be kept within S_i and driven towards S_{i+1}, and so on. Lastly, the state is kept within S_N and driven towards the origin.

2. The number of polyhedral invariant sets $S_i,\ i \in \{1,2,\ldots,N\}$ constructed by the chosen states $x_i, i \in \{1,2,\ldots,N\}$ affects control performance. Although the state feedback gain K_i is guaranteed to drive all states within S_i towards the origin, it is not necessary to keep this state feedback gain constant. By increasing the number of polyhedral invariant sets (This is done by increasing the number of chosen states N), the control performance is improved due to the fact that we have more freedom to adopt varying state feedback gains based on the distance between the state and the origin.

3. In an off-line step 1, a sequence of state feedback gains corresponding to a sequence of ellipsoidal invariant sets is calculated. Although each state feedback gain calculated guarantees robust stability within the corresponding ellipsoidal invariant set, this ellipsoidal invariant set is only an approximation of the true polyhedral invariant set. By using only an off-line step 1, the conservative result is obtained because the stabilizable region of the ellipsoidal invariant set is significantly smaller than the polyhedral counterpart. This problem is especially severe in the case of tight constraints. Thus, for a given sequence of state feedback gains calculated from an off-line step 1, a sequence of true polyhedral invariant sets is calculated in an off-line step 2. By using an off-line step 2, the conservativeness is reduced because the stabilizable region of each state feedback gain is substantially expanded.

An overall algorithm is proved to guarantee robust stability in Theorem 3.1.

Theorem 3.1 *Given the initial measured state* $x(k) \in S_1$, *the control law provided by algorithm 3.1 assures robust stability to the closed-loop system.*

Proof. The satisfaction of (3.2) for the state feedback gain $K_i = Y_i Q_i^{-1}$ ensures that

$$\begin{aligned}&x(k+i/k)^T \left\{[A(p(k+i)) + B(p(k+i))K_i]^T P_i [A(p(k+i)) + B(p(k+i))K_i] - P_i\right\} x(k+i/k) \\ &\leq -\left\{x(k+i/k)^T \Theta x(k+i/k) + x(k+i/k)^T K_i^T R K_i x(k+i/k)\right\}\end{aligned}$$

Thus, $x(k+i/k)^T P_i x(k+i/k)$ is a strictly decreasing Lyapunov function and the closed-loop system is robustly stabilized by the state feedback gain K_i.

By solving (3.5) and iteratively adding non-redundant constraints $M_{i,m}(A_j+B_jK_i)x \le d_{i,m}$ to (M_i,d_i) by assigning $M_i=[M_i^T,(M_{i,m}(A_j+B_jK_i))^T]^T$ and $d_i=[d_i^T,d_{i,m}^T]^T$, we can find the set of initial states x defined by $S_i=\{x/M_ix\le d_i\}$ such that all future states are guaranteed to stay within this set without input and output constraints violation. Any initial states outside S_i lead to the future states that violate input and output constraints for at least one realization of the uncertainty.

Thus, the set S_i is polyhedral invariant set and the corresponding state feedback control law $u(k+i/k)=K_ix(k+i/k)$ assures robust stability to the closed-loop system. More proof details can be found in Appendix A.

Example 3.1:

In the first example, we will consider an application of our approach to an uncertain non-isothermal CSTR where an exothermic reaction $A \longrightarrow B$ takes place. The reaction is irreversible and the rate of reaction is first order with respect to component A. A cooling coil is used to remove heat that is released in the exothermic reaction. The reaction rate constant k_o and the heat of reaction ΔH_{rxn} are considered to be the uncertain parameters. They are assumed to be arbitrarily time-varying in the indicated range of variation. The linearized model based on the component balance and the energy balance is given as follows

$$\begin{bmatrix}\dot{C}_A\\ \dot{T}\end{bmatrix}=\begin{bmatrix}-\frac{F}{V}-k_oe^{-\frac{E}{RT_{eq}}} & -\frac{E}{RT_{eq}^2}k_oe^{-\frac{E}{RT_{eq}}}C_{A,eq}\\ \frac{-\Delta H_{rxn}k_oe^{-\frac{E}{RT_{eq}}}}{\rho C_p} & -\frac{F}{V}-\frac{UA}{V\rho C_p}-\Delta H_{rxn}\frac{E}{\rho C_pRT_{eq}^2}k_oe^{-\frac{E}{RT_{eq}}}C_{A,eq}\end{bmatrix}\begin{bmatrix}C_A\\ T\end{bmatrix}+\begin{bmatrix}\frac{F}{V} & 0\\ 0 & -2.098x10^5\frac{T_{eq}-365}{V\rho C_p}\end{bmatrix}\begin{bmatrix}C_{A,F}\\ F_C\end{bmatrix}$$

$$y=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}C_A\\ T\end{bmatrix} \tag{3.6}$$

where C_A is the concentration of A in the reactor, $C_{A,F}$ is the feed concentration of A, T is the reactor temperature and F_c is the coolant flow. The operating parameters are shown in table 3.1.

Table 3.1 The operating parameters of non-isothermal CSTR in example 3.1.

Parameter	Value	Unit
F	1	m^3/min
V	1	m^3
ρ	10^6	g/m^3
C_p	1	cal/g.K
$-\Delta H_{rxn}$	10^7-10^8	cal/kmol
E/R	8330.1	K
k_o	10^9-10^{10}	min^{-1}
UA	5.34x10^6	cal/K.min
$C_{A,eq}$	0.265	kmol/m^3
T_{eq}	394	K

Let $\overline{C_A} = C_A - C_{A,eq}$, $\overline{T} = T - T_{eq}$, $\overline{C}_{A,F} = C_{A,F} - C_{A,F,eq}$ and $\overline{F_c} = F_c - F_{c,eq}$ where the subscript eq is used to denote the corresponding variable at equilibrium condition. The discrete-time model (3.7) is obtained by discretizing (3.6) using Euler first-order approximation with a sampling time of 0.15 min.

$$\begin{bmatrix} \overline{C}_A(k+1) \\ \overline{T}(k+1) \end{bmatrix} = \begin{bmatrix} 0.85 - 0.0986\alpha(k) & -0.0014\alpha(k) \\ 0.9864\alpha(k)\beta(k) & 0.0487 + 0.01403\alpha(k)\beta(k) \end{bmatrix} \begin{bmatrix} \overline{C}_A(k) \\ \overline{T}(k) \end{bmatrix} + \begin{bmatrix} 0.15 & 0 \\ 0 & -0.912 \end{bmatrix} \begin{bmatrix} \overline{C}_{A,F}(k) \\ \overline{F_C}(k) \end{bmatrix}$$

$$\overline{y}(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \overline{C}_A(k) \\ \overline{T}(k) \end{bmatrix} \quad (3.7)$$

where $1 \le \alpha(k) = k_o / 10^9 \le 10$ and $1 \le \beta(k) = -\Delta H_{rxn} / 10^7 \le 10$. Because two uncertain parameters $\alpha(k)$ and $\beta(k)$ are independent of each other, we have to consider the polytopic uncertain model with its four vertices representing all the possible combinations of the two uncertain parameters. The polytopic uncertain set is given as follows

$$\Omega = Co\left\{ \begin{bmatrix} 0.751 & -0.0014 \\ 0.986 & 0.063 \end{bmatrix}, \begin{bmatrix} 0.751 & -0.0014 \\ 9.864 & 0.189 \end{bmatrix}, \begin{bmatrix} -0.136 & -0.014 \\ 9.864 & 0.189 \end{bmatrix}, \begin{bmatrix} -0.136 & -0.014 \\ 98.644 & 1.451 \end{bmatrix} \right\} \quad (3.8)$$

The objective is to regulate $\overline{C}_A$ and $\overline{T}$ by manipulating $\overline{C}_{A,F}$ and $\overline{F}_c$, respectively. The input constraints are $\left|\overline{C}_{A,F}\right| \le 0.5\,\text{kmol/m}^3$ and $\left|\overline{F}_c\right| \le 1.5\,\text{m}^3/\text{min}$. Here $J_\infty(k)$ is given by $J_\infty(k) = \sum_{i=0}^{\infty} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}^T \begin{bmatrix} \Theta & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}$ with $\Theta = I$ and $R = 0.1I$.

Next, we will illustrate the step-by-step implementation of the proposed algorithm 3.1. In an off-line step 1, a sequence of states $x_i = \{(0.0525,0.0525),(0.0475,0.0475),(0.0425,0.0425),$ $(0.0375,0.0375),(0.0325,0.0325),(0.0275,0.0275)\}$ is chosen to calculate the corresponding state feedback gains K_i. Note that a sequence of states x_i is chosen such that the distance between x_{i+1} and the origin is less than the distance between x_i and the origin. This is to ensure that the polyhedral invariant sets constructed are nested ($S_{i+1} \subset S_i$). In this example, only six feedback gains $K_i,\ i = 1,\ldots,6$ are computed off-line because the feedback gains K_i are almost constant beyond $i = 6$.

After a sequence of state feedback gains $K_i,\ i = 1,\ldots,6$ is computed in an off-line step 1, the corresponding polyhedral invariant sets $S_i,\ i = 1,\ldots,6$ will be constructed in an off-line step 2. Let us begin with the first feedback gain K_1. In an off-line step 2.1, we first set $M_i = [K_i^T, -K_i^T]^T$, $d_i = [u_{\max}^T, u_{\min}^T]^T$ (There is no output constraint in this example) and $m = 1$. Then, M_1 and d_1 can be written as follows

$$M_1 = [K_1^T, -K_1^T]^T = \begin{bmatrix} -1.34 & -0.01 \\ 24.79 & 0.14 \\ 1.34 & 0.01 \\ -24.79 & -0.14 \end{bmatrix}, \quad d_1 = [u_{\max}^T, u_{\min}^T]^T = \begin{bmatrix} 0.5 \\ 1.5 \\ -0.5 \\ -1.5 \end{bmatrix}$$

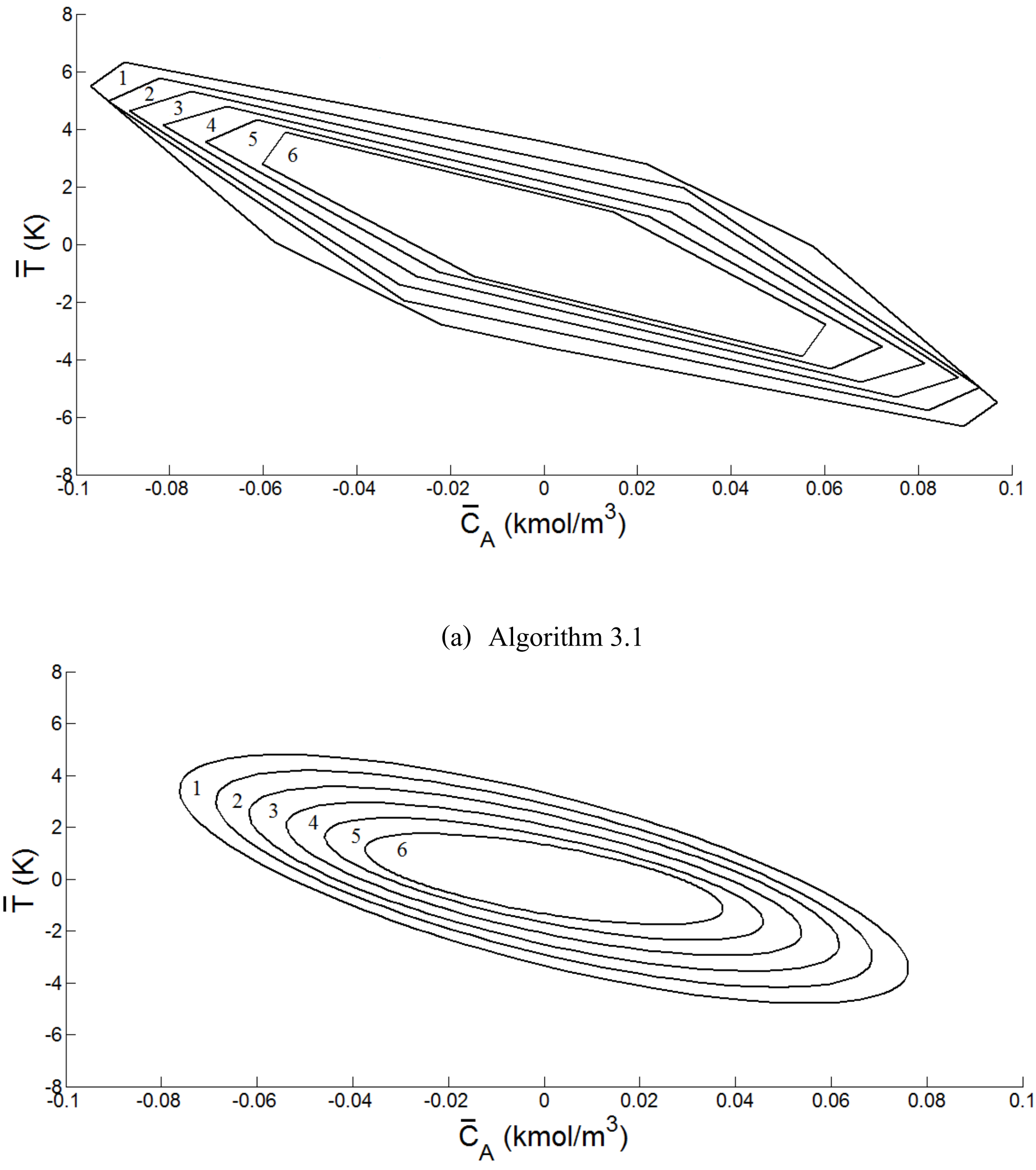
where $K_1 = \begin{bmatrix} -1.34 & -0.01 \\ 24.79 & 0.14 \end{bmatrix}$ corresponding to the chosen state $x_1 = (0.0525,0.0525)$ is calculted from an off-line step 1, $u_{\max} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$ and $u_{\min} = \begin{bmatrix} -0.5 \\ -1.5 \end{bmatrix}$ are the input constraints.

In an off-line step 2.2, we select row m=1 from (M_1, d_1), which is an input constraint $K_1(1,:)x \le u_{\max}(1,1)$, and find whether the constaint $K_1(1,:)(A_1 + B_1K_1)x \le u_{\max}(1,1)$ is redundant with respect to the constraints defined by (M_1, d_1). By setting $W_{1,1,1} = K_1(1,:)(A_1 + B_1K_1)x - u_{\max}(1,1)$ and solving (3.5), we found that $W_{1,1,1} > 0$. Thus, an input constraint $K_1(1,:)(A_1 + B_1K_1)x \le u_{max}(1,1)$ is non-redundant and it has to be included in the construction of polyhedral invariant set S_1 by adding it to the constraints defined by (M_1, d_1). Without this constraint, the state $x(k+1)$ can violate an input constraint $K_1(1,:)x(k+1) \le u_{max}(1,1)$ for at least one realization of the uncertainty. M_1 and d_1 now can be written as $M_1 = [K_1^T, -K_1^T, (K_1(1,:)(A_1 + B_1K_1))^T]^T$ and $d_1 = [u_{max}^T, u_{min}^T, u_{max}(1,1)^T]^T$. Then an off-line step 2.2 continues for $W_{1,1,2}$, $W_{1,1,3}$ and $W_{1,1,4}$, respectively. We found that $W_{1,1,2} < 0$, $W_{1,1,3} > 0$, $W_{1,1,4} < 0$ and hence the constraint $K_1(1,:)(A_3 + B_3K_1)x \le u_{max}(1,1)$ also has to be included in the construction of the polyhedral invariant set S_1.

In an off-line step 2.3, by setting $m = 2, 3$ and 4, (3.5) is repeatly solved to find the set of initial states such that the constraints $K_1(2,:)x(k+1) \le u_{max}(2,1)$, $K_1(1,:)x(k+1) \ge u_{min}(1,1)$ and $K_1(2,:)x(k+1) \ge u_{min}(2,1)$ are guaranteed to be satisfied, respectively. Then the procedure is continued for $m > 4$ until there is no non-redundant constraint. Note that the number of contraints defining the polyhedral invariant set S_1 is finite because the closed-loop system is robustly stabilized by the state feedback gain K_1 calculated from an off-line step 1 (The satisfaction of the Lyapunov stability constraint (3.2) for the state feedback gain K_1 guarantees that the closed-loop system is robustly stabilized.). Finally, in this example, the algorithm terminates at $m = 16$.

The polyhedral invariant sets $S_i,\ i = 2,\ldots,6$ corresponding to the state feedback gains $K_i,\ i = 2,\ldots,6$ can be constructed by following the same procedure as S_1. After the constructions of all S_i are completed, the polyhedral invariant sets $S_i,\ i = 1,\ldots,6$ as shown in Fig. 3.1 (a) are obtained.

A sequence of ellipsoidal invariant sets constructed off-line by an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003) by choosing the same sequence of states $x_i,\ i = 1,\ldots,6$ is shown in Fig. 3.1 (b). It can be observed from the figure that for each chosen state x_i, the stabilizable region of polyhedral invariant set constructed by algorithm 3.1 is significantly larger than the stabilizable region of an ellipsoidal invariant set constructed by an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003). This is due to the fact that an ellipsoidal invariant set constructed by an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003) is only an approximation of the true polyhedral invariant set.



(a) Algorithm 3.1

(b) Wan and Kothare (2003)

Figure 3.1 The comparison between a sequence of polyhedral invariant sets constructed off-line by algorithm 3.1 and a sequence of ellipsoidal invariant sets constructed off-line by an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003) in example 3.1.

The comparison between the stabilizable regions of feedback gains K_1 and K_2 is shown in Fig. 3.2. In this example, K_2 is larger than K_1 because K_2 is computed by using the state which is closer to the origin than K_1. It can be observed from the figure that an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003) cannot stabilize the states at point A. This is due to the fact that the states at point A are not contained in the largest invariant ellipsoid $x_{point\,A} \notin \varepsilon_1$. In comparison, algorithm 3.1 can regulate the states from point A to the origin by using K_1 because the states are contained in the largest polyhedral invariant set $x_{point\,A} \in S_1$. It can also be observed from the figure that if we start at point B, an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003) can regulate the states to the origin by using the lowest feedback gain K_1. In comparison, algorithm 3.1 can regulate the states to the origin by using higher feedback gain K_2 due to the fact that $x_{point\,B} \in S_2$. In this circumstance, algorithm 3.1 can adopt higher feedback gain as compared to an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003). Thus, algorithm 3.1 can achieve less conservative result as compared to an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003).

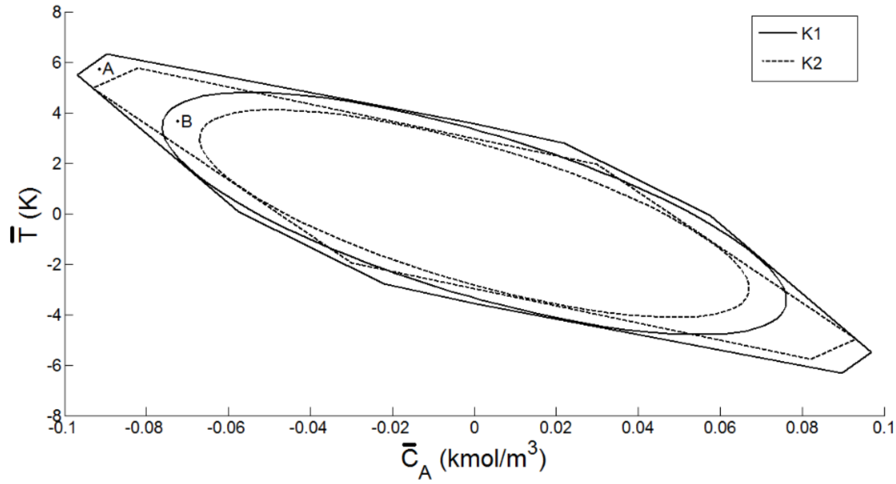
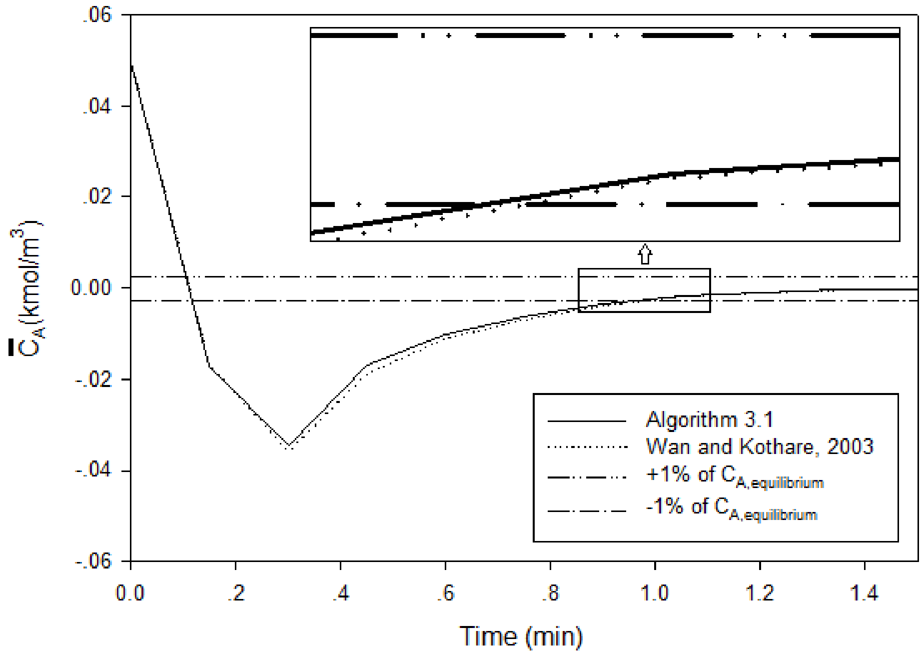
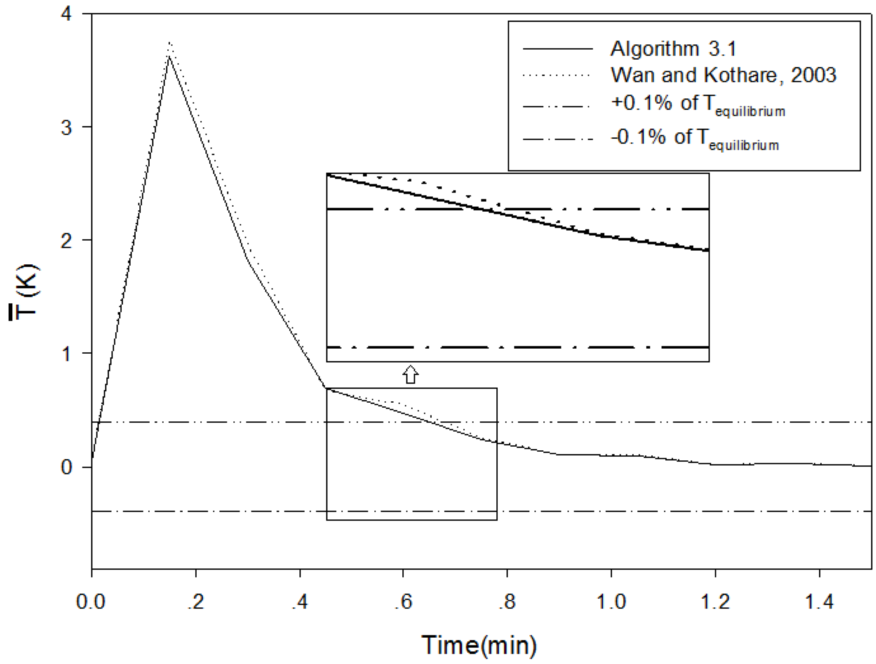


Figure 3.2 The stabilizable regions of feedback gains K_1 and K_2 in example 3.1.

Figure 3.3 shows the regulated output and Fig. 3.4 shows the control input. In this example, two uncertain parameters $\alpha(k)$ and $\beta(k)$ are randomly time-varying between $10^9 \le \alpha(k) = k_o \le 10^{10}$ and $10^7 \le \beta(k) = -\Delta H_{rxn} \le 10^8$. It can be observed from the figure that algorithm 3.1 can achieve less conservative results as compared to an off-line MPC algorithm of Wan and Kothare (2003). Moreover, algorithm 3.1 takes less time than an off-line MPC algorithm of Wan and Kothare (2003) to reach and remain inside the settling band which is properly chosen as ±1% of $C_{A,\text{equilibrium}}$ and ±0.1% of $T_{\text{equilibrium}}$. Thus, it can be concluded that algorithm 3.1 has less settling time than an off-line MPC algorithm of Wan and Kothare (2003).

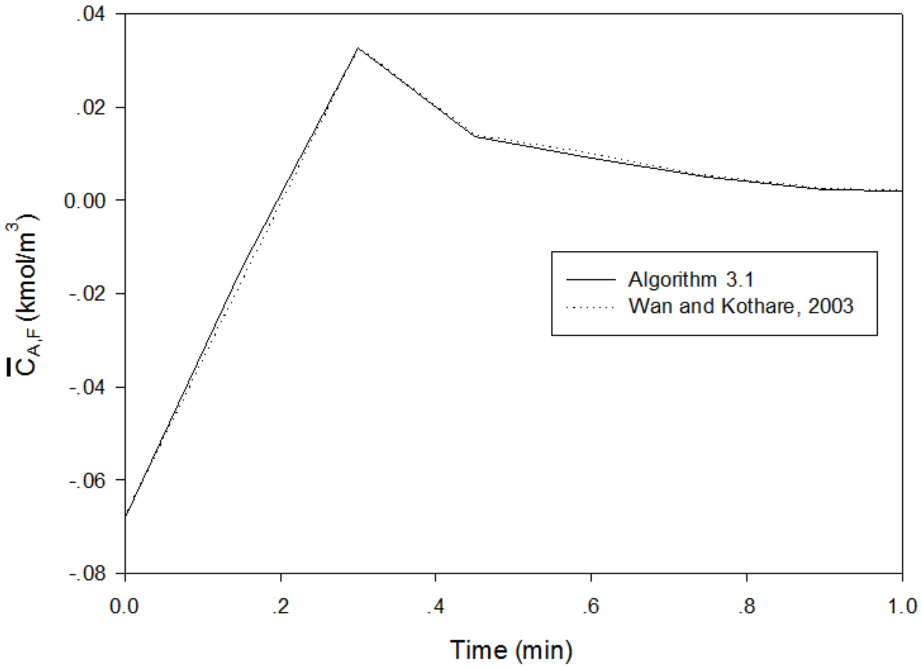


(a) The concentration of *A* in the reactor

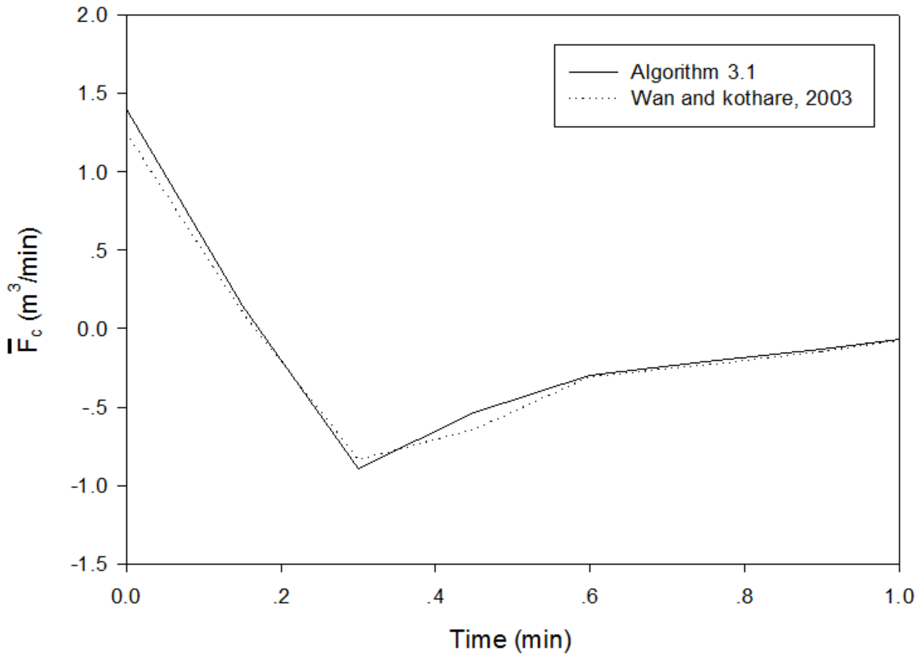


(b) The reactor temperature

Figure 3.3 The regulated output in example 3.1 (a) The concentration of *A* in the reactor (b) The reactor temperature.



(a) The feed concentration of A



(b) The coolant flow

Figure 3.4 The control input in example 3.1 (a) The feed concentration of A (b) The coolant flow.

The cumulative cost $\sum_{i=0}^{\infty} x(i)^T \Theta x(i) + u(i)^T Ru(i)$ is shown in table 3.2. It can be observed from the table that algorithm 3.1 has less cumulative cost than an off-line robust MPC algorithm of Wan and Kothre (2003) so algorithm 3.1 can achieve better control performance as compared to an off-line robust MPC algorithm of Wan and Kothre (2003).

Table 3.2 The cumulative cost in example 3.1 .

Algorithm	Cumulative Cost
Algorithm 3.1	17.51
Wan and Kothare (2003)	19.12

Table 3.3 shows the overall numerical burdens in example 3.1. Although the construction of polyhedral invariant sets is more computationally demanding than the construction of ellipsoidal invariant sets, this is done off-line and hence the on-line computation is tractable. All of the numerical simulations have been performed in Intel Core i-5 (2.4GHz), 2 GB RAM, using SeDuMi (Sturm, 1999) and YALMIP (Löfberg, 2004) within Matlab R2008a environment.

Table 3.3 The overall numerical burdens in example 3.1.

Algorithm	Overall off-ine computational time	On-line computational time per prediction
Wan and Kothare (2003)	3.672 s	0.001 s
Algorithm 3.1	4.372 s	0.001 s

Example 3.2:

In the second example, we will consider an application of our approach to an angular positioning system. The system consists of an electric motor driving a rotating antenna so that it always points in the direction of a moving object. The motion of the antenna can be described by the following discrete-time equation

$$\begin{bmatrix} \theta(k+1) \\ \dot{\theta}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1-0.1\alpha(k) \end{bmatrix} \begin{bmatrix} \theta(k) \\ \dot{\theta}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta(k) \\ \dot{\theta}(k) \end{bmatrix} \tag{3.9}$$

where $\theta(k)$ is the angular position of the antenna, $\dot{\theta}(k)$ is the angular velocity of the antenna and $u(k)$ is the input voltage to the motor. The uncertain parameter $\alpha(k)$ is proportional to the coefficient of viscous friction in the rotating parts of the antenna. It is assumed to be arbitrarily time-varying in the range of $0.1 \le \alpha(k) \le 10$. Let $\bar{\theta} = \theta - \theta_{eq}$, $\bar{\dot{\theta}} = \dot{\theta} - \dot{\theta}_{eq}$ and $\bar{u} = u - u_{eq}$ where the subscript eq is used to denote the corresponding variable at equilibrium condition. The discrete-time model (3.9) can be written as follows

$$\begin{bmatrix} \bar{\theta}(k+1) \\ \bar{\dot{\theta}}(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.1 \\ 0 & 1-0.1\alpha(k) \end{bmatrix} \begin{bmatrix} \bar{\theta}(k) \\ \bar{\dot{\theta}}(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0.0787 \end{bmatrix} \bar{u}(k)$$
$$\bar{y}(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{\theta}(k) \\ \bar{\dot{\theta}}(k) \end{bmatrix} \tag{3.10}$$

Because the uncertain parameter $\alpha(k)$ is varied between 0.1 and 10, we conclude that $A(k) \in \Omega$ where Ω is given as follows

$$\Omega = Co\left\{ \begin{bmatrix} 1 & 0.1 \\ 0 & 0.99 \end{bmatrix}, \begin{bmatrix} 1 & 0.1 \\ 0 & 0 \end{bmatrix} \right\} \tag{3.11}$$

The objective is to regulate $\bar{\theta}$ from 0.2 to the origin by manipulating $\bar{u}$. The input constraint is $|\bar{u}(k)| \le 2$ volts. Here $J_\infty(k)$ is given by $J_\infty(k) = \sum_{i=0}^{\infty} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}^T \begin{bmatrix} \Theta & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}$ with $\Theta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $R = 0.00002$.

Next, we will illustrate the step-by-step implementation of the proposed algorithm 3.1. In an off-line step 1, a sequence of states $x_i = \{(0.35,0.35), (0.30,0.30), (0.25,0.25), (0.20,0.20), (0.15,0.15), (0.10,0.10), (0.05,0.05)\}$ is chosen to calculate the corresponding state feedback gains K_i. A sequence of states x_i is chosen such that the distance between x_{i+1} and the origin is less than the distance between x_i and the origin. This is to ensure that the polyhedral invariant sets constructed are nested ($S_{i+1} \subset S_i$). In this example, only seven state feedback gains K_i, $i = 1,...,7$ are computed off-line because the state feedback gains K_i are almost constant beyond $i = 7$.

After a sequence of state feedback gains $K_i,\ i=1,...,7$ is computed in an off-line step 1, the corresponding polyhedral invariant sets $S_i,\ i=1,...,7$ will be constructed in an off-line step 2. Let us begin with the first feedback gain K_1. In an off-line step 2.1, we first set $M_i=[K_i^T,-K_i^T]^T$, $d_i=[u_{\max}^T,u_{\min}^T]^T$ (There is no output constraint in this example) and $m=1$. Then, M_1 and d_1 can be written as follows

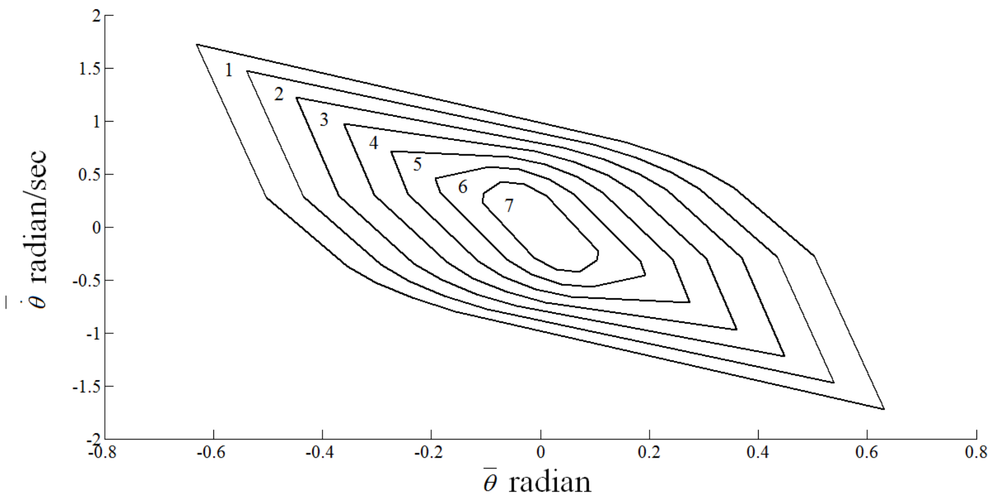
$$M_1=[K_1^T,-K_1^T]^T=\begin{bmatrix}-4.55 & -1.04\\ 4.55 & 1.04\end{bmatrix},\ \ d_1=[u_{\max}^T,u_{\min}^T]^T=\begin{bmatrix}2\\ -2\end{bmatrix}$$

where $K_1=\begin{bmatrix}-4.55 & -1.04\end{bmatrix}$ corresponding to the chosen state $x_1=(0.35,0.35)$ is calculated from an off-line step 1, $u_{\max}=2$ and $u_{\min}=-2$ are the input constraints.

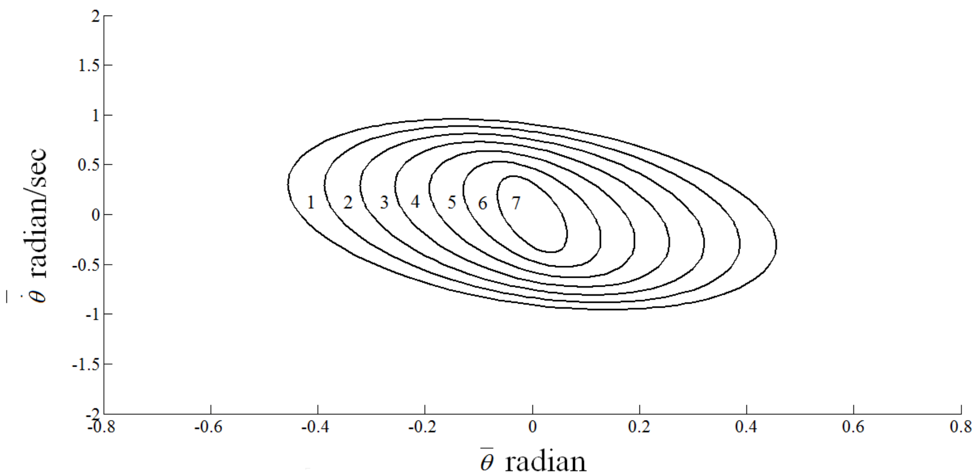
In an off-line step 2.2, we select row *m*=1 from (M_1,d_1), which is an input constraint $K_1x\le u_{\max}$, and find whether the constaint $K_1(A_1+B_1K_1)x\le u_{\max}$ is redundant with respect to the constraints defined by (M_1,d_1). By setting $W_{1,1,1}=K_1(A_1+B_1K_1)x-u_{\max}$ and solving (3.5), we found that $W_{1,1,1}>0$. Thus, an input constraint $K_1(A_1+B_1K_1)x\le u_{\max}$ is non-redundant and it has to be included in the construction of polyhedral invariant set S_1 by adding it to the constraints defined by (M_1,d_1). Without this constraint, the state $x(k+1)$ can violate an input constraint $K_1x(k+1)\le u_{\max}$ for at least one realization of the uncertainty. M_1 and d_1 now can be written as $M_1=[K_1^T,-K_1^T,(K_1(A_1+B_1K_1))^T]^T$ and $d_1=[u_{\max}^T,u_{\min}^T,u_{\max}^T]^T$. Then an off-line step 2.2 continues for $W_{1,1,2}$. We found that $W_{1,1,2}>$ 0 and hence the constraints $K_1(A_2+B_2K_1)x\le u_{\max}$ also has to be included in the construction of the polyhedral invariant set S_1.

In an off-line step 2.3, by setting $m=2$, (3.5) is repeatly solved to find the set of initial states such that the constraint $K_1x(k+1)\ge u_{\min}$ is guaranteed to be satisfied. Then the procedure is continued for $m>2$ until there is no non-redundant constraint. Note that the number of contraints defining the polyhedral invariant set S_1 is finite because the closed-loop system is robustly stabilized by the state feedback gain K_1 calculated from an off-line step 1 (The satisfaction of the Lyapunov stability constraint (3.2) for the state feedback gain K_1 guarantees that the closed-loop system is robustly stabilized.). Finally, in this example, the algorithm terminates at $m=12$.

The polyhedral invariant sets $S_i,\ i=2,\ldots,7$ corresponding to the state feedback gains $K_i,\ i=2,\ldots,7$ can be constructed by following the same procedure as S_1. After the constructions of all S_i are completed, the polyhedral invariant sets $S_i,\ i=1,\ldots,7$ as shown in Fig. 3.5 (a) are obtained.



(a) Algorithm 3.1



(b) Wan and Kothare (2003)

Figure 3.5 The comparison between a sequence of polyhedral invariant sets constructed off-line by algorithm 3.1 and a sequence of ellipsoidal invariant sets constructed off-line by an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003) in example 3.2.

The comparison between the polyhedral invariant sets constructed by algorithm 3.1 and the ellipsoidal invariant sets constructed by an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003) is shown in Fig. 3.5. Note that for both algorithms, the invariant sets are constructed by choosing the same sequence of states x_i, $i = 1,\ldots,7$. For each chosen state x_i, the polyhedral invariant set has a significantly larger stabilizable region as compared to an ellipsoidal invariant set. This is due to the fact that an ellipsoidal invariant set constructed by an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003) is only an approximation of the true polyhedral invariant set constructed by algorithm 3.1.

The comparison between the stabilizable regions of feedback gains K_1 and K_2 is shown in Fig. 3.6. In this example, K_2 is larger than K_1 because K_2 is computed by using the state which is closer to the origin than K_1. It can be observed from the figure that an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003) cannot stabilize the states at point A because the states at point A are not contained in the largest invariant ellipsoid $x_{point\,A} \notin \varepsilon_1$. In comparison, algorithm 3.1 can stabilize the states at point A by using K_1 because the states are contained in the largest polyhedral invariant set $x_{point\,A} \in S_1$. It can also be observed from the figure that an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003) can stabilize the states at point B by using the lowest feedback gain K_1. In comparison, algorithm 3.1 can stabilize the states at point B by using higher feedback gain K_2 due to the fact that $x_{point\,B} \in S_2$. In this circumstance, algorithm 3.1 can adopt higher feedback gain as compared to an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003). Thus, algorithm 3.1 can achieve less conservative result as compared to an ellipsoidal off-line robust MPC algorithm of Wan and Kothare (2003).

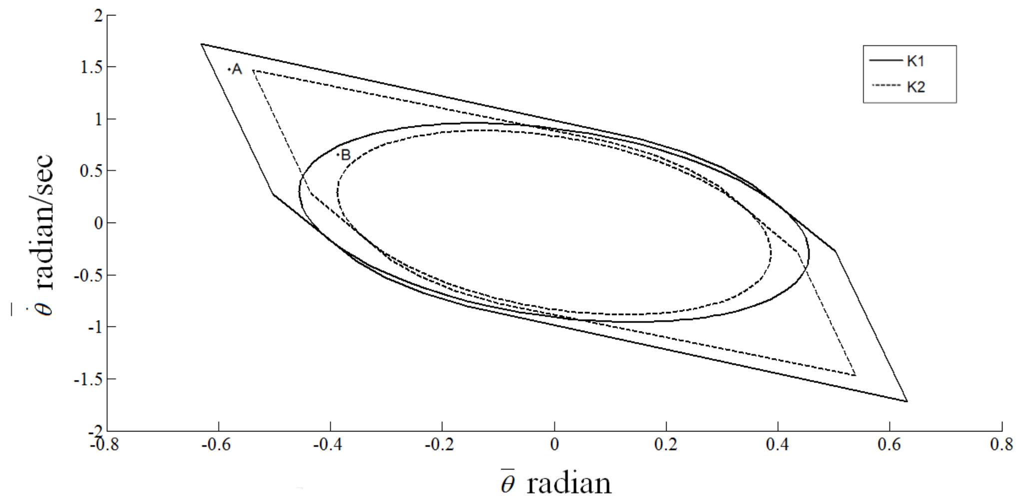
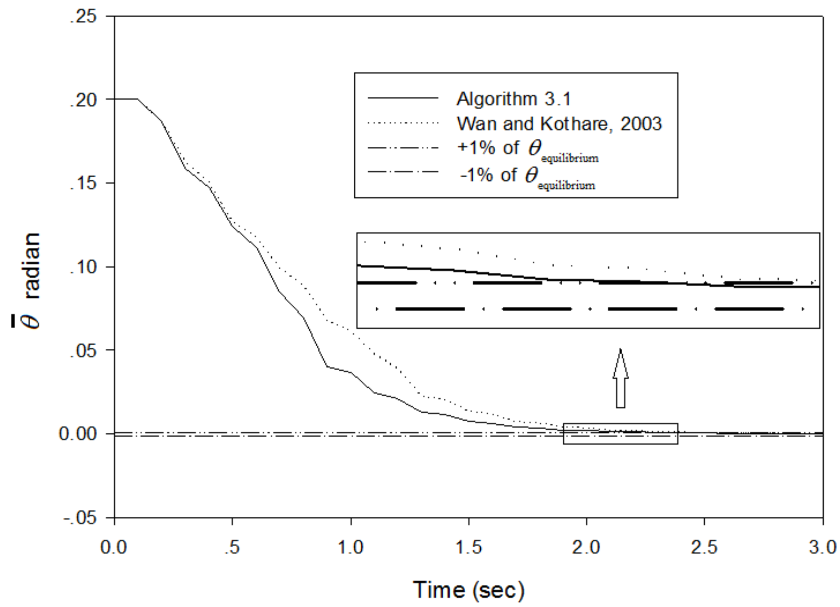
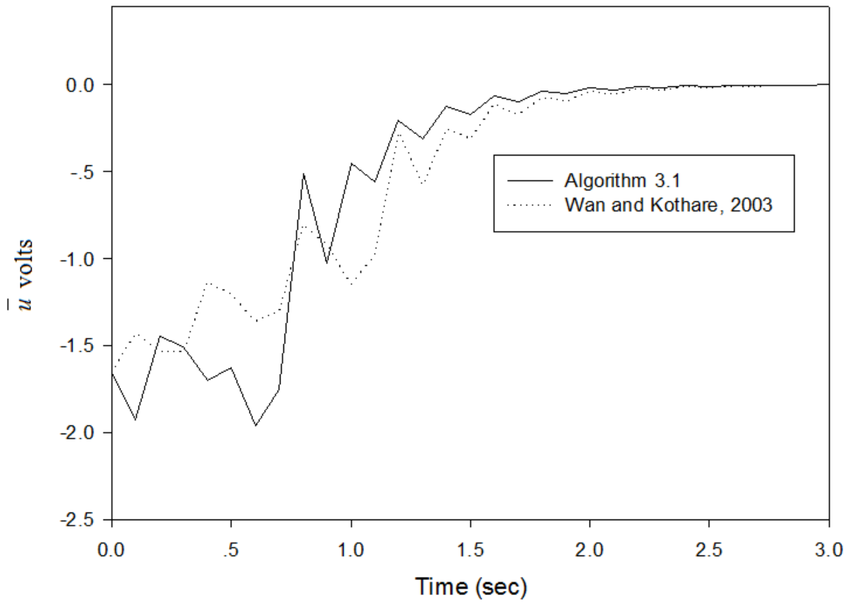


Figure 3.6 The stabilizable regions of feedback gains K_1 and K_2 in example 3.2.

Figure 3.7 shows the closed-loop responses of the system when $\alpha(k)$ is randomly time-varying between $0.1 \le \alpha(k) \le 10$. It can be observed from the figure that algorithm 3.1 can achieve less conservative results as compared to an off-line robust MPC algorithm of Wan and Kothare (2003). Moreover, it is seen that algorithm 3.1 settles within ±1% of $\theta_{equilibrium}$ faster than an off-line robust MPC algorithm of Wan and Kothare (2003). Thus, it is concluded that algorithm 3.1 has less settling time than an off-line MPC algorithm of Wan and Kothare (2003).



a) The regulated output



b) The control input

Figure 3.7 The closed-loop responses of the system in example 3.2 when $\alpha(k)$ is randomly time-varying between $0.1 \le \alpha(k) \le 10$ (a) The regulated output (b) The control input.

The cumulative cost $\sum_{i=0}^{\infty} x(i)^T \Theta x(i) + u(i)^T Ru(i)$ is shown in table 3.4. It can be observed that algorithm 3.1 has less cumulative cost than an off-line robust MPC algorithm of Wan and Kothare (2003) so algorithm 3.1 can achieve better control performance as compared to an off-line robust MPC algorithm of Wan and Kothare (2003). This is due to the fact that for each chosen state x_i, the stabilizable region of polyhedral invariant set constructed by algorithm 3.1 is significantly larger than the stabilizable region of an ellipsoidal invariant set constructed by an off-line robust MPC algorithm of Wan and Kothare (2003). As previously discussed in example 3.1, algorithm 3.1 can adopt higher feedback gain as compared to an off-line robust MPC algorithm of Wan and Kothare (2003).

Table 3.4 The cumulative cost in example 3.2.

Algorithm	Cumulative Cost
Algorithm 3.1	0.21
Wan and Kothare (2003)	0.23

Table 3.5 shows the overall numerical burdens in example 3.2. For both algorithms, most of the computational burdens are moved off-line and hence on-line computations are tractable. However, it can be observed that by using the same amount of on-line computational time, the proposed algorithm can achieve better control performance as shown in Fig. 3.7.

Table 3.5 The overall numerical burdens in example 3.2.

Algorithm	Overall off-ine computational time	On-line computational time per prediction
Wan and Kothare (2003)	2.831 s	0.001 s
Algorithm 3.1	3.541 s	0.001 s

3.3 An Extension of An Off-line Robust MPC Algorithm 3.1 to Linear Parameter Varying (LPV) Systems

By assuming that the time-varying parameter can be measured on-line at each sampling instant, the control performance of robust MPC is improved because the first control input can be calculated without any model uncertainty. In this section, an off-line MPC algorithm for LPV systems using polyhedral invariant sets is proposed. LPV systems are linear systems whose dynamics depend on the scheduling parameters that can be measured on-line. The analysis and synthesis of LPV systems play an important role in control theory since nonlinear systems can be dealt within the framework of LPV systems. Algorithm 3.1 will be extended by precomputing off-line the sequences of state feedback gains corresponding to the sequences of nested polyhedral invariant sets. Instead of constructing only a sequence of nested polyhedral invariant sets as proposed in algorithm 3.1, the number of sequences of nested polyhedral invaraint sets constructed is equal to the number of the vertices of polytope. At each sampling instant, the smallest polyhedral invariant set containing the currently measured state is determined in each sequence and the scheduling parameter is measured. The real-time state feedback gain is then calculated by linear interpolation between the corresponding off-line state feedback gains.

Algorithm 3.2

Off-line step 1: Choose a sequence of states $x_i, i \in \{1,2,\ldots,N\}$ and solve the following problem to obtain the corresponding state feedback gains $K_{i,j} = Y_{i,j}G_{i,j}^{-1}, \forall i = 1,2,\ldots,N, \forall j = 1,2,\ldots,L$. The states x_i should be chosen such that the distance between x_{i+1} and the origin is less than the distance between x_i and the origin.

$$\min_{Y_{i,j},G_{i,j},Q_{i,j}} \gamma_i$$

$$\text{s.t.} \begin{bmatrix} 1 & * \\ x_i & Q_{i,j} \end{bmatrix} \geq 0 \text{ }, \forall i = 1,2,\ldots,N, \forall j = 1,2,\ldots,L \tag{3.12}$$

$$\begin{bmatrix} G_{i,j} + G_{i,j}^T - Q_{i,j} & * & * & * \\ A_j G_{i,j} + BY_{i,j} & Q_{i,l} & * & * \\ \Theta^{\frac{1}{2}} G_{i,j} & 0 & \gamma_i I & * \\ R^{\frac{1}{2}} Y_{i,j} & 0 & 0 & \gamma_i I \end{bmatrix} \geq 0, \tag{3.13}$$
$$\forall i = 1,2,\ldots,N, \forall j = 1,2,\ldots,L, \forall l = 1,2,\ldots,L$$

$$\begin{bmatrix} X & * \\ Y_{i,j}^T & G_{i,j} + G_{i,j}^T - Q_{i,j} \end{bmatrix} \geq 0,$$
$$\forall i = 1,2,\ldots,N, \forall j = 1,2,\ldots,L, X_{hh} \leq u_{h,\max}^2, h = 1,2,\ldots,n_u \tag{3.14}$$

$$\begin{bmatrix} S & * \\ (A_j G_{i,j} + BY_{i,j})^T C^T & G_{i,j} + G_{i,j}^T - Q_{i,j} \end{bmatrix} \geq 0,$$
$$\forall i = 1,2,\ldots,N, \forall j = 1,2,\ldots,L, S_{rr} \leq y_{r,\max}^2, r = 1,2,\ldots,n_y \tag{3.15}$$

Off-line step 2: Given the state feedback gains $K_{i,j} = Y_{i,j}G_{i,j}^{-1}, \forall i = 1,2,\ldots,N, \forall j = 1,2,\ldots,L$ from step 1. For each $K_{i,j}$, the corresponding polyhedral invariant sets $S_{i,j} = \{x / M_{i,j} x \leq d_{i,j}\}$ is constructed by following these steps:

2.1) Set $M_{i,j} = [C^T, -C^T, K_{i,j}^T, -K_{i,j}^T]^T$, $d_{i,j} = [y_{\max}^T, y_{\min}^T, u_{\max}^T, u_{\min}^T]^T$ and $m = 1$.

2.2) Select row m from $(M_{i,j}, d_{i,j})$ and check $\forall l,\ l = 1,\ldots L$ whether $M_{i,j,m}(A_l + B_l K_{i,j})x \leq d_{i,j,m}$ is redundant with respect to the constraints defined by $(M_{i,j}, d_{i,j})$ by solving the following problem:

$$\begin{aligned} &\max_x W_{i,j,m,l} \\ &\text{s.t. } W_{i,j,m,l} = M_{i,j,m}(A_l + B_l K_{i,j})x - d_{i,j,m} \\ &M_{i,j} x \le d_{i,j} \end{aligned} \tag{3.16}$$

If $W_{i,j,m,l} > 0$, the constraint $M_{i,j,m}(A_l + B_l K_{i,j})x \le d_{i,j,m}$ is non-redundant with respect to $(M_{i,j}, d_{i,j})$. Then, add non-redundant constraints to $(M_{i,j}, d_{i,j})$ by assigning $M_{i,j} = [M_{i,j}^T, (M_{i,j,m}(A_l + B_l K_{i,j}))^T]^T$ and $d_{i,j} = [d_{i,j}^T, d_{i,j,m}^T]^T$.

2.3) Let $m = m+1$ and return to step 2.2. If m is strictly larger than the number of rows in (M_i, d_i) then terminate.

On-line: At each sampling time, measure $x(k)$, $p_j(k)$ and determine the smallest polyhedral invariant set $S_{i,j} = \{x / M_{i,j} x \le d_{i,j}\}$ containing the measured state in each sequence. Then implement the corresponding state feedback control law $u(k) = (\sum_{j=1}^{L} p_j(k) K_{ij})\, x(k)$ to the process.

The satisfaction of (3.13) for the state feedback gain $K(p(k+i)) = (\sum_{j=1}^{L} p_j(k+i) K_{i,j})$, $K_{i,j} = Y_{i,j} G_{i,j}^{-1}$ ensures that (Wada et al., 2006)

$$\begin{aligned} &\left\{[A(p(k+i)) + BK(p(k+i))]^T P(i+1,k)[A(p(k+i)) + BK(p(k+i))] - P(i,k)\right\} \\ &\le -\left\{\Theta + K(p(k+i))^T R K(p(k+i))\right\}, \forall i \ge 0. \end{aligned} \tag{3.17}$$

Thus, $V(i,k) = x(k+i/k)^T P(i,k) x(k+i/k)$ is a strictly decreasing Lyapunov function and the closed-loop system is robustly stabilized by the state feedback gain $K(p(k+i))$.

By solving (3.16) and iteratively adding non-redundant constraints $M_{i,j,m}(A_l + B_l K_{i,j})x \le d_{i,j,m}$ to $(M_{i,j}, d_{i,j})$ by assigning $M_{i,j} = [M_{i,j}^T, (M_{i,j,m}(A_l + B_l K_{i,j}))^T]^T$ and $d_{i,j} = [d_{i,j}^T, d_{i,j,m}^T]^T$, we can find the set of initial states x defined by $S_{i,j} = \{x / M_{i,j} x \le d_{i,j}\}$ such that all future states are guaranteed to stay within this set without input and output constraints violation. Thus, the set $S_{i,j}$ is polyhedral invariant set and any convex combination of the corresponding state feedback gain $K(p(k+i)) = (\sum_{j=1}^{L} p_j(k+i) K_{i,j})$ guarantees robust constraint satisfaction.

Examples 3.3

In this example, we will consider the application of our approach to the nonlinear two-tank system (Angeli et al., 2000) which is described by the following equation

$$\begin{aligned} \rho S_1 \dot{h}_1 &= -\rho A_1 \sqrt{2gh_1} + u \\ \rho S_2 \dot{h}_2 &= \rho A_1 \sqrt{2gh_1} - \rho A_2 \sqrt{2gh_2} \end{aligned} \tag{3.18}$$

where h_1 is the water level in tank 1, h_2 is the water level in tank 2 and u is the water flow. The operating parameters are shown in table 3.6.

Table 3.6 The operating parameters of the nonlinear two-tank system in example 3.3.

Parameter	Value	Unit
S_1	2500	cm^2
S_2	1600	cm^2
A_1	9	cm^2
A_2	4	cm^2
g	980	cm/s^2
ρ	0.001	kg/cm^3
$h_{1,eq}$	14	cm
$h_{2,eq}$	70	cm

Let $\overline{h_1} = h_1 - h_{1,eq}$, $\overline{h_2} = h_2 - h_{2,eq}$ and $\bar{u} = u - u_{eq}$ where subscript eq is used to denote the corresponding variable at equilibrium condition, the objective is to regulate $\overline{h_2}$ to the origin by manipulating $\bar{u}$. The input and output constraints are given as follows

$$\left|\bar{u}\right| \le 1.5 \text{ kg/s}, \ \left|\overline{h_1}\right| \le 13 \text{ cm}, \ \left|\overline{h_2}\right| \le 50 \text{ cm} \tag{3.19}$$

By evaluating the Jacobian matrix of (3.18) along the vertices of the constraints set (3.19), we have that all the solutions of (3.18) are also the solution of the following differential inclusion

$$\begin{bmatrix} \rho S_1 \dot{\bar{h}}_1 \\ \rho S_2 \dot{\bar{h}}_2 \end{bmatrix} \in \left(\sum_{j=1}^{4} p_j A_j \right) \begin{bmatrix} \bar{h}_1 \\ \bar{h}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{u} \tag{3.20}$$

where $A_j, j = 1,\ldots,4$ are given by

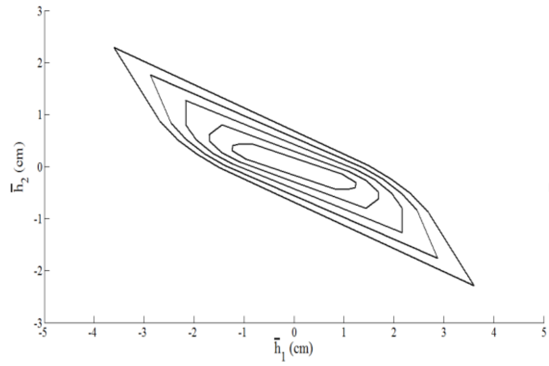
$$A_1 = \begin{bmatrix} -\rho A_1 \sqrt{\frac{2g}{h_{1,\min}}} & 0 \\ \rho A_1 \sqrt{\frac{2g}{h_{1,\min}}} & -\rho A_2 \sqrt{\frac{2g}{h_{2,\min}}} \end{bmatrix}, A_2 = \begin{bmatrix} -\rho A_1 \sqrt{\frac{2g}{h_{1,\max}}} & 0 \\ \rho A_1 \sqrt{\frac{2g}{h_{1,\max}}} & -\rho A_2 \sqrt{\frac{2g}{h_{2,\min}}} \end{bmatrix}$$
$$A_3 = \begin{bmatrix} -\rho A_1 \sqrt{\frac{2g}{h_{1,\min}}} & 0 \\ \rho A_1 \sqrt{\frac{2g}{h_{1,\min}}} & -\rho A_2 \sqrt{\frac{2g}{h_{2,\max}}} \end{bmatrix}, A_4 = \begin{bmatrix} -\rho A_1 \sqrt{\frac{2g}{h_{1,\max}}} & 0 \\ \rho A_1 \sqrt{\frac{2g}{h_{1,\max}}} & -\rho A_2 \sqrt{\frac{2g}{h_{2,\max}}} \end{bmatrix} \tag{3.21}$$

and $p_j, j = 1,\ldots,4$ are given by

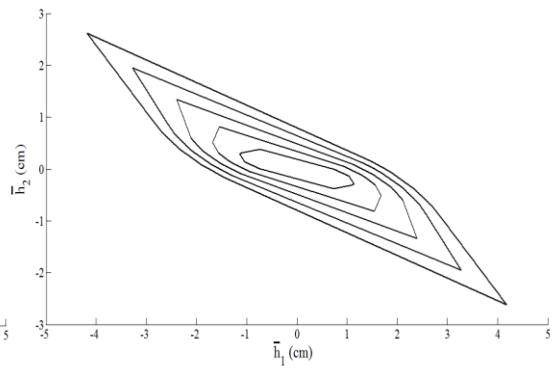
$$p_1 = \left[\frac{(1/\sqrt{h_{1,\max}}) - (1/\sqrt{h_1})}{(1/\sqrt{h_{1,\max}}) - (1/\sqrt{h_{1,\min}})} \right] \cdot \left[\frac{(1/\sqrt{h_{2,\max}}) - (1/\sqrt{h_2})}{(1/\sqrt{h_{2,\max}}) - (1/\sqrt{h_{2,\min}})} \right]$$
$$p_2 = \left[\frac{(1/\sqrt{h_1}) - (1/\sqrt{h_{1,\min}})}{(1/\sqrt{h_{1,\max}}) - (1/\sqrt{h_{1,\min}})} \right] \cdot \left[\frac{(1/\sqrt{h_{2,\max}}) - (1/\sqrt{h_2})}{(1/\sqrt{h_{2,\max}}) - (1/\sqrt{h_{2,\min}})} \right]$$
$$p_3 = \left[\frac{(1/\sqrt{h_{1,\max}}) - (1/\sqrt{h_1})}{(1/\sqrt{h_{1,\max}}) - (1/\sqrt{h_{1,\min}})} \right] \cdot \left[\frac{(1/\sqrt{h_2}) - (1/\sqrt{h_{2,\min}})}{(1/\sqrt{h_{2,\max}}) - (1/\sqrt{h_{2,\min}})} \right]$$
$$p_4 = \left[\frac{(1/\sqrt{h_1}) - (1/\sqrt{h_{1,\min}})}{(1/\sqrt{h_{1,\max}}) - (1/\sqrt{h_{1,\min}})} \right] \cdot \left[\frac{(1/\sqrt{h_2}) - (1/\sqrt{h_{2,\min}})}{(1/\sqrt{h_{2,\max}}) - (1/\sqrt{h_{2,\min}})} \right] \tag{3.22}$$

The discrete-time model is obtained by discretization of (3.20) using Euler first-order approximation with a sampling period of 0.5 s and it is omitted here for brevity. Here $J_\infty(k)$ is given by $J_\infty(k) = \sum_{i=0}^{\infty} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}^T \begin{bmatrix} \Theta & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}$ with $\Theta = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $R = 0.01$.

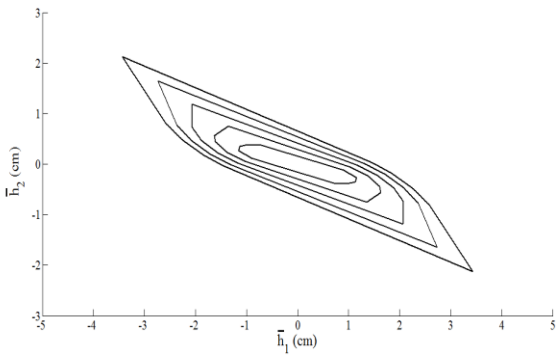
Figure 3.8 shows the polyhedral invariant sets constructed off-line by algorithm 3.2 and algorithm 3.1. For both algorithms, the polyhedral invariant sets are constructed by choosing the same sequence of state $x_i, i \in \{1,2,\ldots 5\}$. Note that with the same number of chosen states, algorithm 3.2 requires larger number of polyhedral invariant sets than algorithm 3.1. This is due to the fact that for algorithm 3.2, the number of sequences of nested polyhedral invaraint sets constructed is equal to the number of the vertices of the polytope.



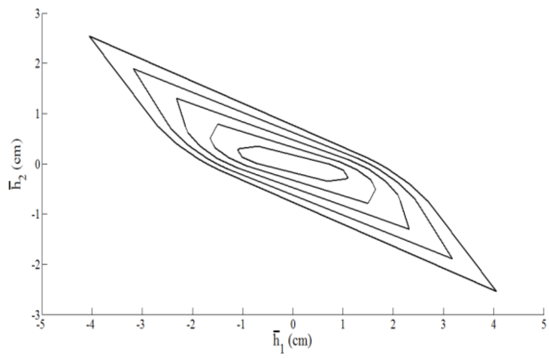
a.1) $S_{i,1}, i \in \{1,2,\ldots 5\}$



a.2) $S_{i,2}, i \in \{1,2,\ldots 5\}$

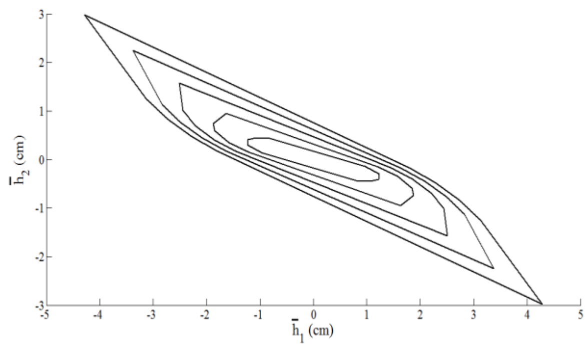


a.3) $S_{i,3}, i \in \{1,2,\ldots 5\}$



a.4) $S_{i,4}, i \in \{1,2,\ldots 5\}$

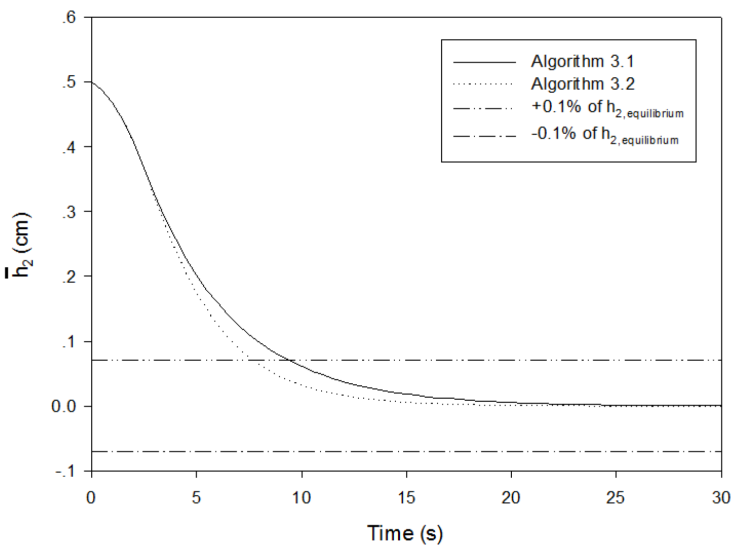
(a) Algorithm 3.2



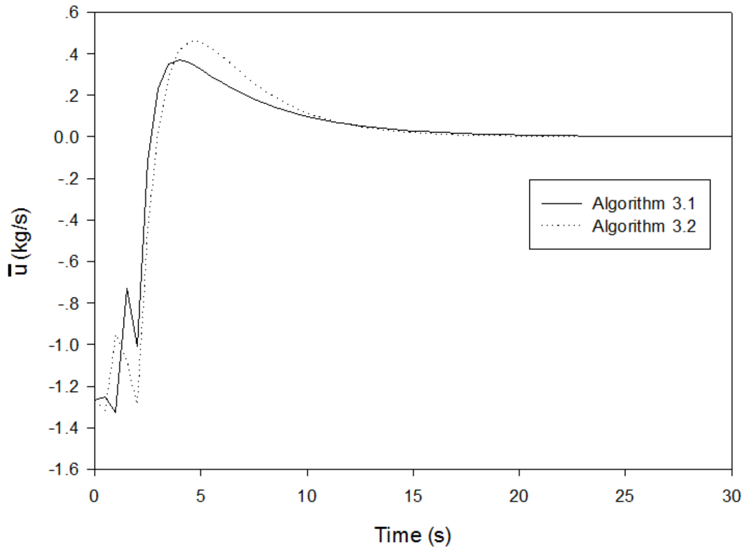
(b) Algorithm 3.1

Figure 3.8 The polyhedral invariant sets constructed off-line by (a) Algorithm 3.2 and (b) Algorithm 3.1 in example 3.3.

Figure 3.9 (a) shows the regulated output. For algorithm 3.2, the scheduling parameter is measured on-line at each sampling time so it can achieve less conservative result as compared to algorithm 3.1. Moreover, algorithm 3.2 has less settling time than algorithm 3.1 because algorithm 3.2 requires less time to enter and remain within the settling band ($\pm 0.1\%$ of $h_{2,\text{equilibrium}}$). The control input is shown in Fig. 3.9 (b), the input discontinuities are caused by the switching of state feedback gains based on the distance between the state and the origin.



(a) The regulated output



(b) The control input

Figure 3.9 The closed-loop responses in example 3.3 a) The regulated output b) The control input.

The overall off-line computational burdens are shown in table 3.7. Although algorithm 3.2 requires larger off-line computational time than algorithm 3.1, the on-line computation is tractable because there is no optimization problem needed to be solved on-line.

Table 3.7 The overall off-line computational burdens in example 3.3.

Algorithm	Off-line CPU time (s)
Algorithm 3.1	3.612
Algorithm 3.2	6.738

3.4 Conclusions

In this chapter, we have presented an off-line synthesis approach to robust MPC using polyhedral invariant sets. The proposed algorithm precomputes off-line a sequence of state feedback control laws corresponding to a sequence of polyhedral invariant sets. At each sampling time, the smallest polyhedral invariant set that the currently measured state can be embedded is determined. The corresponding state feedback control law is then implemented to the process. Several examples that illustrate the implementation of the proposed off-line robust MPC algorithm is presented. The results show that the control performance of our proposed algorithm is better than an ellipsoidal off-line robust MPC algorithm. Moreover, a significantly larger stabilizable region is obtained. Finally, an off-line MPC algorithm for LPV systems is proposed. The scheduling parameter is measured on-line at each sampling instant so the control performance is improved.

CHAPTER IV

AN INTERPOLATION-BASED MPC STRATEGY FOR LINEAR PARAMETER VARYING SYSTEMS

One of the main important problems in implementation of MPC is the on-line computational complexity. Although the significant advances of modern computers over the past few years have alleviated the computational problem of MPC, the application of MPC is rather restricted due to its on-line computational requirements. Moreover, the size of the optimization problem grows respectively with the number of independent uncertain process parameters.

This chapter presents a strategy to alleviate the problem of on-line computational complexity of MPC. The on-line computational burdens are reduced by precomputing off-line the sequences of state feedback gains corresponding to the sequences of nested ellipsoids. At each sampling time, the scheduling parameter is measured and the real-time state feedback gain is calculated by linear interpolation between the precomputed state feedback gains.

4.1 Introduction

MPC based on linear model is typically used in many industrial processes because the on-line optimization problem can be formulated as the convex optimization problem by either linear programming or quadratic programming. However, most of the chemical processes are nonlinear. The performance of linear MPC can deteriorate drastically when the operating conditions undergo significant changes. Moreover, the stability of nonlinear system cannot be guaranteed.

Linear parameter varying (LPV) systems are linear systems whose dynamics depend on the scheduling parameter that can be measured on-line. The analysis and synthesis of LPV systems play an important role in control theory since both nonlinear systems and linear systems with model uncertainties can be dealt within the framework of LPV systems. At each sampling instant, the scheduling parameter can be measured on-line. Its future behavior is considered to be uncertain and contained in a polytope.

Quasi-min-max MPC algorithm for LPV systems was proposed by Lu and Arkun (2000). The algorithm is seen as an extension of the algorithm presented by Kothare et al. (1996) by keeping the first control input as a free decision variable. Since the optimization problem is really solved on-line at each sampling instant, the algorithm requires high on-line computational time. Moreover, the algorithm turns out to be conservative because it is derived by using a single Lyapunov function. Two-stage scheduling quasi-min-max MPC algorithm was presented by Lu and Arkun (2002). The algorithm can achieve less conservative result as compared to Quasi-min-max MPC algorithm of Lu and Arkun (2000) because more control moves are relaxed from the feedback control law. However, it is computationally prohibitive in the practical situations because the size of the on-line optimization problem grows significantly with respect to the size of the polytope.

In order to reduce the conservativeness arising from the use of a single Lyapunov function, MPC for LPV systems using parameter-dependent Lyapunov function was proposed by Wada et al. (2006). As compared with a single Lyapunov function, the use of parameter-dependent Lyapunov function can reduce the conservativeness because there are more degrees of freedom in solving the optimization problem. However, the algorithm requires high on-line computational time because the on-line optimization problem contains many decision variables and constraints.

Since the application of on-line MPC is restricted to only a slow dynamic process, some researchers have begun to study off-line MPC (Wan and Kothare, 2003; Ding et al., 2007). The on-line computational time is reduced by precomputing off-line a sequence of explicit control laws corresponding to a sequence of invariant ellipsoids. At each sampling time, the smallest ellipsoid containing the measured state is determined and the real-time control law is calculated by linear interpolation between control laws of two adjacent invariant ellipsoids. Although the on-line computational time is reduced, the conservativeness is obtained due to the fact that the scheduling parameter is not taken into account in the controller design. Moreover, the algorithm is still derived by using only a single Lyapunov function.

From the preceding review, we can see that on-line robust MPC usually requires high computational time. Thus, it is computationally prohibitive in practical situations. For off-line robust MPC, the conservative result is usually obtained because the scheduling parameter is not taken into account in the control synthesis. In this chapter, an off-line synthesis approach to MPC for LPV systems is presented. The on-line computational burdens are reduced by precomputing off-line the sequences of state feedback gains corresponding to the sequences of nested ellipsoids. At each sampling time, the scheduling parameter is measured and the real-time state feedback gain is calculated by linear interpolation between the state feedback gains of the smallest ellipsoid containing the measured state in each sequence.

4.2 An Interpolation-based MPC Strategy for LPV Systems

In this section, a strategy to reduce the on-line computational complexity of MPC for LPV systems is presented. Most of the computational burdens are moved off-line. Thus, the on-line computation is tractable. The on-line computation is reduced by precomputing off-line the sequences of state feedback gains corresponding to the sequences of nested ellipsoids. At each sampling instant, the smallest ellipsoid containing the currently measured state is determined in each sequence of ellipsoids and the scheduling parameter is measured. The real-time state feedback gain is then calculated by linear interpolation between the corresponding state feedback gains.

Algorithm 4.1

Off-line Step 1: Choose a sequence of states $x_i, i \in \{1,2,....,N\}$ and solve the following problem to obtain the corresponding state feedback gains $K_{i,j} = Y_{i,j} G_{i,j}^{-1}$ and ellipsoids $\varepsilon_{i,j} = \{x \in \Re^n / x^T Q_{i,j}^{-1} x \le 1\}$, $\forall i = 1,2,\ldots,N$, $\forall j = 1,2,\ldots,L$

$$\min_{\gamma_i, Y_{i,j}, G_{i,j}, Q_{i,j}} \gamma_i$$

$$\text{s.t.} \begin{bmatrix} 1 & x_i^T \\ x_i & Q_{i,j} \end{bmatrix} \ge 0, \forall j = 1,2,\ldots,L \tag{4.1}$$

$$\begin{bmatrix} G_{i,j} + G_{i,j}^T - Q_{i,j} & * & * & * \\ A_j G_{i,j} + B Y_{i,j} & Q_{i,l} & * & * \\ \Theta^{\frac{1}{2}} G_{i,j} & 0 & \gamma_i I & * \\ R^{\frac{1}{2}} Y_{i,j} & 0 & 0 & \gamma_i I \end{bmatrix} \ge 0,\ \forall j = 1,2,\ldots,L, \forall l = 1,2,\ldots,L \tag{4.2}$$

$$\begin{bmatrix} X & * \\ Y_{i,j}^{\ T} & G_{i,j} + G_{i,j}^{\ T} - Q_{i,j} \end{bmatrix} \geq 0, \quad (4.3)$$
$$\forall j = 1,\ldots,L, X_{hh} \leq u_{h,\max}^2,\ h = 1,2,\ldots,n_u$$

$$\begin{bmatrix} T & * \\ (A_j G_{i,j} + BY_{i,j})^T C^T & G_{i,j} + G_{i,j}^{\ T} - Q_{i,j} \end{bmatrix} \geq 0, \quad (4.4)$$
$$\forall j = 1,2.\ldots.L, T_{rr} \leq y_{r,\max}^2,\ r = 1,2,..n_y$$

Off-line Step 2: For each $i \neq N$, check if the following inequality is satisfied

$$Q_{i,j}^{-1} - (A_j + BK_{i+1,j})^T Q_{i,l}^{-1} (A_j + BK_{i+1,j}) > 0, \quad (4.5)$$
$$\forall j = 1,2,\ldots,L, \forall l = 1,2,\ldots,L$$

Online: At each sampling time k, measure $x(k), p(k)$ and adopt the following state feedback control law

$$u(k) = \begin{Bmatrix} K(\alpha_i(k))x(k) / x(k) \in \varepsilon_{i,j}, x(k) \notin \varepsilon_{i+1,j}, \forall j = 1,2.\ldots.L, i \neq N \\ K_N x(k) / x(k) \in \varepsilon_{N,j}, \forall j = 1,2.\ldots.L \end{Bmatrix} \quad (4.6)$$

where
$$K(\alpha_i(k)) = \alpha_i(k)[\sum_{j=1}^{L} p_j(k) K_{i,j}] + (1 - \alpha_i(k))[\sum_{j=1}^{L} p_j(k) K_{i+1,j}] \quad (4.7)$$

$$K_N x(k) = \sum_{j=1}^{L} p_j(k) K_{N,j} \quad (4.8)$$

If (4.5) is satisfied, then $\alpha_i(k) \in (0,1]$ is calculated from

$$x(k)^T (\alpha_i(k)[\sum_{j=1}^{L} p_j(k) Q_{i,j}^{-1}] + (1 - \alpha_i(k))[\sum_{j=1}^{L} p_j(k) Q_{i+1,j}^{-1}]) x(k) = 1 \quad (4.9)$$

If (4.5) is not satisfied, then $\alpha_i(k) = 1$.

An overall algorithm is proved to guarantee robust stability in Theorem 4.1.

Theorem 4.1 *Given an initial measured state* $x(k)$ *satisfying* $\|x(k)\|_{Q_{1,j}^{-1}}^2 \leq 1, \forall j = 1,2,\ldots,L$, *algorithm 4.1 asymptotically stabilizes the closed-loop system.*

Proof.

When an initial measured state $x(k)$ satisfies $\|x(k)\|^2_{Q^{-1}_{i,j}} \leq 1$, $\|x(k)\|^2_{Q^{-1}_{i+1,j}} > 1$, $\forall j = 1,2\ldots L$, the satisfaction of (4.5) ensures that $\begin{bmatrix} Q^{-1}_{i,j} & (A_j + BK_{i+1,j})^T Q^{-T}_{i,l} \\ Q^{-1}_{i,l}(A_j + BK_{i+1,j}) & Q^{-1}_{i,l} \end{bmatrix} > 0$ and the satisfaction of (4.2) ensures that $\begin{bmatrix} Q^{-1}_{i,j} & (A_j + BK_{i,j})^T Q^{-T}_{i,l} \\ Q^{-1}_{i,l}(A_j + BK_{i,j}) & Q^{-1}_{i,l} \end{bmatrix} > 0$. The satisfaction of both (4.2) and (4.5) ensures that $(A(p(k+i)) + BK(\alpha_i(k)))^T P(i+1,k)(A(p(k+i)) + BK(\alpha_i(k))) - P(i,k) < 0$ must be satisfied and the state feedback gain $K(\alpha_i(k))$ is guaranteed to asymptotically stabilize the closed-loop system. More proof details can be found in Appendix B.

Example 4.1: Consider the following nonlinear model for CSTR (Magni et al., 2001) where the exothermic reaction $A \longrightarrow B$ takes place.

$$\begin{aligned} \dot{C}_A &= \frac{q}{V}(C_{AF} - C_A) - k_o e^{\left(-\frac{Ea}{RT}\right)} C_A \\ \dot{T} &= \frac{q}{V}(T_f - T) + \frac{-\Delta H}{\rho C_p} k_o e^{\left(-\frac{Ea}{RT}\right)} C_A + \frac{UA}{V\rho C_p}(T_c - T) \end{aligned} \tag{4.10}$$

where C_A denotes the concentration of A in the reactor, T denotes the reactor temperature and T_c denoted the temperature of coolant stream. The operating parameters are shown in table 4.1.

Table 4.1 The operating parameters of nonlinear CSTR in example 4.1.

Parameter	Value	Unit
q	100	L/min
T_f	350	K
C_{AF}	1	mol/L
V	100	L
ρ	1,000	g/L
C_p	0.239	J/g K
ΔH	$-5\text{x}10^4$	J/mol
E_a / R	8,750	K
k_o	$7.2\text{x}10^{10}$	min^{-1}
UA	$5\text{x}10^4$	J/min K
$C_{A,eq}$	0.5	mol/L
T_{eq}	350	K

By defining $\overline{C}_A = C_A - C_{A,eq}$, $\overline{T} = T - T_{eq}$, $\overline{T_c} = T_c - T_{c,eq}$, (4.10) can be written as (Ding et al., 2008)

$$\begin{bmatrix} \dot{\overline{C}}_A \\ \dot{\overline{T}} \end{bmatrix} = \begin{bmatrix} -\frac{q}{V} - k_o \exp\left(-\frac{E_a/R}{\overline{T}+T_{eq}}\right) & 0 & -k_o[\exp\left(-\frac{E_a/R}{\overline{T}+T_{eq}}\right) - \exp\left(-\frac{E_a/R}{T_{eq}}\right)]C_{A,eq} \\ \frac{-\Delta H}{\rho C_p} k_o \exp\left(-\frac{E_a/R}{\overline{T}+T_{eq}}\right) & -\frac{q}{V} - \frac{UA}{V\rho C_p} & \frac{-\Delta H}{\rho C_p} k_o[\exp\left(-\frac{E_a/R}{\overline{T}+T_{eq}}\right) - \exp\left(-\frac{E_a/R}{T_{eq}}\right)]C_{A,eq} \end{bmatrix} \begin{bmatrix} \overline{C}_A \\ \overline{T} \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{UA}{V\rho C_p} \end{bmatrix} \overline{T}_c$$

$$= \begin{cases} \begin{bmatrix} -1-\varphi_1(\overline{T}) & -\varphi_2(\overline{T}) \\ 209.2\varphi_1(\overline{T}) & -3.092+209.2\varphi_2(\overline{T}) \end{bmatrix} \begin{bmatrix} \overline{C}_A \\ \overline{T} \end{bmatrix} + \begin{bmatrix} 0 \\ 2.092 \end{bmatrix} \overline{T_c}, & x_2 \neq 0 \\ \begin{bmatrix} -2 & 0 \\ 209.2 & -3.092 \end{bmatrix} \begin{bmatrix} \overline{C}_A \\ \overline{T} \end{bmatrix} + \begin{bmatrix} 0 \\ 2.092 \end{bmatrix} \overline{T_c}, & x_2 = 0 \end{cases}$$

$$= \begin{cases} \begin{bmatrix} -1-\varphi_1^0 - g_1(\overline{T}) & -\varphi_2^0 - g_2(\overline{T}) \\ 209.2\varphi_1^0 + 209.2 g_1(\overline{T}) & -3.092+209.2\varphi_2^0 + 209.2 g_2(\overline{T}) \end{bmatrix} \begin{bmatrix} \overline{C}_A \\ \overline{T} \end{bmatrix} + \begin{bmatrix} 0 \\ 2.092 \end{bmatrix} \overline{T_c}, & x_2 \neq 0 \\ \begin{bmatrix} -2 & 0 \\ 209.2 & -3.092 \end{bmatrix} \begin{bmatrix} \overline{C}_A \\ \overline{T} \end{bmatrix} + \begin{bmatrix} 0 \\ 2.092 \end{bmatrix} \overline{T_c}, & x_2 = 0 \end{cases} \tag{4.11}$$

where

$$\varphi_1(\overline{T}) = k_o \exp\left(-\frac{E_a/R}{\overline{T}+T_{eq}}\right), \varphi_2(\overline{T}) = k_o[\exp\left(-\frac{E_a/R}{\overline{T}+T_{eq}}\right) - \exp\left(-\frac{E_a/R}{T_{eq}}\right)]C_{A,eq}/\overline{T}$$

$$g_1(\overline{T}) = \varphi_1(\overline{T}) - \varphi_1^0, \; g_2(\overline{T}) = \varphi_2(\overline{T}) - \varphi_2^0$$

For $\left|\overline{T}\right| \leq \beta, \varphi_1^0 = [\varphi_1(-\beta) + \varphi_1(\beta)]/2, \varphi_2^0 = [\varphi_2(-\beta) + \varphi_2(\beta)]/2.$

Since $g_1(\overline{T})$ and $g_2(\overline{T})$ can vary between $g_1(-\beta) \leq g_1(\overline{T}) \leq g_1(\beta)$ and $g_2(-\beta) \leq g_2(\overline{T}) \leq g_2(\beta)$, we have that all the solutions of (4.11) are also the solutions of the following differential inclusion

$$\begin{bmatrix} \dot{\overline{C}}_A \\ \dot{\overline{T}} \end{bmatrix} \in \left(\sum_{j=1}^{4} p_j A_j\right) \begin{bmatrix} \overline{C}_A \\ \overline{T} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{UA}{V\rho C_p} \end{bmatrix} \overline{T} \tag{4.12}$$

where A_j is given by

$$A_1 = \begin{bmatrix} -1-\varphi_1^0 - 2g_1(\beta) & -\varphi_2^0 \\ 209.2\varphi_1^0 + 418.4g_1(\beta) & -3.092 + 209.2\varphi_2^0 \end{bmatrix}, A_2 = \begin{bmatrix} -1-\varphi_1^0 - 2g_1(-\beta) & -\varphi_2^0 \\ 209.2\varphi_1^0 + 418.4g_1(-\beta) & -3.092 + 209.2\varphi_2^0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} -1-\varphi_1^0 & -\varphi_2^0 - 2g_2(\beta) \\ 209.2\varphi_1^0 & -3.092 + 209.2\varphi_2^0 + 418.4g_2(\beta) \end{bmatrix}, A_4 = \begin{bmatrix} -1-\varphi_1^0 & -\varphi_2^0 - 2g_2(-\beta) \\ 209.2\varphi_1^0 & -3.092 + 209.2\varphi_2^0 + 418.4g_2(-\beta) \end{bmatrix}.$$

and p_j is given by

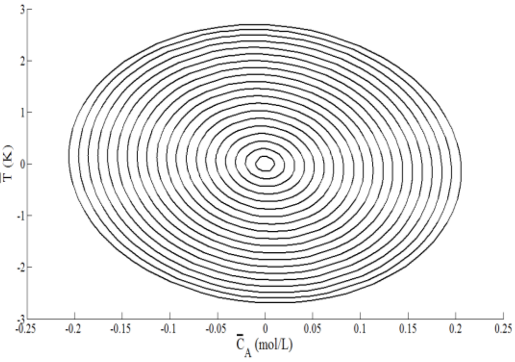
$$p_1 = \frac{1}{2}\frac{(g_1(\overline{T}) - g_1(-\beta))}{(g_1(\beta) - g_1(-\beta))}, p_2 = \frac{1}{2}\frac{(g_1(\beta) - g_1(\overline{T}))}{(g_1(\beta) - g_1(-\beta))}, p_3 = \frac{1}{2}\frac{(g_2(\overline{T}) - g_2(-\beta))}{(g_2(\beta) - g_2(-\beta))}, p_4 = \frac{1}{2}\frac{(g_2(\beta) - g_2(\overline{T}))}{(g_2(\beta) - g_2(-\beta))}$$

The objective is to regulate the concentration $\overline{C}_A$ and the reactor temperature $\overline{T}$ to the origin by manipulating $\overline{T}_c$. The input and output constraints are given as follows

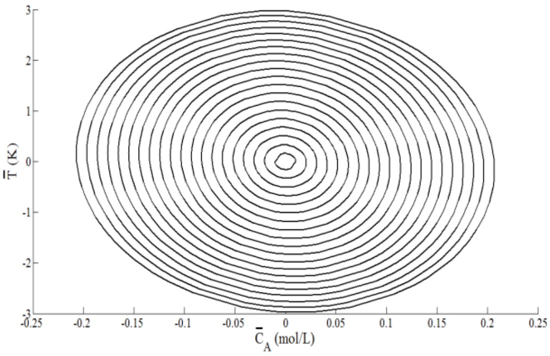
$$\begin{aligned} &\left|\overline{C}_A\right| \le 1 \text{ mol/l} \\ &\left|\overline{T}\right| \le 5\,\text{K} \\ &\left|\overline{T}_c\right| \le 40\,\text{K} \end{aligned} \tag{4.13}$$

The discrete-time model is obtained by discretization of (4.12) with a sampling period of 0.01 min and it is omitted here for brevity. The proposed algorithm will be compared with Quasi-min-max MPC algorithm of Lu and Arkun (2000) and an off-line robust MPC algorithm of Wan and Kothare (2003). Here $J_\infty(k)$ is given by $J_\infty(k) = \sum_{i=0}^{\infty} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}^T \begin{bmatrix} \Theta & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}$ with $\Theta = I$ and $R = 0.01I$.

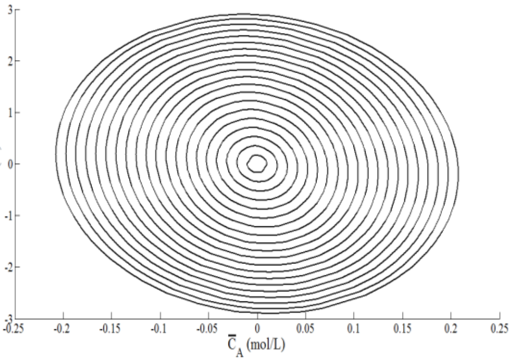
Figure 4.1 shows the ellipsoids constructed off-line by the proposed algorithm and an off-line robust MPC algorithm of Wan and Kothare (2003). For both algorithms, the ellipsoids are constructed by choosing the same sequence of states $x_i, i \in \{1,2,....,20\}$. Note that with the same number of chosen states, the proposed algorithm requires larger number of ellipsoids than an off-line robust MPC algorithm of Wan and Kothare (2003).



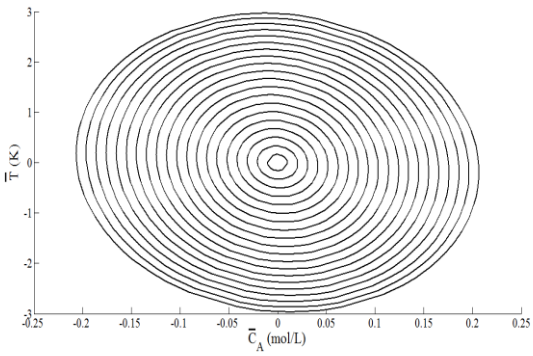
a.1) $\varepsilon_{i,1} = \{x \in \Re^n / x^T Q_{i,1}^{-1} x \le 1\}$, $i \in \{1,2,\ldots,20\}$



a.2) $\varepsilon_{i,2} = \{x \in \Re^n / x^T Q_{i,2}^{-1} x \le 1\}$, $i \in \{1,2,\ldots,20\}$

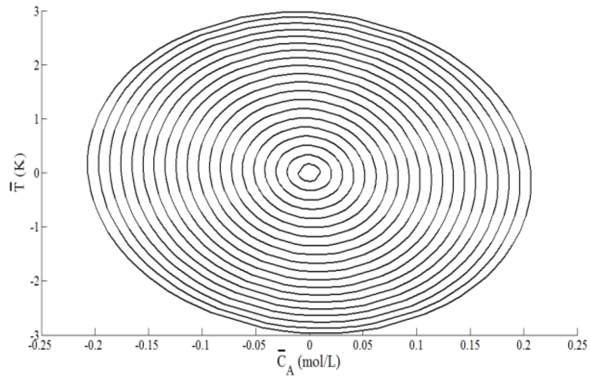


a.3) $\varepsilon_{i,3} = \{x \in \Re^n / x^T Q_{i,3}^{-1} x \le 1\}$, $i \in \{1,2,\ldots,20\}$



a.4) $\varepsilon_{i,4} = \{x \in \Re^n / x^T Q_{i,4}^{-1} x \le 1\}$, $i \in \{1,2,\ldots,20\}$

(a) Algorithm 4.1



(b) An off-line robust MPC algorithm of Wan and Kothare (2003)

Figure 4.1 The ellipsoids constructed off-line by (a) Algorithm 4.1 and (b) An off-line robust MPC algorithm of Wan and Kothare (2003) in example 4.1.

Table 4.2 shows the overall off-line numerical burdens. It can be observed that the proposed algorithm requires larger off-line computational time than an off-line robust MPC algorithm of Wan and Kothare (2003). This is due to the fact that the ellipsoids constructed by the proposed algorithm are derived by using parameter-dependent Lyapunov function. Thus, the optimization problem solved off-line is more complex that that presented in an off-line robust MPC algorithm of Wan and Kothare (2003).

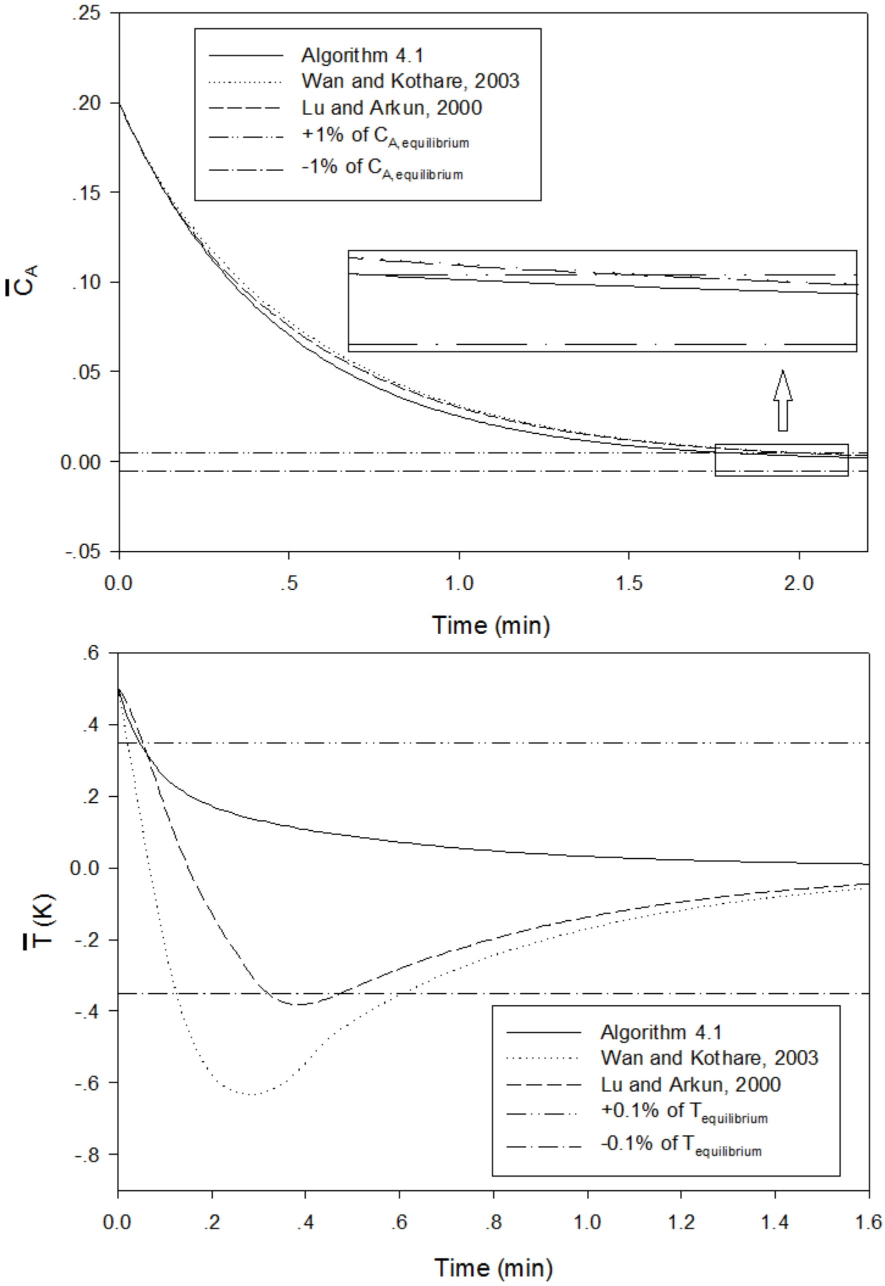
Table 4.2 The overall off-line numerical burdens in example 4.1.

Algorithm	CPU time (s)
Algorithm 4.1	8.174
Wan and Kothare (2003)	5.783

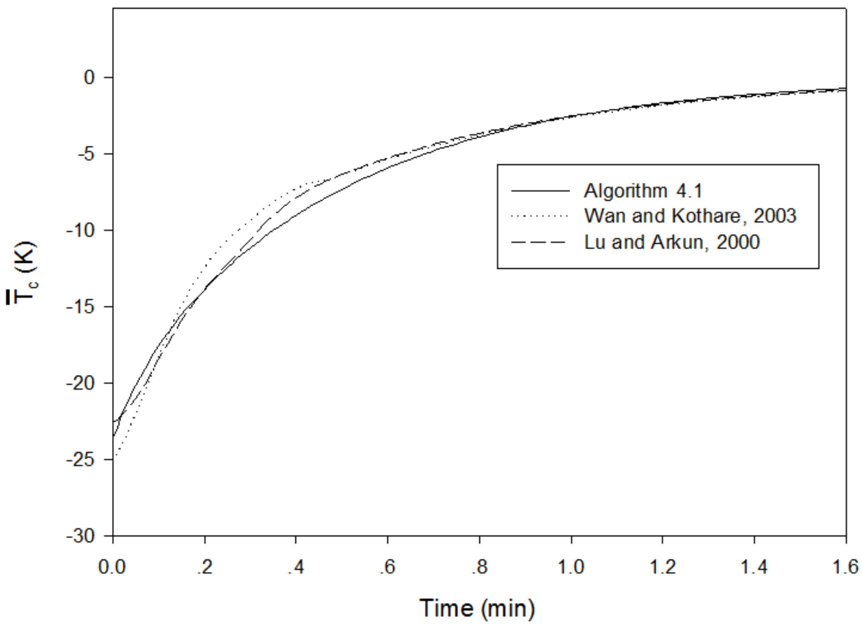
Figure 4.2 shows the closed-loop responses of the system. It is seen from the figure that the proposed algorithm can achieve less conservative result as compared with Quasi-min-max MPC algorithm of Lu and Arkun (2000) and an off-line robust MPC algorithm of Wan and Kothare (2003). Moreover, the proposed algorithm takes less time to reach and remain inside the settling band which is properly chosen as $\pm 1\%$ of $C_{\mathrm{A,equilibrium}}$ and $\pm 0.1\%$ of $T_{\mathrm{equilibrium}}$. Thus, it can be concluded that algorithm 4.1 has less settling time than an off-line MPC algorithm of Wan and Kothare (2003) and Quasi-min-max MPC algorithm of Lu and Arkun (2000). For Quasi-min-max MPC algorithm, although the scheduling parameter is measured and the optimization problem is really solved on-line, the conservative result is obtained because the algorithm is derived by using a single Lyapunov function. Moreover, it requires heavy on-line computational burden as shown in table 4.3.

Table 4.3 The on-line numerical burdens in example 4.1.

Algorithm	CPU time (s) per step
Algorithm 4.1	0.001
Wan and Kothare (2003)	0.001
Lu and Arkun (2000)	0.296



(a) The regulated output



(b) The control input

Figure 4.2 The closed-loop responses in example 4.1 (a) The regulated output (b) The control input.

For an off-line robust MPC algorithm of Wan and Kothare (2003), although the on-line computational burden is significantly reduced, the algorithm turns out to be very conservative as shown in Fig. 4.2. This is due to the fact that the nonlinear system is approximated by the polytopic uncertain system and the scheduling parameter is not taken into account in the controller synthesis. The cumulative cost $\sum_{i=0}^{\infty} x(i)^T \Theta x(i) + u(i)^T R u(i)$ is shown in table 4.4. It is seen that the proposed algorithm can achieve better control performance than an off-line robust MPC algorithm of Wan and Kothare (2003) and an on-line MPC algorithm of Lu and Arkun (2000).

Table 4.4 The cumulative cost in example 4.1.

Algorithm	Cumulative Cost
Algorithm 4.1	118.01
Wan and Kothare (2003)	122.73
Lu and Arkun (2000)	120.66

Example 4.2:

Consider the following nonlinear model for CSTR (García-Sandoval et al., 2008) where the consecutive reaction $A \longrightarrow B \longrightarrow C$ takes place.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 - Da_1 & 0 \\ Da_1 & -1 - Da_2 x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{4.14}$$

where x_1 denotes the dimensionless concentration of A, x_2 denotes the dimensionless concentration of B, the control variable u corresponds to the inlet concentration of A. The operating parameters are shown in table 4.5.

Table 4.5 The operating parameters of nonlinear CSTR in example 4.2.

Operating parameters	Value
Da_1	1
Da_2	2
$x_{2,eq}$	0.8956

Let $\bar{x}_1 = x_1 - x_{1,eq}$, $\bar{x}_2 = x_2 - x_{2,eq}$, $\bar{u} = u - u_{eq}$ where the subscript eq is used to denote the corresponding variable at equilibrium condition. We have that all the solutions of (4.14) are also the solutions of the following differential inclusion

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} \in \left(\sum_{j=1}^{2} p_j A_j \right) \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{u} \tag{4.15}$$

where A_j is given by

$$A_1 = \begin{bmatrix} -1 - Da_1 & 0 \\ Da_1 & -1 - Da_2 x_{2,\min} \end{bmatrix}, \; A_2 = \begin{bmatrix} -1 - Da_1 & 0 \\ Da_1 & -1 - Da_2 x_{2,\max} \end{bmatrix}$$

and p_j is given by

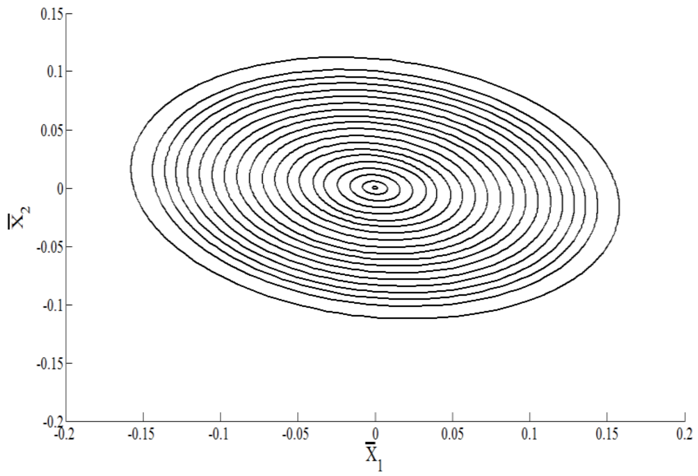
$$p_1 = \frac{x_{2,\max} - x_2}{x_{2,\max} - x_{2,\min}},\ p_2 = \frac{x_2 - x_{2,\min}}{x_{2,\max} - x_{2,\min}}$$

The objective is to regulate $\bar{x}_1$ and $\bar{x}_2$ to the origin by manipulating $\bar{u}$. The input and output constraints are given as follows

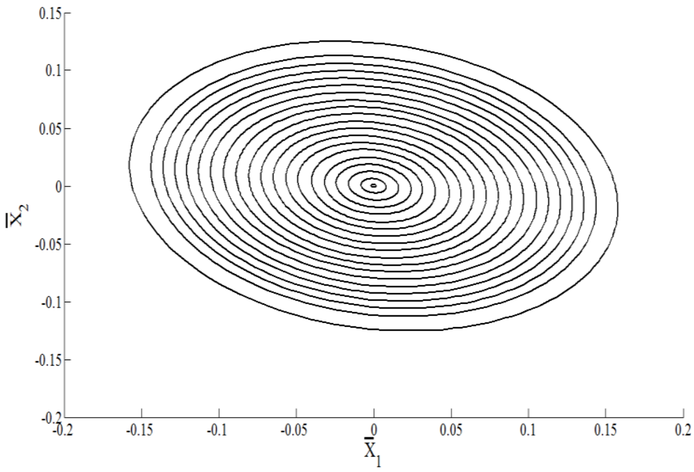
$$\left|\bar{x}_1\right| \le 0.5, \left|\bar{x}_2\right| \le 0.5, \left|\bar{u}\right| \le 0.5 \qquad (4.16)$$

The discrete-time model is obtained by discretizing (4.15) with a sampling period of 0.1 min. Here $J_\infty(k)$ is given by $J_\infty(k) = \sum_{i=0}^{\infty} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}^T \begin{bmatrix} \Theta & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} x(k+i/k) \\ u(k+i/k) \end{bmatrix}$ with $\Theta = I$ and $R = 0.01I$.

For both algorithms, the ellipsoids are constructed by choosing the same sequence of states $x_i, i \in \{1,2,\ldots.20\}$. The resulting ellipsoids are shown in Fig. 4.3.

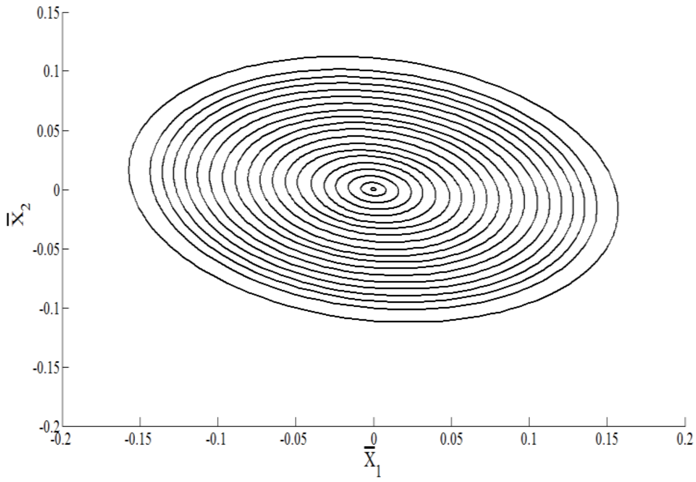


a.1) $\varepsilon_{i,1} = \{x \in \Re^n / x^T Q_{i,1}^{-1} x \le 1\},\ i \in \{1,2,\ldots.20\}$



a.2) $\varepsilon_{i,2} = \{x \in \Re^n / x^T Q_{i,2}^{-1} x \le 1\}$, $i \in \{1,2,\ldots,20\}$

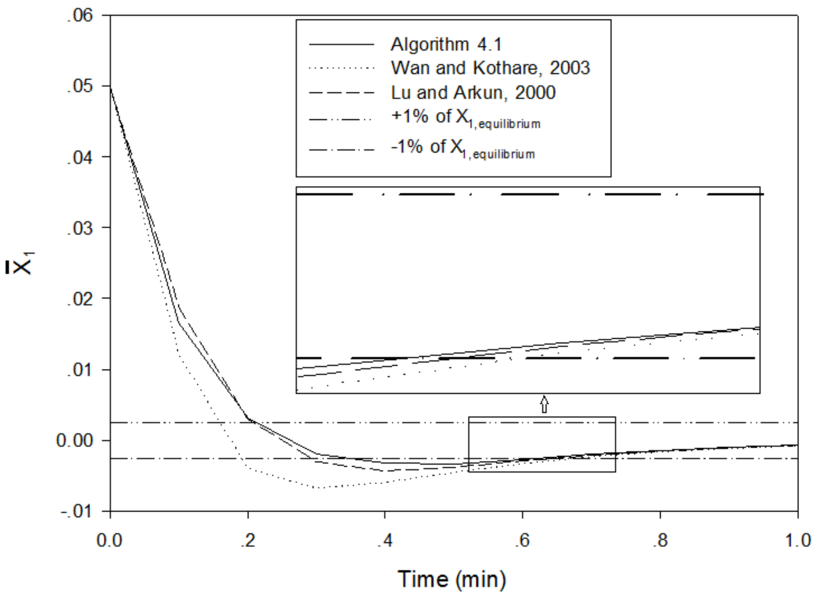
(a) Algorithm 4.1



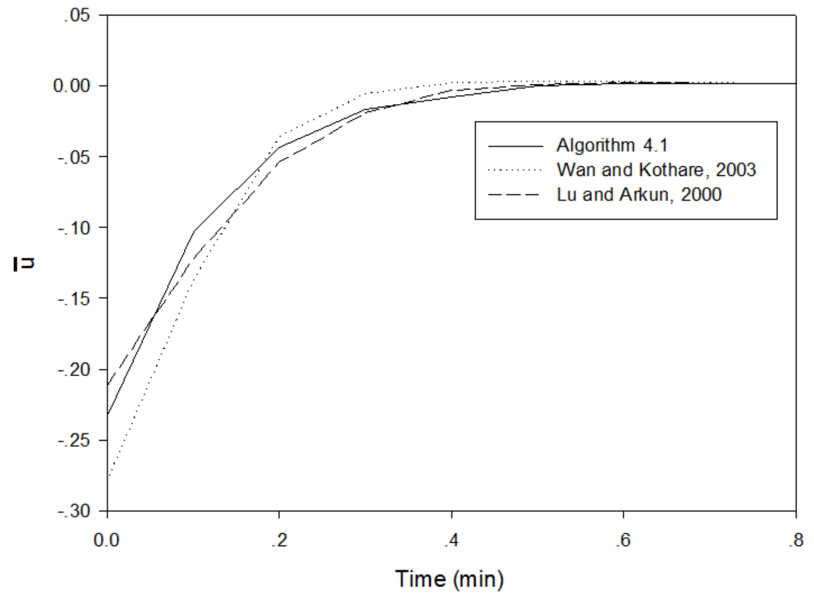
(b) An off-line robust MPC algorithm of Wan and Kothare (2003)

Figure 4.3 The ellipsoids constructed off-line by a) Algorithm 4.1 and b) An off-line robust MPC algorithm of Wan and Kothare (2003) in example 4.2.

Figure 4.4 shows the closed-loop responses of the system. It is seen the proposed algorithm outperforms other algorithms in regulating $\bar{x}_1$. Moreover, the proposed algorithm requires less time to settle within the settling band ($\pm 1\%$ of the equilibrium point). Thus, the settling time of the proposed algorithm is less than the settling times of other algorithms.



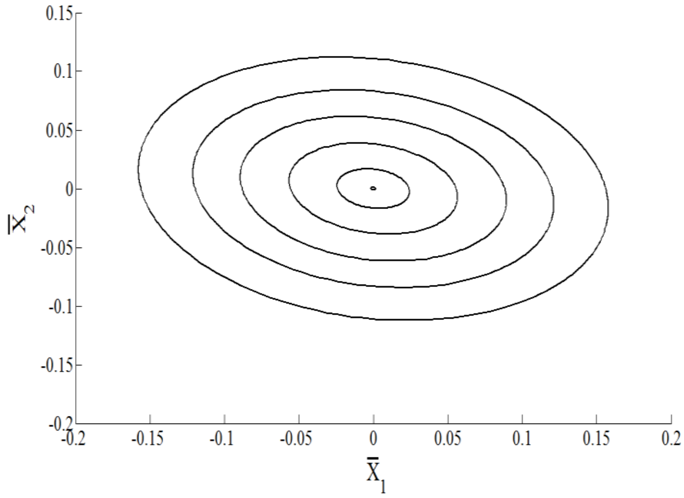
(a) The regulated output



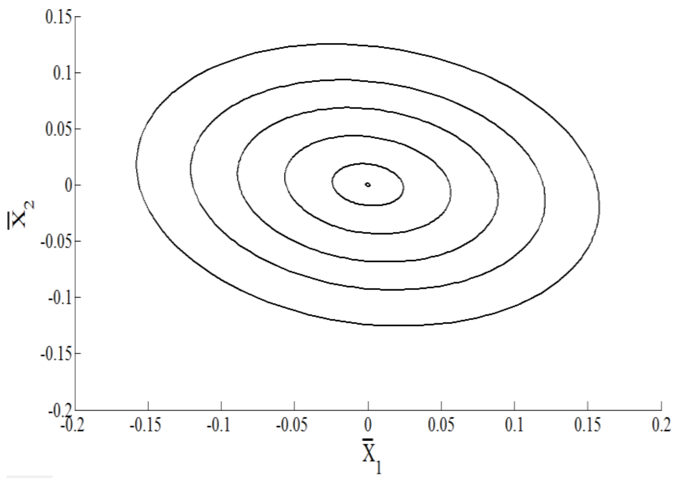
(b) The control input

Figure 4.4 The closed-loop responses in example 4.2 (a) The regulated output (b) The control input.

Then we will study effects of the number of ellipsoids constructed off-line by algorithm 4.1. Figures 4.5, 4.6 and 4.7 show the ellipsoids constructed off-line by algorithm 4.1 when the number of chosen states is varied from N = 6, 11 and 20, respectively.



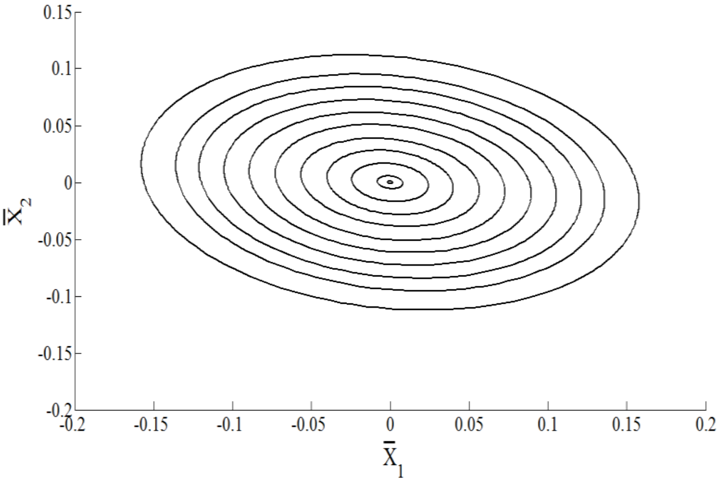
(a) $\varepsilon_{i,1} = \{x \in \Re^n / x^T Q_{i,1}^{-1} x \le 1\}$, $i \in \{1,2,\ldots.6\}$



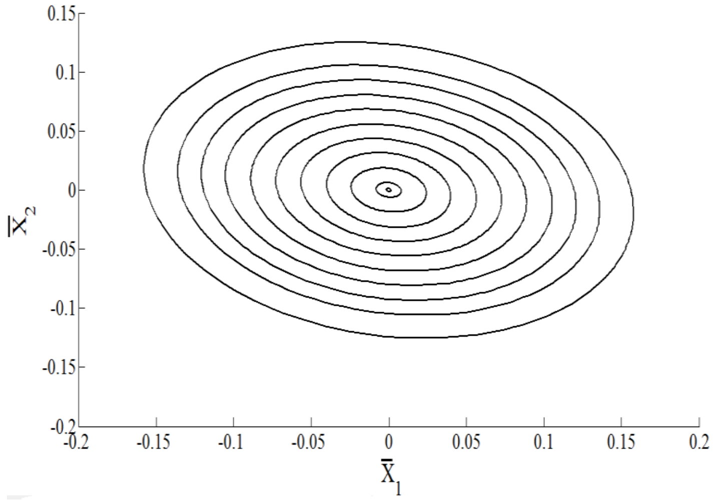
(b) $\varepsilon_{i,2} = \{x \in \Re^n / x^T Q_{i,2}^{-1} x \le 1\}$, $i \in \{1,2,\ldots.6\}$

Figure 4.5 The ellipsoids constructed off-line by algorithm 4.1

when the number of chosen states N = 6.

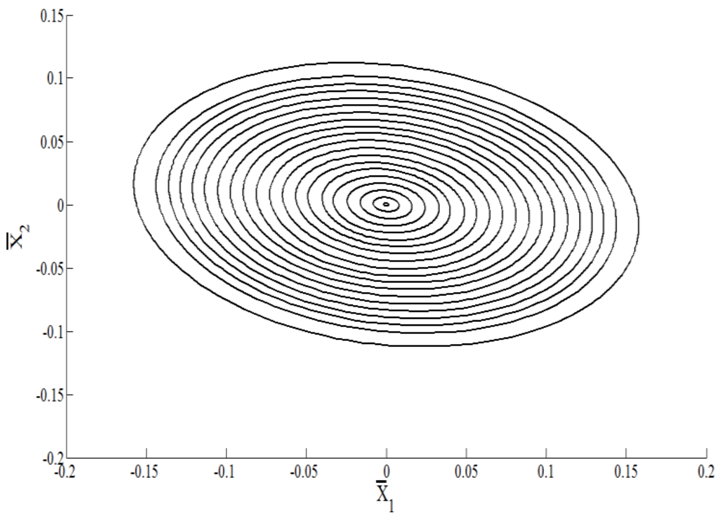


(a) $\varepsilon_{i,1} = \{x \in \Re^n / x^T Q_{i,1}^{-1} x \leq 1\}$, $i \in \{1,2,\ldots,11\}$

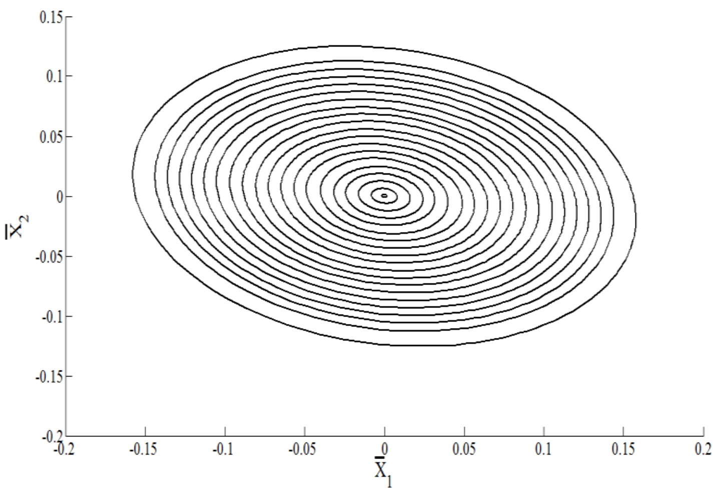


(b) $\varepsilon_{i,2} = \{x \in \Re^n / x^T Q_{i,2}^{-1} x \leq 1\}$, $i \in \{1,2,\ldots,11\}$

Figure 4.6 The ellipsoids constructed off-line by algorithm 4.1 when the number of chosen states $N = 11$.



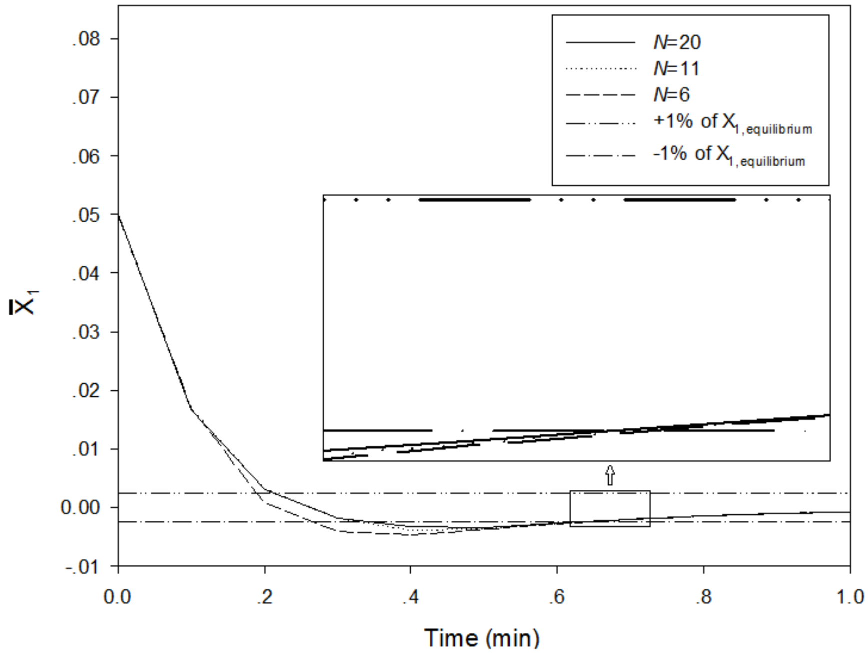
(a) $\varepsilon_{i,1} = \{x \in \Re^n / x^T Q_{i,1}^{-1} x \leq 1\}$, $i \in \{1,2,\ldots.,20\}$



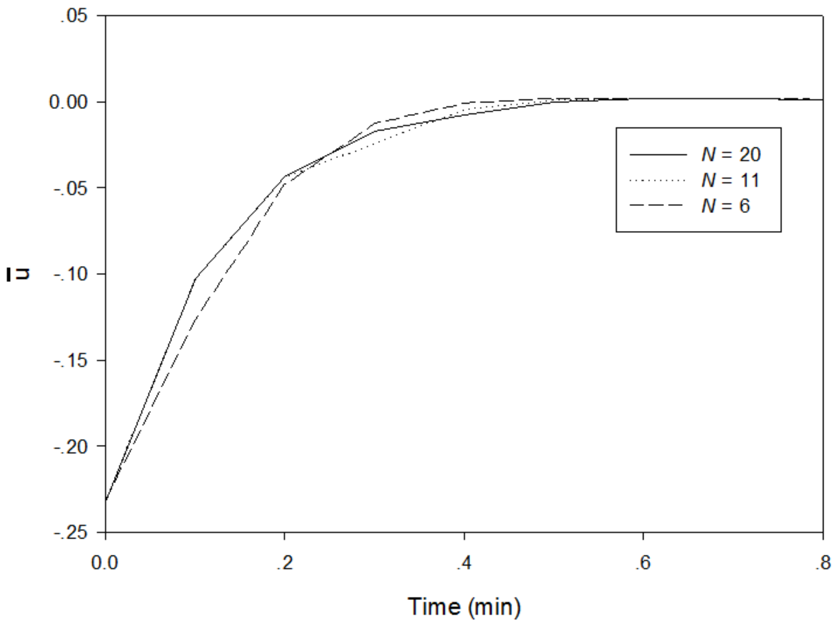
(b) $\varepsilon_{i,2} = \{x \in \Re^n / x^T Q_{i,2}^{-1} x \leq 1\}$, $i \in \{1,2,\ldots.,20\}$

Figure 4.7 The ellipsoids constructed off-line by algorithm 4.1 when the number of chosen states $N = 20$.

Figure 4.8 shows the closed-loop responses of the systems in example 4.2 when the number of chosen states is varied from N = 6, 11 and 20, respectively. It is seen that the control performance improves as the number of chosen states increases.



(a) The regulated output



(b) The control input

Figure 4.8 The closed-loop responses of the systems in example 4.2 when the number of chosen states is varied from N = 6, 11, 20, respectively.

4.3 Conclusions

In this chapter, we have presented an interpolation-based MPC for LPV systems. Most of the computational burdens are moved off-line by precomputing the sequences of state feedback gains corresponding to the sequences of nested ellipsoids. At each sampling instant, the smallest ellipsoid containing the currently measured state is determined in each sequence of ellipsoids and the scheduling parameter is measured. The real-time state feedback gain is calculated by linear interpolation between the corresponding state feedback gains. Comparisons with the existing MPC algorithms for LPV systems have been undertaken. The controller design is illustrated with two examples in chemical processes.

CHAPTER V

IMPROVING CONTROL PERFORMANCE OF OFF-LINE ROBUST MPC BASED ON A ONE-STEP STATE PREDICTION STRATEGY

An important approach to guarantee robust stability of MPC is to impose the state feedback control law on the control input. However, by doing so, the conservative result is obtained because the control input only depends on the evolution of state. Moreover, the saturation at one point in an input horizon will require a small or zero gain for all steps in the horizon.

In this chapter, a strategy to improve control performance based on a one-step state prediction strategy is presented. The conservativeness arising from imposing only the state feedback control law on the control input in chapters 3 and 4 is reduced by an addition of an element of free control input. By using the proposed strategy, the control performance is improved because the number of degrees of freedom in adjusting the plant is increased.

5.1 Introduction

In the presence of plant uncertainty, one of the main approaches to guatantee robust stability of MPC is to impose the state feedback control law on the control input. Robust MPC synthesis that allows an explicit incorporation of plant uncertainty in the problem formulation was proposed by Kothare et al. (1996). The goal is to design the state feedback gain that minimizes the worst-case performance cost. Since the state feedback control law is imposed on the control input, the conservativeness is obtained because the control input only depends on the evolution of state.

In order to reduce the conservativeness, in Casavola et al. (2000), the control performance of the robust MPC algorithm proposed by Kothare et al. (1996) is improved by using a sequence of deterministic control inputs in the N-step state prediction instead of a state feedback control law. Although the control performance is improved, the algorithm cannot guarantee robust stability because there are not enough degrees of freedom in the optimization to be able to guarantee robust stability. A correction to the algorithm of Casavola et al. (2000) was proposed by Casavola et al. (2007). Robust stability of the closed-loop system is recovered by using a control policy $u = Kx + c$ in the N-step state prediction instead of a sequence of deterministic inputs. A larger N implies better control performance at a price of higher computational load so a suitable tradeoff is required in practice.

The idea to reduce the conservativeness by using the perturbation of free control input was also presented by Schuurmans and Rossiter (2000). A sequence of free control inputs is added to the state feedback control law in order to improve the control performance. However, all of the optimization problems are solved on-line so the algorithm requires high on-line computational time.

In this chapter, a strategy to improve control performance based on a one-step state prediction strategy is presented. First of all, the proposed strategy is applied to an off-line robust MPC algorithm using polyhedral invariant sets in chapter III. Instead of implementing only a state feedback gain corresponding to the smallest polyhedral invariant set that the currently measured state can be embedded, the control performance is improved by an addition of an element of free control input calculated by minimizing a one-step state prediction cost function. At each sampling instant, only a numerically low-demanding optimization problem is needed to be solved on-line.

Then the proposed strategy is applied to an off-line MPC algorithm for LPV systems in chapter IV. Instead of implementing only a real-time state feedback gain calculated by linear interpolation between the off-line state feedback gains corresponding to the sequences of nested ellipsoids, the control performance is improved by an addition of an element of free control input. At each sampling instant, only a numerically low-demanding optimization problem is needed to be solved on-line. Moreover, the number of LMI constraints grows up only linearly with the number of vertices of the polytope.

5.2 Improving Control Performance of A Polyhedral Off-line Robust MPC Algorithm Based on A One-step State Prediction Strategy

In this section, a strategy to improve control performance of an off-line robust MPC algorithm proposed in chapter III is presented. Instead of implementing only a state feedback gain K_i corresponding to the smallest polyhedral invariant set S_i that the currently measured state can be embedded, the control performance is improved by an addition of an element of free control input c_k calculated by minimizing a one-step state prediction cost function. At each sampling time, given a state feedback gain K_i as proposed in algorithm 3.1, an element of free control input c_k is calculated by solving the following problem

$$\min_{c_k} \; x_j(k+1)^T \Theta x_j(k+1) + c_k{}^T R c_k \tag{5.1}$$

$$\text{s.t.}\, x_j(k+1) = (A_j + B_j K_i)x(k) + Bc_k, \forall j = 1,2,\ldots,L \tag{5.2}$$

$$x_j(k+1) \in S_i \tag{5.3}$$

$$\left|(K_i x(k) + c_k)_h\right| \le u_{h,\max}, \forall h \in \{1,2,\ldots,n_u\} \tag{5.4}$$

(5.1) is the one-step state prediction cost function. (5.2) is the one-step state prediction. (5.3) is for guaranteeing robust stability and (5.4) is for guaranteeing that an input constraint is satisfied. Note that the output constraint does not need to be incorporated in the problem formulation because the predicted state $x_j(k+1)$ is restricted to lie in S_i by (5.3) so it must also satisfy an output constraint $\left|(Cx_j(k+1))_r\right| \le y_{r,\max}$.

Since the optimization problem (5.1) has to be solved on-line at each sampling instant, it will be formulated as the convex optimization involving linear matrix inequalities (LMIs) that can be solved in polynomial time. The optimization problem (5.1) can be written in the form of LMIs as follows

Algorithm 5.1

$$\min_{C_k} J_0 \tag{5.5}$$

$$\text{s.t.}\, x_j(k+1) = (A_j + B_j K_i)x(k) + Bc_k, \forall j = 1,2,\ldots,L \tag{5.6}$$

$$x_j(k+1) \in S_i \tag{5.7}$$

$$\begin{bmatrix} I & * & * \\ \Theta^{\frac{1}{2}} x_j(k+1) & J_0 I_{n_x} & * \\ R^{\frac{1}{2}} c_k & 0 & J_0 I_{n_u} \end{bmatrix} \ge 0 \tag{5.8}$$

$$\begin{bmatrix} u_{h,\max}^{2} & \\ (K_i x(k) + c_k)_h & 1 \end{bmatrix} \geq 0, \forall h \in \{1,2,\ldots,n_u\} \tag{5.9}$$

Implement $u(k) = K_i x(k) + c_k$ to the process.

By applying Schur complement to (5.8), we obtain $J_o \geq x_j(k+1)^T \Theta x_j(k+1) + c_k^{\ T} R c_k$. Thus, minimizing J_o in (5.5) is equivalent to (5.1). By applying Schur complement to (5.9), (5.4) is obtained.

Example 5.1

In this example, we will consider an application of our approach to a continuous bioreactor in fermentation process. Biochemical reactors are used to produce a large number of products including pharmaceuticals, food and beverages. In this bioreactor model, only two components are considered including biomass X_B and substrate S. A fermentation process is assumed to occur in an isothermal continuous bioreactor with constant volume and constant physical-chemical properties. The maximum growth rate $\mu_{\max}$ of biomass is considered to be an uncertain parameter. The dynamic model based on the component balance is given as follows (Galluzzo et al., 2008)

$$\begin{aligned} \frac{dX_B}{dt} &= \mu(S) X_B - X_B D \\ \frac{dS}{dt} &= -\frac{\mu(S) X_B}{Y} + (S_F - S) D \\ \mu(S) &= \mu_{\max} \frac{S}{K_2 S^2 + S + K_1} \\ y &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_B \\ S \end{bmatrix} \end{aligned} \tag{5.10}$$

where X_B is the biomass concentration, S is the substrate concentration and D is the dilution rate. The operating parameters are shown in table 5.1.

Table 5.1 The operating parameters of continuous bioreactor in example 5.1.

Parameter	Value	Unit
S_F	4	kg/m^3
Y	0.4	-
$\mu_{\max}$	0.01-0.99	hr^{-1}
K_1	0.12	kg/m^3
K_2	0.45	m^3/kg

Let $\overline{X}_B = X_B - X_{B,eq}$, $\overline{S} = S - S_{eq}$ and $\overline{D} = D - D_{eq}$ where the subscript *eq* is used to denote the corresponding variable at equilibrium condition. The discrete-time model (5.11) is obtained by linearization and discretization of (5.10) using Euler first-order approximation with a sampling time of 0.2 hr.

$$\begin{bmatrix} \overline{X}_B(k+1) \\ \overline{S}(k+1) \end{bmatrix} = \begin{bmatrix} 0.1132\mu_{\max}(k) + 0.94 & 0.3414\mu_{\max}(k) \\ -0.283\mu_{\max}(k) & -0.8536\mu_{\max}(k) + 0.94 \end{bmatrix} \begin{bmatrix} \overline{X}_B(k) \\ \overline{S}(k) \end{bmatrix} + \begin{bmatrix} -0.3060 \\ 0.7651 \end{bmatrix} \overline{D}(k) \tag{5.11}$$

where $0.01 \le \mu_{\max}(k) \le 0.99$. Since the uncertain parameter $\mu_{\max}(k)$ varies between 0.01 and 0.99, we conclude that $A(k) \in \Omega$ where Ω is given as follows

$$\Omega = Co\left\{ \begin{bmatrix} 0.9411 & 0.0034 \\ -0.0028 & 0.9314 \end{bmatrix}, \begin{bmatrix} 1.0521 & 0.3380 \\ -0.2802 & 0.0949 \end{bmatrix} \right\} \tag{5.12}$$

The objective is to regulate $\overline{X}_B$ and $\overline{S}$ by manipulating $\overline{D}$. The input constraint is $\left|\overline{D}(k)\right| \le 0.015\,\text{hr}^{-1}$. The symmetric weighting matrices in (5.1) are given by $\Theta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $R = 0.1$.

Figure 5.1 shows the polyhedral invariant sets constructed off-line by algorithm 3.1. In this example, a sequence of nine polyhedral invariant sets is constructed off-line.

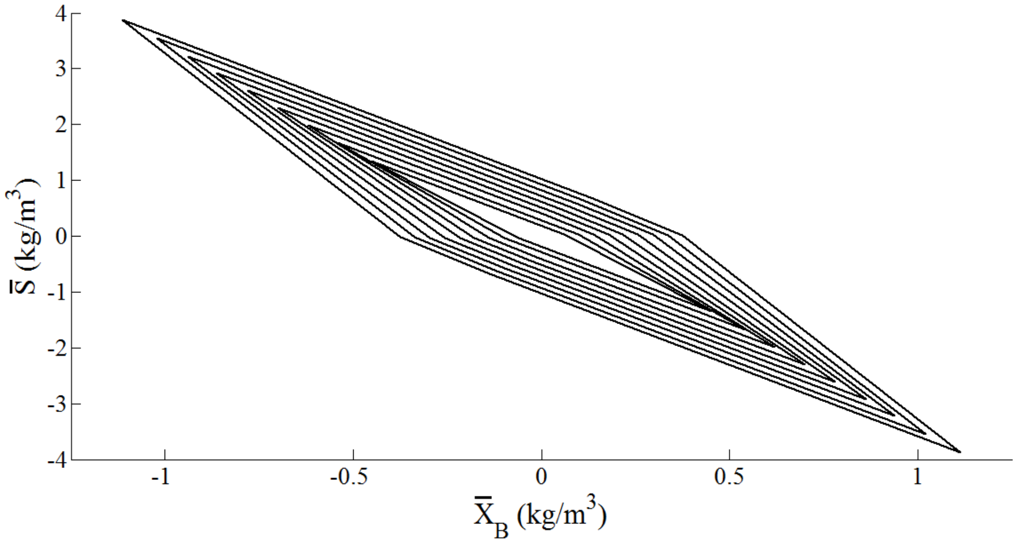
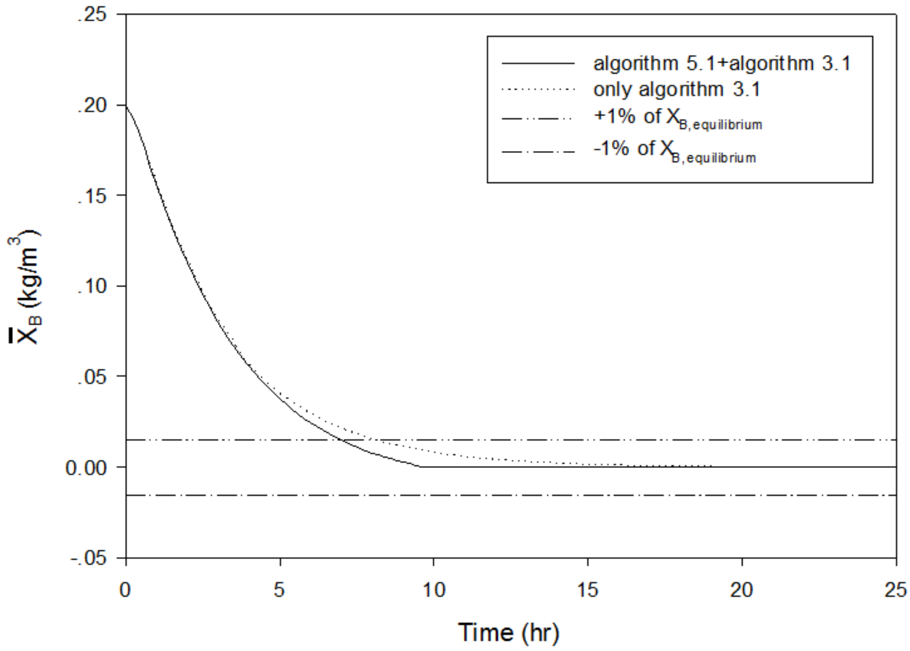
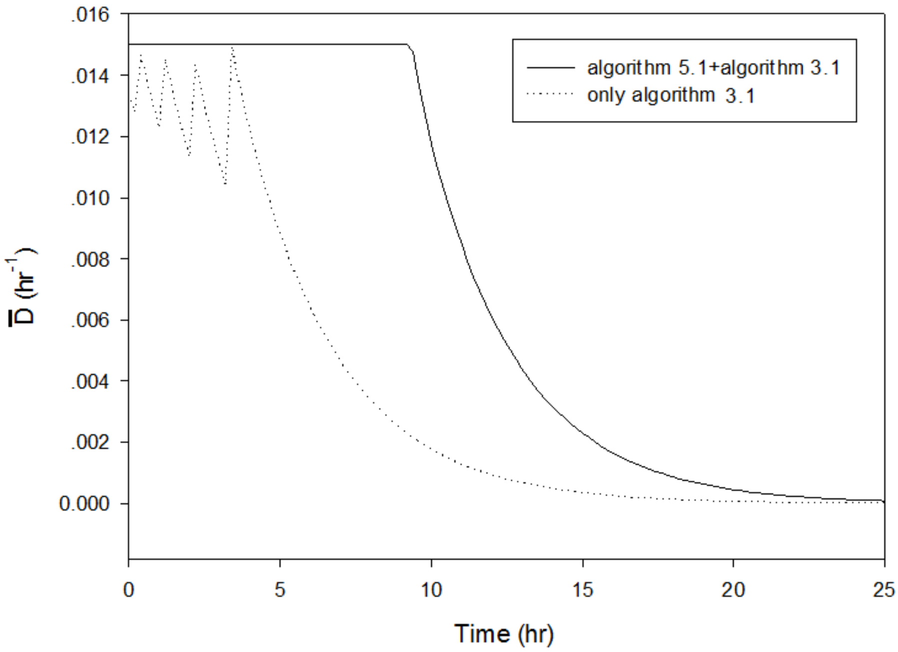


Figure 5.1 A sequence of nine polyhedral invariant sets constructed off-line in example 5.1.

Figure 5.2 (a) shows the regulated output when $\mu_{\max} = 0.5\,hr^{-1}$. It can be observed that by using the proposed strategy together with algorithm 3.1, we can achieve less conservative result as compared with using only algorithm 3.1. This is due to the fact that by adding an element of free control input to the state feedback control law $(u(k) = Kx(k) + c_k)$, we have more degrees of freedom to adjust the plant. Moreover, by using the proposed strategy, less settling time is required for the output to reach and remain inside the settling band ($\pm 1\%$ of the equilibrium point). Figure 5.2 (b) shows the control input. By using only algorithm 3.1, the input discontinuities are caused by the switching between the state feedback gains based on the distance between the state and the origin. However, the input becomes continuous by an addition of an element of free control input as proposed.



(a) The regulated output



(b) The control input

Figure 5.2 The closed-loop responses of the continuous bioreactor in example 5.1

(a) The regulated output (b) The control input.

The cumulative cost $\sum_{i=0}^{\infty} x(i)^T \Theta x(i)^T + u(i)^T Ru(i)$ is shown in table 5.2. It is seen that the cumulative cost is reduced by an addition of an element of free control input as proposed in algorithm 5.1.

Table 5.2 The cumulative cost in example 5.1.

Algorithm	The values of $\mu_{\max}$		
	0.01	0.5	0.99
algorithm 5.1+algorithm 3.1	0.227	0.361	0.404
only algorithm 3.1	0.233	0.372	0.411

5.3 Improving Control Performance of An Off-line MPC Algorithm for LPV Systems Based on A One-step State Prediction Strategy

In chapter 4, an off-line MPC algorithm for LPV systems is presented. The on-line computational time is reduced by precomputing off-line the sequences of state feedback gains $K_{i,j}, \forall i = 1,2,\ldots,N, \forall j = 1,2,\ldots,L$ corresponding to the sequences of nested ellipsoids $\varepsilon_{i,j} = \{x / x^T Q_{i,j}^{-1} x \leq 1\}$ where N is the number of ellipsoids in each sequence and L is the number of vertices of the polytope. At each sampling instant, the smallest ellipsoid containing the currently measured state is determined in each sequence of ellipsoids and the scheduling parameter $p(k)$ is measured. The real-time state feedback gain is then calculated by linear interpolation between the corresponding off-line state feedback gains $K(\alpha_i(k)) = \alpha_i(k)[\sum_{j=1}^{L} p_j(k) K_{i,j}] + (1-\alpha_i(k))[\sum_{j=1}^{L} p_j(k) K_{i+1,j}]$ where $\alpha_i(k) \in (0,1]$.

In this section, a strategy to improve control performance of an off-line MPC algorithm for LPV systems (algorithm 4.1) is presented. Instead of implementing only a real-time feedback gain $K(\alpha_i(k))$ calculated by linear interpolation between the off-line feedback gains, the control performance is improved by an addition of an element of free control input c_k. At each sampling instant, when the measured state satisfies $x(k)^T Q_{i,j} x(k) \leq 1$ and $x(k)^T Q_{i+1,j} x(k) > 1, \forall j = 1,2,\ldots,L, i \neq N$, an element of free control input c_k is calculated based on a one-step state prediction strategy as follows

$$\min_{c_k} \; x(k+1)^T \Theta x(k+1) + c_k^{\;T} R c_k \tag{5.13}$$

$$\text{s.t. } x(k+1) = (A(p(k)) + BK(\alpha_i(k)))x(k) + Bc_k \tag{5.14}$$

$$x(k+1)^T Q_{i,j}^{-1} x(k+1) \le 1, \forall j = 1,2,\ldots,L \tag{5.15}$$

$$\left|(K(\alpha_i(k))x(k) + c_k)_h\right| \le u_{h,\max}, \forall h \in \{1,2,\ldots,n_u\} \tag{5.16}$$

(5.13) is the one-step state prediction cost function, (5.14) is the one-step ahead state prediction, (5.15) is for guaranteeing robust stability and (5.16) is an input constraint. Note that the output constraint $|(Cx(k+1))_r| \le y_{r,\max}$ does not need to be incorporated in the problem formulation because the state $x(k+1)$ is restricted to lie in $\varepsilon_{i,j}$ by (5.15) and hence it must also satisfy an output constraint $|(Cx(k+1))_r| \le y_{r,\max}$.

Since the optimization problem (5.13) has to be solved on-line at each sampling instant, it will be formulated as the convex optimization involving linear matrix inequalities (LMIs) that can be solved in polynomial time. The optimization problem (5.13) can be written in the form of LMIs as follows

Algorithm 5.2

$$\min_{c_k} J_o \tag{5.17}$$

$$\text{s.t. } x(k+1) = (A(p(k)) + BK(\alpha_i(k)))x(k) + Bc_k \tag{5.18}$$

$$\begin{bmatrix} I & * & * \\ \Theta^{\frac{1}{2}} x(k+1) & J_o I_{n_x} & * \\ R^{\frac{1}{2}} c_k & 0 & J_o I_{n_u} \end{bmatrix} \ge 0 \tag{5.19}$$

$$\begin{bmatrix} 1 & * \\ x(k+1) & Q_{i,j} \end{bmatrix} \ge 0, \; \forall j = 1,2,\ldots,L \tag{5.20}$$

$$\begin{bmatrix} u_{h,\max}^2 & * \\ (K(\alpha_i(k))x(k) + c_k)_h & 1 \end{bmatrix} \ge 0, \forall h \in \{1,2,\ldots,n_u\} \tag{5.21}$$

Implement $u(k) = K(\alpha_i(k))x(k) + c_k$ to the process.

By applying Schur complement to (5.19), we obtain $J_o \geq x(k+1)^T \Theta x(k+1) + c_k^{\ T} R c_k$. Thus, minimizing J_o in (5.17) is equivalent to (5.13). By applying Schur complement to (5.20), (5.15) is obtained. Finally, by applying Schur complement to (5.21), (5.16) is obtained.

By applying the proposed strategy, the state is driven from $\varepsilon_{i,j}$ towards $\varepsilon_{i+1,j}$, and so on. Finally, the state is kept within $\varepsilon_{N,j}$ and driven towards the origin. Thus, robust stability is guaranteed. Moreover, it can be observed that only (5.20) depends on the number of vertices of the polytope. The size of the optimization problem of the proposed algorithm grows up only linearly with the number of vertices of the polytope.

Example 5.2

Consider the following nonlinear model for CSTR where the consecutive reaction $A \longrightarrow B \longrightarrow C$ takes place.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 - Da_1 & 0 \\ Da_1 & -1 - Da_2 x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \tag{5.22}$$

where x_1 denotes the dimensionless concentration of A, x_2 denotes the dimensionless concentration of B and the control variable u corresponds to the inlet concentration of A. The operating parameters are shown in table 5.3. It is assumed that $A \longrightarrow B$ is a first order chemical reaction whereas $B \longrightarrow C$ is a second order chemical reaction.

Table 5.3 The operating parameters of nonlinear CSTR in example 5.2.

Parameter	Value
Da_1	1
Da_2	2
$x_{2,eq}$	0.8956

Let $\bar{x}_1 = x_1 - x_{1,eq}$, $\bar{x}_2 = x_2 - x_{2,eq}$, $\bar{u} = u - u_{eq}$, we have that all the solutions of (5.22) are also the solutions of the following differential inclusion

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} \in \left(\sum_{j=1}^{2} p_j A_j \right) \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \bar{u} \tag{5.23}$$

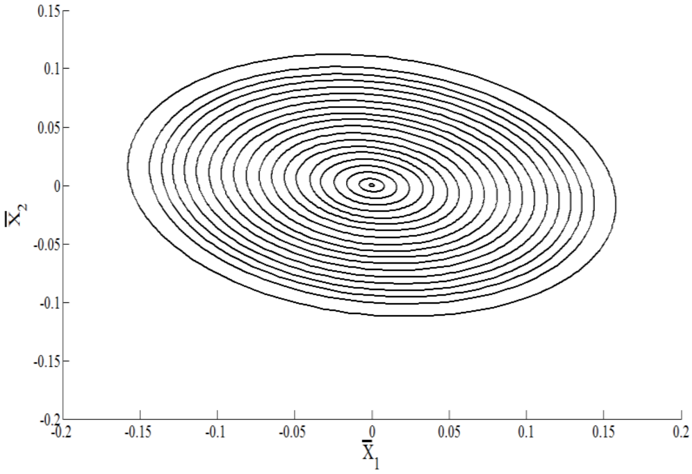
where A_j is given by $A_1 = \begin{bmatrix} -1 - Da_1 & 0 \\ Da_1 & -1 - Da_2 x_{2,\min} \end{bmatrix}$, $A_2 = \begin{bmatrix} -1 - Da_1 & 0 \\ Da_1 & -1 - Da_2 x_{2,\max} \end{bmatrix}$ and p_j is given by $p_1 = \dfrac{x_{2,\max} - x_2}{x_{2,\max} - x_{2,\min}}$, $p_2 = \dfrac{x_2 - x_{2,\min}}{x_{2,\max} - x_{2,\min}}$.

The objective is to regulate $\bar{x}_1$ and $\bar{x}_2$ to the origin by manipulating $\bar{u}$. The input and output constraints are given as follows

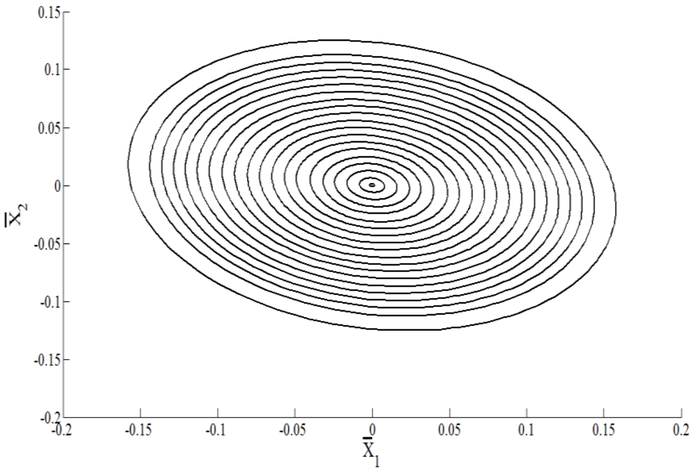
$$\begin{aligned} |\bar{x}_1| &\le 0.5 \\ |\bar{x}_2| &\le 0.5 \\ |\bar{u}| &\le 0.5 \end{aligned} \tag{5.24}$$

The discrete-time model is obtained by discretizing (5.23) using Euler first-order approximation with a sampling period of 0.1 min and it is omitted here for brevity. Here the symmetric weighting matrices in (5.13) are given by $\Theta = I$ and $R = 0.01$.

Figure 5.3 shows the ellipsoids $\varepsilon_{i,j}$ where $i = 1,2,\ldots,20$ and $j = 1,2$ constructed off-line by algorithm 4.1. In this example, two sequences of ellipsoids are constructed because the polytope has two vertices $j = 1,2$. Note that a sequence of states $x_i = 1,2,\ldots,20$ chosen to constructed the ellipsoids should be chosen such that the distance between x_{i+1} and the origin is less than the distance between x_i and the origin. This is to ensure that the ellipsoids constructed in each sequence are nested $\varepsilon_{i+1,j} \subset \varepsilon_{i,j}$.



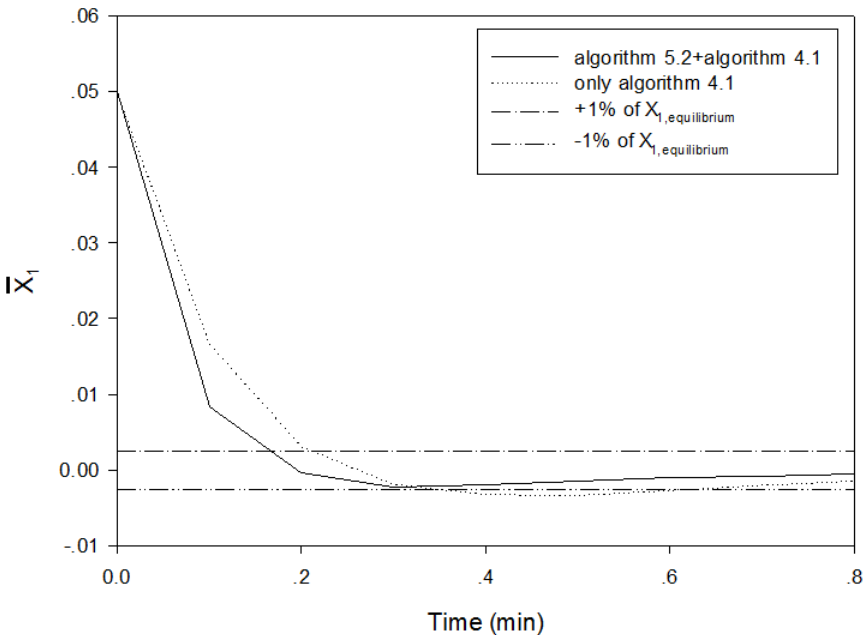
a.1) $\varepsilon_{i,1} = \{x \in \Re^n / x^T Q_{i,1}^{-1} x \le 1\}$, $i = 1,2,\ldots,20$



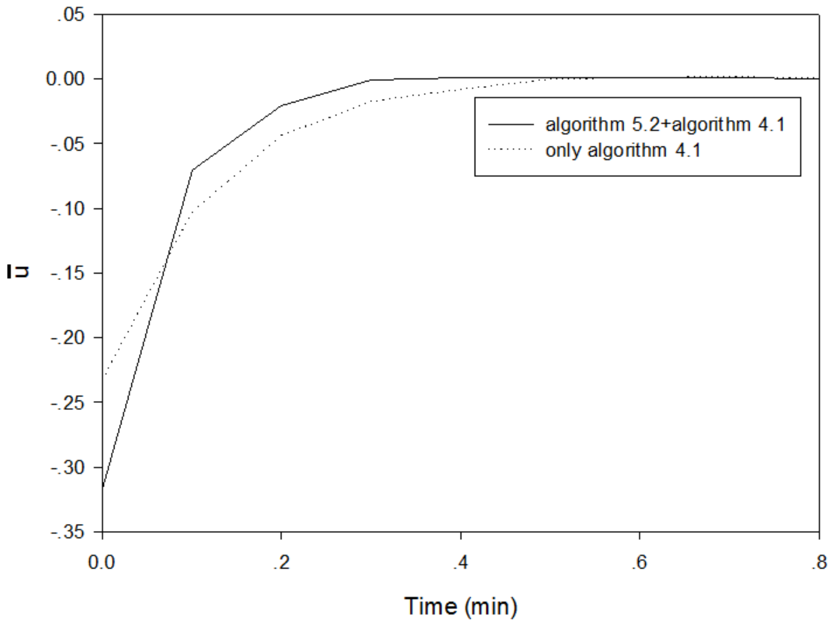
a.2) $\varepsilon_{i,2} = \{x \in \Re^n / x^T Q_{i,2}^{-1} x \le 1\}$, $i = 1,2,\ldots,20$

Figure 5.3 Two sequences of ellipsoids constructed off-line in example 5.2.

The closed-loop responses of the system are shown in Fig. 5.4. It is seen that by using the proposed strategy together with algorithm 4.1, we can achieve better control performance than using only algorithm 4.1. This is due to the fact that the degree of freedom to adjust the plant is increased. Moreover, the settling time, which is the time required for the output to enter and remain inside the settling band ($\pm 1\%$ of the equilibrium point), is also reduced. In this example, the on-line computational time at each sampling instant of the proposed strategy is only 0.01 s.



a) The regulated output



b) The control input

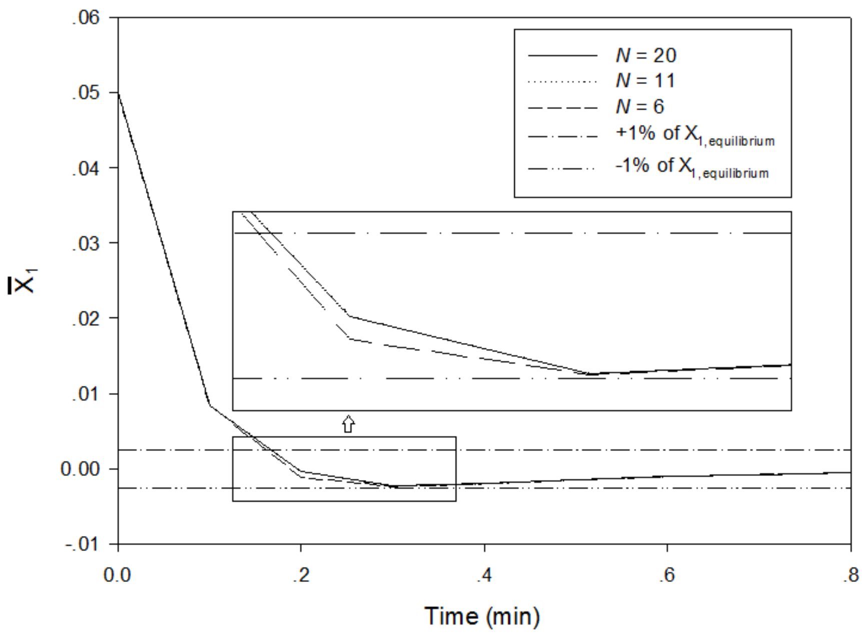
Figure 5.4 The closed-loop responses in example 5.2 a) The regulated output b) The control input.

The cumulative cost $\sum_{i=0}^{\infty} x(i)^T \Theta x(i)^T + u(i)^T Ru(i)$ is shown in table 5.4. It is seen that the cumulative cost is reduced by an addition of an element of free control input as proposed in algorithm 5.2.

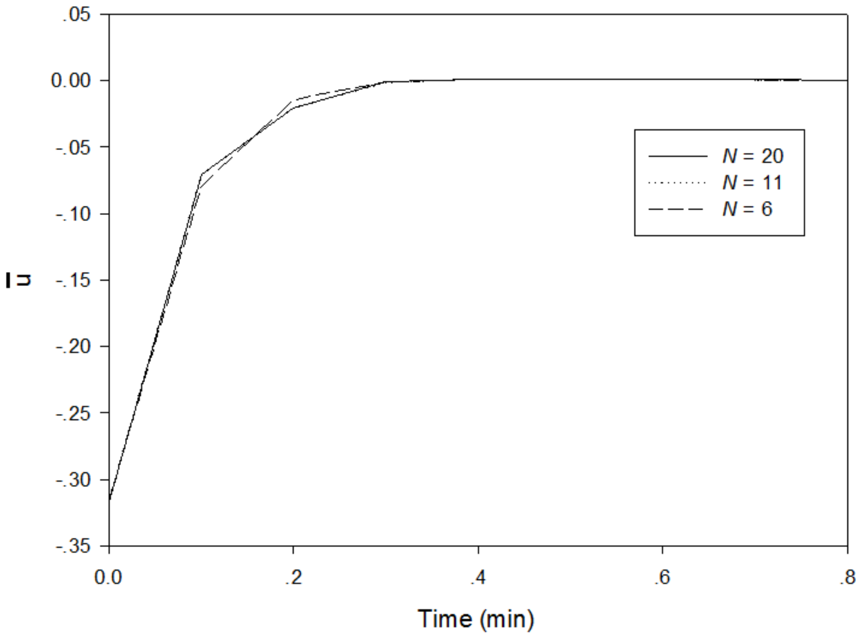
Table 5.4 The cumulative cost in example 5.2.

Algorithm	Cumulative Cost
Algorithm 5.2 + Algorithm 4.1	0.023
Only algorithm 4.1	0.024

Figure 5.5 shows the closed-loop responses when the number of ellipsoids in each sequence is varied from N = 6, 11 and 20, respectively. It is seen that by an addition of an element of free control input, almost identical behaviours are obtained for all cases so the numbers of ellipsoids constructed off-line have less effect on the control performance.



a) The regulated output



b) The control input

Figure 5.5 The closed-loop responses in example 5.2 when the number of ellipsoids constructed off-line is varied a) The regulated output b) The control input.

5.4 Conclusions

In this chapter, a strategy to improve control performance based on a one-step state prediction strategy has been presented. The conservativeness arising from imposing only a state feedback gain on the control input is reduced by an addition of an element of free control input. By using the proposed strategy, the control performance is improved because the number of degrees of freedom to adjust the plant is increased.

CHAPTER VI

CONCLUSIONS

6.1 Summary of Results

In this research, three synthesis approaches for robust model predictive control have been proposed in order to solve three important issues including the size of stabilizable region, the on-line computational complexity and the conservativeness. For all algorithms, all of the on-line computational burndens are moved off-line so the on-line computation is tractable.

Firstly, an off-line formulation of robust MPC using polyhedral invariant sets is proposed in order to deal with the problem of the size of stabilizable region. The algorithm precomputes off-line a sequence of state feedback gains corresponding to a sequence of polyhedral invariant sets. At each sampling time, the smallest polyhedral invariant set containing the measured state is determined and the corresponding state feedback gain is then implemented to the process. As compared with an off-line formulation of robust model predictive control using ellipsoidal invariant sets of Wan and Kothare (2003), the proposed algorithm gives a significantly larger stabilizable region because the true stabilizable region is calculated. Moreover, the proposed strategy can achieve better control performance. The proposed strategy also solves the problem of on-line computational complexity because all of the optimization problems are solved off-line and no optimization problem is needed to be solved on-line.

Secondly, an interpolation-based MPC strategy for LPV systems is proposed to alleviate the problem of on-line computational complexity. The on-line computational burdens are reduced by precomputing off-line the sequences of state feedback gains corresponding to the sequences of nested ellipsoids. At each sampling instant, the real-time state feedback gain is calculated by linear interpolation between the precomputed state feedback gains and no optimization problem is needed to be solved on-line. As compared with an on-line MPC algorithm for LPV systems of Lu and Arkun (2000), the proposed strategy gives the same control performance with a significantly smaller on-line computational time. Moreover, the proposed strategy can achieve better control performance as compared with an ellipsoidal off-line robust model predictive control strategy with no interpolation between state feedback gains of Wan and Kothare (2003).

Finally, the conservativeness arising from imposing the state feedback control law on the control input in order to guarantee robust stability is reduced by using a one-step state prediction strategy. The conservativeness is reduced by an addition of an element of free control input in order to increase the number of degrees of freedom to adjust the plant. At each sampling instant, only a computationally low-demanding optimization problem is needed to be solved on-line so the on-line computation is tractable. By using the proposed strategy, the conservativeness is reduced because we have more degrees of freedom to adjust the plant.

6.2 Limitations and Future Works

For an off-line formulation of robust MPC using polyhedral invariant sets proposed in chapter 3, the input discontinuities usually occur because the state feedback gains are constant in the regions between two adjacent polyhedral invariant sets. This problem can be solved by developing the technique to interpolate the state feedback gains. Another issue is the construction of polyhedral invariant set. In algorithm 3.1, non-redundant constraints are iteratively added to find the region that all future states are guaranteed to stay within this set without violation of input and output constraints. For large system with large number of vertices, more efficient approach needs to be developed in order to reduce the complexity in construction of polyhedral invariant set.

For an interpolation-based MPC strategy for LPV systems proposed in chapter 4, the stabilizable region of the algorithm is quite small in the case of asymmetric input and output constraints because it is constructed based on the ellipsoidal approximation of the true polyhedral invariant set. The size of the stabilizable region can be enlarged by calculating the true polyhedral invariant set. However, new interpolation technique suitable for the true polyhedral invariant set also needs to be developed.

Finally, in chapter 5, an element of free control input calculated on-line is added to the state feedback control law calculated off-line in order to increase the degrees of freedom in adjusting the plant. This strategy can be further improved by developing the technique to calculate both state feedback control law and an element of free control input off-line.

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APPENDICES

APPENDIX A

PROOF DETAILS OF THEOREM 3.1

In order to prove that algorithm 3.1 assures robust stability to the closed-loop, we have to prove that the state feedback gain K_i satisfies the Lyapunov stability constraint

$$\begin{aligned} & x(k+i/k)^T \left\{ [A(p(k+i)) + B(p(k+i))K_i]^T P_i [A(p(k+i)) + B(p(k+i))K_i] - P_i \right\} x(k+i/k) \\ & \leq -\left\{ x(k+i/k)^T \Theta x(k+i/k) + x(k+i/k)^T K_i^T R K_i x(k+i/k) \right\} \end{aligned} \quad \text{(A.1)}$$

From (A.1), by pre-multiplying by $x(k+i/k)^{-T}$ and post-multiplying by $x(k+i/k)^{-1}$, we obtain

$$\left\{ [A(p(k+i)) + B(p(k+i))K_i]^T P_i [A(p(k+i)) + B(p(k+i))K_i] - P_i \right\} \leq -\left\{ \Theta + K_i^T R K_i \right\} \quad \text{(A.2)}$$

By substituting $P_i = \gamma_i Q_i^{-1}$, pre-multiplying by Q_i^T, post-multiplying by Q_i, substituting $Y_i = K_i Q_i$ and applying Schur complement to the resulting inequality, we obtain

$$\begin{bmatrix} Q_i & Q_i A(p(k+i))^T + Y_i^T B(p(k+i))^T & Q_i \Theta^{\frac{1}{2}} & Y_i^T R^{\frac{1}{2}} \\ A(p(k+i))Q_i + B(p(k+i))Y_i & Q_i & 0 & 0 \\ \Theta^{\frac{1}{2}} Q_i & 0 & \gamma_i I & 0 \\ R^{\frac{1}{2}} Y_i & 0 & 0 & \gamma_i I \end{bmatrix} \geq 0 \quad \text{(A.3)}$$

This inequality is affine in $[A(p(k)), B(p(k))] = \sum_{j=1}^{L} p_j(k)[A_j, B_j]$. Thus, it is satisfied for

$$\begin{bmatrix} Q_i & * & * & * \\ A_j Q_i + B_j Y_i & Q_i & * & * \\ \Theta^{\frac{1}{2}} Q_i & 0 & \gamma_i I & * \\ R^{\frac{1}{2}} Y_i & 0 & 0 & \gamma_i I \end{bmatrix} \geq 0, \forall j = 1,2,..,L \quad \text{(A.4)}$$

From the proof, it is seen that (A.1) is equivalent to (A.4) so the state feedback gain K_i is guaranteed to satisfy the Lyapunov stability constraint (A.1) by imposing (A.4) in the optimization problem.

Since the state is guaranteed to be driven towards the origin and the input and output constraints are the closed convex set containing the origin, by iteratively adding non-redundant constraints $M_{i,m}(A_j + B_j K_i)x \le d_{i,m}$ to (M_i, d_i) by assigning $M_i = [M_i^T, (M_{i,m}(A_j + B_j K_i))^T]^T$ and $d_i = [d_i^T, d_{i,m}^T]^T$ as proposed in algorithm 3.1, we can find the set of initial states x defined by $S_i = \{x / M_i x \le d_i\}$ such that all future states are guaranteed to stay within this set without input and output constraints violation. Thus, robust constraint satisfaction is guaranteed. Note that the iteratively adding non-redundant constraints terminates in a finite number of steps because the closed-loop system is guaranteed to be robustly stabilized by the state feedback gain K_i.

APPENDIX B

PROOF DETAILS OF THEOREM 4.1

In order to prove that algorithm 4.1 assures robust stability to the closed-loop system, we have to prove that (a) the state feedback control law $K_i = \sum_{j=1}^{L} p_j(k) K_{ij}$ robustly stabilizes the closed-loop system and (b) the interpolation between the state feedback control laws $K(\alpha_i(k)) = \alpha_i(k)K_i + (1-\alpha_i(k))K_{i+1}$, $K_i = \sum_{j=1}^{L} p_j(k)K_{ij}$, $K_{i+1} = \sum_{j=1}^{L} p_j(k)K_{i+1,j}$ also robustly stabilizes the closed-loop system.

Firstly, we will prove that (a) the state feedback control law $K_i = \sum_{j=1}^{L} p_j(k)K_{ij}$ robustly stabilizes the closed-loop system. From the inequality

$$\begin{bmatrix} G_{i,j} + G_{i,j}{}^T - Q_{i,j} & * & * & * \\ A_j G_{i,j} + BY_{i,j} & Q_{i,l} & * & * \\ \Theta^{\frac{1}{2}} G_{i,j} & 0 & \gamma_i I & * \\ R^{\frac{1}{2}} Y_{i,j} & 0 & 0 & \gamma_i I \end{bmatrix} \geq 0,\ \forall j = 1,2,\ldots,L, \forall l = 1,2,\ldots,L \qquad \text{(B.1)}$$

$Q_{i,j}$ has to be positive definite. Thus, $(G_{i,j}\text{-}Q_{i,j})^T Q_{i,j}^{-1}(G_{i,j} - Q_{i,j})$ has to be non-negative. $(G_{i,j}\text{-}Q_{i,j})^T Q_{i,j}^{-1}(G_{i,j} - Q_{i,j}) \geq 0$ is equivalent to $G_{i,j}^T Q_{i,j}^{-1} G_{i,j} \geq G_{i,j}^T + G_{i,j} - Q_{i,j}$. By substituting $Y_{i,j} = K_{i,j} G_{i,j}$ and $G_{i,j}^T Q_{i,j}^{-1} G_{i,j} \geq G_{i,j}^T + G_{i,j} - Q_{i,j}$, (B.1) can be written as

$$\begin{bmatrix} G_{i,j}^T Q_{i,j}^{-1} G_{i,j} & * & * & * \\ (A_j + BK_{i,j})G_{i,j} & Q_{i,l} & * & * \\ \Theta^{\frac{1}{2}} G_{i,j} & 0 & \gamma_i I & * \\ R^{\frac{1}{2}} K_{i,j} G_{i,j} & 0 & 0 & \gamma_i I \end{bmatrix} \geq 0,\ \forall j = 1,2,\ldots,L, \forall l = 1,2,\ldots,L \qquad \text{(B.2)}$$

(B.2) is equivalent to

$$diag(\gamma_i^{-\frac{1}{2}} G_{i,j}^T, \gamma_i^{-\frac{1}{2}} Q_{i,l}, I, I) \begin{bmatrix} \gamma_i Q_{i,j}^{-1} & * & * & * \\ \gamma_i Q_{i,l}^{-1}(A_j + BK_{i,j}) & \gamma Q_{i,l}^{-1} & * & * \\ \Theta^{\frac{1}{2}} & 0 & I & * \\ R^{\frac{1}{2}} K_{i,j} & 0 & 0 & I \end{bmatrix} diag(\gamma_i^{-\frac{1}{2}} G_{i,j}, \gamma_i^{-\frac{1}{2}} Q_{i,l}, I, I) \geq 0 \qquad \text{(B.3)}$$

By substituting $P_{i,j} = \gamma_i Q_{i,j}^{-1}$ and $P_{i,l} = \gamma_i Q_{i,l}^{-1}$, we will obtain

$$\begin{bmatrix} P_{i,j} & * & * & * \\ P_{i,l}(A_j + BK_{i,j}) & P_{i,l} & * & * \\ \Theta^{\frac{1}{2}} & 0 & I & * \\ R^{\frac{1}{2}}K_{i,j} & 0 & 0 & I \end{bmatrix} \geq 0,\ \forall j = 1,2,\ldots,L, \forall l = 1,2,\ldots,L \qquad (B.4)$$

For each j, multiply the corresponding inequalities by $p_j(k+i)$ and sum for $j = 1,..,L$ to get

$$\begin{bmatrix} P(i,k) & * & * & * \\ P_{i,l}(A(p(k+i)) + BK_i(p(k+i)) & P_{i,l} & * & * \\ \Theta^{\frac{1}{2}} & 0 & I & * \\ R^{\frac{1}{2}}K_i(p(k+i)) & 0 & 0 & I \end{bmatrix} \geq 0, \forall l = 1,2,\ldots,L \qquad (B.5)$$

where $P(i,k) = \sum_{j=1}^{L} p_j(k+i)P_{i,j}$, $A(p(k+i)) = \sum_{j=1}^{L} p_j(k+i)A_j$ and $K_i(p(k+i)) = \sum_{j=1}^{L} p_j(k+i)K_{i,j}$.

For each l, multiply the corresponding inequalities by $p_l(k+i+1)$ and sum for $l = 1,..,L$ to get

$$\begin{bmatrix} P(i,k) & * & * & * \\ P(i+1,k)(A(p(k+i)) + BK_i(p(k+i)) & P(i+1,k) & * & * \\ \Theta^{\frac{1}{2}} & 0 & I & * \\ R^{\frac{1}{2}}K_i(p(k+i)) & 0 & 0 & I \end{bmatrix} \geq 0 \qquad (B.6)$$

where $P(i+1,k) = \sum_{l=1}^{L} p_l(k+i+1)P_{i,l}$. By applying twice the Schur Complement to the resulting inequality, we obtain

$$\begin{aligned} &(A(p(k+i)) + BK_i(p(k+i)))^T P(i+1,k)(A(p(k+i)) + BK_i(p(k+i))) - P(i,k) \leq \\ &-K_i(p(k+i))^T RK_i(p(k+i)) - \Theta \end{aligned} \qquad (B.7)$$

Thus, the Lyapunov function $V(i,k) = x(k+i/k)^T P(i,k)x(k+i/k)$ is a strictly decreasing function and robust stability is guaranteed.

Secondly, we will prove that (b) the interpolation between the state feedback control laws $K(\alpha_i(k)) = \alpha_i(k)K_i + (1-\alpha_i(k))K_{i+1},\ K_i = \sum_{j=1}^{L} p_j(k)K_{ij},\ K_{i+1} = \sum_{j=1}^{L} p_j(k)K_{i+1,j}$ also robustly stabilizes the closed-loop system. (B.7) is equivalent to

$$(A(p(k+i)) + BK_i(p(k+i)))^T P(i+1,k)(A(p(k+i)) + BK_i(p(k+i))) - P(i,k) < 0 \quad \text{(B.8)}$$

By substituting $P(i,k) = \sum_{j=1}^{L} p_j(k+i)P_{i,j}$, $P(i+1,k) = \sum_{l=1}^{L} p_l(k+i+1)P_{i,l}$ $A(p(k+i)) = \sum_{j=1}^{L} p_j(k+i)A_j$ and $K_i(p(k+i)) = \sum_{j=1}^{L} p_j(k+i)K_{ij}$. (B.8) can be written as

$$\begin{bmatrix} P_{i,j} & (A_j + BK_{i,j})^T P_{i,l}^T \\ P_{i,l}(A_j + BK_{i,j}) & P_{i,l} \end{bmatrix} > 0,\ \forall j = 1,2,\ldots,L, \forall l = 1,2,\ldots,L \quad \text{(B.9)}$$

By substituting $P_{i,j} = \gamma_i Q_{i,j}^{-1}$ and $P_{i,l} = \gamma_i Q_{i,l}^{-1}$, we obtain

$$\begin{bmatrix} Q_{i,j}^{-1} & (A_j + BK_{i,j})^T Q_{i,l}^{-T} \\ Q_{i,l}^{-1}(A_j + BK_{i,j}) & Q_{i,l}^{-1} \end{bmatrix} > 0,\ \forall j = 1,2,\ldots,L, \forall l = 1,2,\ldots,L \quad \text{(B.10)}$$

By applying the Schur Complement to (4.5), we obtain

$$\begin{bmatrix} Q_{i,j}^{-1} & (A_j + BK_{i+1,j})^T Q_{i,l}^{-T} \\ Q_{i,l}^{-1}(A_j + BK_{i+1,j}) & Q_{i,l}^{-1} \end{bmatrix} > 0,\ \forall j = 1,2,\ldots,L, \forall l = 1,2,\ldots,L \quad \text{(B.11)}$$

Since (B.10) is affine in $K_{i,j}$ and (B.11) is also affine in $K_{i+1,j}$, any linear combination $K(\alpha_i(k)) = \alpha_i(k)K_i + (1-\alpha_i(k))K_{i+1},\ K_i = \sum_{j=1}^{L} p_j(k)K_{ij},\ K_{i+1} = \sum_{j=1}^{L} p_j(k)K_{i+1,j}$ also satisfies

$$\begin{bmatrix} Q_{i,j}^{-1} & (A_j + BK(\alpha_i(k)))^T Q_{i,l}^{-T} \\ Q_{i,l}^{-1}(A_j + BK(\alpha_i(k))) & Q_{i,l}^{-1} \end{bmatrix} > 0,\ \forall j = 1,2,\ldots,L, \forall l = 1,2,\ldots,L \quad \text{(B.12)}$$

By substituting $P_{i,j} = \gamma_i Q_{i,j}^{-1}$ and $P_{i,l} = \gamma_i Q_{i,l}^{-1}$, we obtain

$$\begin{bmatrix} \dfrac{P_{i,j}}{\gamma_i} & (A_j + BK(\alpha_i(k)))^T (\dfrac{P_{i,l}}{\gamma_i})^T \\ \dfrac{P_{i,l}}{\gamma_i}(A_j + BK(\alpha_i(k))) & \dfrac{P_{i,l}}{\gamma_i} \end{bmatrix} > 0,\ \forall j = 1,2,\ldots,L, \forall l = 1,2,\ldots,L \quad \text{(B.13)}$$

For each j, multiply the corresponding inequalities by $p_j(k+i)$ and sum for $j = 1,..,L$. Then, for each l, multiply the corresponding inequalities by $p_l(k+i+1)$ and sum for $l = 1,..,L$. We obtain

$$\begin{bmatrix} \dfrac{P(i,k)}{\gamma_i} & (A(p(k+i)) + BK(\alpha_i(k)))^T (\dfrac{P(i+1,k)}{\gamma_i})^T \\ \dfrac{P(i+1,k)}{\gamma_i}(A(p(k+i)) + BK(\alpha_i(k))) & \dfrac{P(i+1,k)}{\gamma_i} \end{bmatrix} > 0 \quad \text{(B.14)}$$

By applying the Schur complement to (B.14), we obtain

$$(A(p(k+i)) + BK(\alpha_i(k)))^T P(i+1,k)(A(p(k+i)) + BK(\alpha_i(k))) - P(i,k) < 0 \quad \text{(B.15)}$$

Thus, $V(i,k) = x(k+i/k)^T P(i,k)x(k+i/k)$ is a strictly decreasing Lyapunov function and the closed-loop system is robustly stabilized by the state feedback gain $K(\alpha_i(k))$.

From the proof in steps (a) and (b), we conclude that algorithm 4.1 assures robust stability to the closed-loop system.

Next, we will prove that the state feedback gain $K(\alpha_i(k)) = \alpha_i(k)K_i + (1-\alpha_i(k))K_{i+1}$ assures robust constraint satisfaction. Let us begin with the input constraint

$$|u_h(k+i/k)| \le u_{h,\max}, h = 1,2,3,....,n_u \quad \text{(B.16)}$$

From algorithm 4.1, the state feedback gain K_i must satisfy

$$\begin{bmatrix} X & * \\ {Y_{i,j}}^T & G_{i,j} + {G_{i,j}}^T - Q_{i,j} \end{bmatrix} \ge 0,\ \forall j = 1,\ldots,L, X_{hh} \le u_{h,\max}^2,\ h = 1,2,\ldots,n_u \quad \text{(B.17)}$$

By substituting $G_{i,j}^T Q_{i,j}^{-1} G_{i,j} \geq G_{i,j}^T + G_{i,j} - Q_{i,j}$, (B.17) can be written as

$$\begin{bmatrix} X & * \\ Y_{i,j}{}^T & G_{i,j}^T Q_{i,j}^{-1} G_{i,j} \end{bmatrix} \geq 0,\ \forall j = 1,\ldots,L, X_{hh} \leq u_{h,\max}^2,\ h = 1,2,\ldots,n_u \qquad \text{(B.18)}$$

(B.18) is equivalent to

$$diag(I, G_{i,j}^T) \begin{bmatrix} X & * \\ G_{i,j}^{-T} Y_{i,j}{}^T & Q_{i,j}^{-1} \end{bmatrix} diag(I, G_{i,j}) \geq 0 \qquad \text{(B.19)}$$

By substituting $K_{i,j} = Y_{i,j} G_{i,j}^{-1}$, (B.19) can be written as

$$\begin{bmatrix} X & * \\ K_{i,j}{}^T & Q_{i,j}^{-1} \end{bmatrix} \geq 0 \qquad \text{(B.20)}$$

For each j, multiply the corresponding inequalities by $p_j(k+i)$ and sum for $j = 1,..,L$ to obtain

$$\begin{bmatrix} X & * \\ K_i^T & Q_i^{-1} \end{bmatrix} \geq 0 \qquad \text{(B.21)}$$

From (B.17), by following the same procedures, the state feedback gain K_{i+1} must also satisfy

$$\begin{bmatrix} X & * \\ K_{i+1}{}^T & Q_{i+1}^{-1} \end{bmatrix} \geq 0 \qquad \text{(B.22)}$$

From (B.21) and (B.22), the following inequality must be satisfied

$$\begin{bmatrix} X & * \\ K^T(\alpha_i(k)) & Q^{-1}(\alpha_i(k)) \end{bmatrix} \geq 0 \qquad \text{(B.23)}$$

where $K(\alpha_i(k)) = \alpha_i(k)K_i + (1-\alpha_i(k))K_{i+1}$ and $Q^{-1}(\alpha_i(k)) = \alpha_i(k)Q_i^{-1} + (1-\alpha_i(k))Q_{i+1}^{-1}$.

By applying the Schur complement to (B.23), we obtain

$$X \geq K(\alpha_i(k))Q(\alpha_i(k))K^T(\alpha_i(k)) \tag{B.24}$$

Since $X_{hh} \leq u^2_{h,\max},\ h = 1,2,\ldots,n_u$, (B.24) can be written as

$$u^2_{h,\max} \geq \left\|(K(\alpha_i(k))Q^{\frac{1}{2}}(\alpha_i(k)))_h\right\|^2 \tag{B.25}$$

Since $\left\|(Q^{-\frac{1}{2}}(\alpha_i(k))x(k+i/k))\right\|^2 \leq 1$, (B.25) can be written as

$$u^2_{h,\max} \geq \left|(K(\alpha_i(k))x(k+i/k))_h\right|^2 \tag{B.26}$$

By substituting $u_h(k+i/k) = (K(\alpha_i(k))x(k+i/k))_h$, we can conclude that $u_{h,\max} \geq \left|u_h(k+i/k)\right|, h = 1,2,3,\ldots,n_u$. Thus, the state feedback gain $K(\alpha_i(k)) = \alpha_i(k)K_i + (1-\alpha_i(k))K_{i+1}$ assures input constraint satisfaction.

Then we will prove that the state feedback gain $K(\alpha_i(k)) = \alpha_i(k)K_i + (1-\alpha_i(k))K_{i+1}$ assures the following output constraint satisfaction.

$$\left|y_r(k+i+1/k)\right| \leq y_{r,\max}, r = 1,2,3,\ldots,n_y \tag{B.27}$$

From algorithm 4.1, the state feedback gain K_i must satisfy

$$\begin{bmatrix} T & * \\ (A_jG_{i,j} + BY_{i,j})^TC^T & G_{i,j} + G_{i,j}{}^T - Q_{i,j} \end{bmatrix} \geq 0, \forall j = 1,2.\ldots.L, T_{rr} \leq y^2_{r,\max},\ r = 1,2,..n_y \tag{B.28}$$

By substituting $G^T_{i,j}Q^{-1}_{i,j}G_{i,j} \geq G^T_{i,j} + G_{i,j} - Q_{i,j}$, (B.28) can be written as

$$\begin{bmatrix} T & * \\ (A_jG_{i,j} + BY_{i,j})^TC^T & G^T_{i,j}Q^{-1}_{i,j}G_{i,j} \end{bmatrix} \geq 0, \forall j = 1,2.\ldots.L, T_{rr} \leq y^2_{r,\max},\ r = 1,2,..n_y \tag{B.29}$$

(B.29) is equivalent to

$$diag(I, G_{i,j}^T)\begin{bmatrix} T & * \\ (A_j + BY_{i,j}G_{i,j}^{-1})^T C^T & Q_{i,j}^{-1} \end{bmatrix} diag(I, G_{i,j}) \geq 0 \tag{B.30}$$

By substituting $K_{i,j} = Y_{i,j}G_{i,j}^{-1}$, (B.30) can be written as

$$\begin{bmatrix} T & * \\ (A_j + BK_{i,j})^T C^T & Q_{i,j}^{-1} \end{bmatrix} \geq 0 \tag{B.31}$$

For each j, multiply the corresponding inequalities by $p_j(k+i)$ and sum for $j = 1,..,L$ to obtain

$$\begin{bmatrix} T & * \\ (A + BK_i)^T C^T & Q_i^{-1} \end{bmatrix} \geq 0 \tag{B.32}$$

From (B.28), by following the same procedures, the state feedback gain K_{i+1} must also satisfy

$$\begin{bmatrix} T & * \\ (A + BK_{i+1})^T C^T & Q_{i+1}^{-1} \end{bmatrix} \geq 0 \tag{B.33}$$

From (B.32) and (B.33), the following inequality must be satisfied

$$\begin{bmatrix} T & * \\ (A + BK(\alpha_i(k)))^T C^T & Q^{-1}(\alpha_i(k)) \end{bmatrix} \geq 0 \tag{B.34}$$

where $K(\alpha_i(k)) = \alpha_i(k)K_i + (1-\alpha_i(k))K_{i+1}$ and $Q^{-1}(\alpha_i(k)) = \alpha_i(k)Q_i^{-1} + (1-\alpha_i(k))Q_{i+1}^{-1}$.

By applying the Schur complement to (B.34), we obtain

$$T \geq C(A + BK(\alpha_i(k)))Q(\alpha_i(k))(A + BK(\alpha_i(k)))^T C^T \tag{B.35}$$

Since $T_{rr} \le y_{r,\max}^2,\ r = 1,2,\ldots,n_y$, (B.35) can be written as

$$y_{r,\max}^2 \ge \left\| (C(A + BK(\alpha_i(k)))Q^{\frac{1}{2}}(\alpha_i(k)))_r \right\|^2 \tag{B.36}$$

Since $\left\| (Q^{-\frac{1}{2}}(\alpha_i(k))x(k+i/k)) \right\|^2 \le 1$, (B.36) can be written as

$$y_{r,\max}^2 \ge \left| (C(A + BK(\alpha_i(k)))x(k+i/k))_r \right|^2 \tag{B.37}$$

By substituting $y_r(k+i+1/k) = (C(A+BK(\alpha_i(k)))x(k+i/k))_r$, we can conclude that $y_{r,\max} \ge \left| y_r(k+i+1/k) \right|, r = 1,2,3,\ldots,n_y$. Thus, the output constraint is guaranteed to be satisfied.

BIOGRAPHY

Mister Pornchai Bumroongsri was born in Thailand on the 31th of January 1985. He received both his Bachelor's degree and Master's degree in Chemical Engineering from Chulalongkorn University in 2007 and 2008, respectively. From June 2003 to present, he is a Ph.D. student at Chulalongkorn University in the Department of Chemical Engineering. His main research interests are in robust model predictive control and control of linear parameter varying systems.

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