CHAPTER IV

HAMILTONIAN, ACTION AND A BROKEN SYMMETTRY

4.1 BOGOLIUBOV IDEA AND METHOD[15]

This chapter is concerned with a system of N interacting with boson particles in a volume V where both N and V are large. A Hamiltonian of this system is,

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \sum_{i \neq j}^{N} V\left(\bar{x}_i - \bar{x}_j\right) \qquad (4.1)$$

where, $V\left(\bar{x}_i - \bar{x}_j\right)$ is a pair interaction between particle i and particle j. By confining the particles in an external potential an additional external potential term is added into the above Hamiltonian. Thus, the Hamiltonian changes to a new form,

$$H_{total} = H + V_{ext} \qquad (4.2)$$

Applying the method of second quantization, the basic equation is presented in the form,

$$i\hbar \frac{\partial \psi(\bar{x})}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi(\bar{x}) + \int V(\bar{x} - \bar{y}) \psi^{+}(\bar{y}) \psi(\bar{y}) d\bar{y} \cdot \psi(\bar{x}) \qquad (4.3)$$

whereby,

$$\psi(\vec{x}) = \frac{1}{\sqrt{V}}\sum_k a_k e^{i\vec{k}\cdot\vec{x}}$$
$$\psi^+(\vec{x}) = \frac{1}{\sqrt{V}}\sum_k a_k^+ e^{-i\vec{k}\cdot\vec{x}}. \tag{4.4}$$

Here a_k and a_k^+ are conjugate operators with commutation rules of the well known type

$$[a_k, a_{k'}] = [a_k^+, a_{k'}^+] = 0 \tag{4.5}$$

and
$$[a_k, a_{k'}^+] = \delta(k - k') \tag{4.6}$$

and $\{e^{i\vec{k}\cdot\vec{x}}\}$ is a complete orthonormal set of functions

$$\frac{1}{(2\pi\hbar)^{3/2}}\int e^{-i\vec{k}'\cdot\vec{x}} e^{i\vec{k}\cdot\vec{x}} d\vec{x} = \delta(k - k'). \tag{4.7}$$

The operator $n_k = a_k^+ a_k$ represents the number of particles with momentum $\vec{k}$. For finite volume V, vector $\vec{k}$ is obviously quantized under usual boundary conditions of the periodicity type,

$$k = \frac{2\pi n}{L}, \tag{4.8}$$

where, n are integer and L denotes the length of side of a cube of the volume V.

However, since thermodynamic volume properties of the system, such as when $N \to \infty$ is to be examined, the boundary of the volume tends to infinity, $V \to \infty$, but the volume per particle, $v = \frac{V}{N}$, remains constant. Therefore, the final results to a continuous spectrum shall be passed over, replacing the summations $\sum_k F(k)$ by integrals

$$\frac{V}{(2\pi)^3}\int F(k)d\bar{k} \tag{4.9}$$

If, when turning to the formulation of an approximation method, there is no interaction at all, zero temperature could be put at $n_0 = N, n_k = 0, (k \neq 0)$. However, in the case under consideration, the system is in a weakly excited state with low energy, and, therefore, these relationships are only approximately valid because zero momentum is the limiting factor for particles in the ground state.

Thus, the approximation method, based on the above considerations, are:

(1) Since $n_0 = a_0^+ a_0$ is very large compared with unity, the expression

$$a_0 a_0^+ - a_0 a_0^+ = 1, \tag{4.10}$$

is small compared with a_0^+, a_0. However, using the Bogoliubov postulate they can be treated as ordinary numbers and their non-commutability can be neglected.

(2) Replacing

$$\psi = \frac{a_0}{\sqrt{V}} + \theta \tag{4.11}$$

where, $\theta = \frac{1}{\sqrt{V}}\sum_{k \neq 0} a_k e^{i\bar{k}\cdot x}$

and considing θ as a correction term of the first order. If ψ is substituted into equation (4.3), and all the terms involve the second and higher powers of θ are neglected, this is permissible since the condition is a weak excitation. Following basic approximation ideas these equations were obtained:

$$i\hbar\frac{\partial\theta}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\theta+\frac{n_0}{V}\phi_0\theta+\frac{n_0}{V}\int\phi(|\bar{x}-\bar{x}'|)\theta(\bar{x}')d\bar{x}'+\frac{a_0^2}{V}\int\phi(|\bar{x}-\bar{x}'|)\theta^+(\bar{x}')d\bar{x}', \quad (4.12)$$

and

$$i\hbar\frac{\partial a_0}{\partial t}=\frac{n_0}{V}\phi_0 a_0, \quad (4.13)$$

where,

$$\phi_0=\int\phi(|\bar{x}|)d\bar{x}. \quad (4.14)$$

Passing from the operator wave function θ to operator amplitudes allows for the application of Fourier's expansion,

$$\phi(|\bar{x}-\bar{x}'|)=\frac{1}{V}\sum_k e^{i\bar{k}\cdot(x-x')}\theta(k) \quad (4.15)$$

From the radial symmetry of this potential function its follows that the amplitudes of this expansion

$$\theta(k)=\int\phi(|\bar{x}|)e^{-i\bar{k}\cdot x}d\bar{x} \quad (4.16)$$

depend upon the length $|\bar{k}|$ of vector $\bar{k}$ only.

Substitution of equation (4.16) into equation (4.12) gives,

$$i\hbar\frac{\partial a_k}{\partial t}=\left\{-\frac{\hbar^2}{2m}\nabla^2+\frac{n_0}{V}\phi_0+\frac{n_0}{V}\theta(k)\right\}a_k+\frac{a_0^2}{V}\theta(k)a_{-k}^+ \tag{4.17}$$

If a new variable, b_k is set, that is time evolution of a_k,

$$a_k=Exp\left(-\frac{i}{\hbar}E_0t\right)b_k,\quad a_0=Exp\left(-\frac{i}{\hbar}E_0t\right)b_0 \tag{4.18}$$

where, $E_0=\frac{n_0}{V}\phi_0$

which results in

$$i\hbar\frac{\partial b_k}{\partial t}=\left\{T(k)+\frac{n_0}{V}\theta(k)\right\}b_k+\frac{b_0^2}{V}\theta(k)b_{-k}^+ \tag{4.19}$$

$$-i\hbar\frac{\partial b_{-k}^+}{\partial t}=\left(\frac{b_0^+}{V}\right)\theta(k)b_k+\left\{T(k)+\frac{n_0}{V}\theta(k)\right\}b_{-k}^+ \tag{4.20}$$

where, $T(k)=-\frac{\hbar^2}{2m}\nabla^2$

Solving the system of two differential equations with constant coefficients, it was found that the operators b_k, b_k^+ depend upon time by means of a linear combination of exponentials, $Exp\left[\pm(i/\hbar)E(k)t\right]$

where,

$$E(k)=\sqrt{2T(k)\frac{n_0}{V}\theta(k)+(T(k))^2} \tag{4.21}$$

observing that the inequality

$$\theta(0)=\int \phi(|\bar{x}|)d\bar{x} \quad > 0 \tag{4.22}$$

implies the positiveness of the expression under the sign of the radical $E(k)$. If $\theta(k)<0$, the expression of $E(k)$ is negative for small momenta and, therefore, $E(k)$ receives complex value. b_k and b_k^+, increasing with time, whence it follows that the states with small $n_k = b_k^+ b_k$ are unstable. In order to be sure of excited states stability, restrict the class of possible types of interaction forces and the inequality equation (4.22) will be satisfied for all types. It is interesting to note that the inequality equation (4.22) represents the condition of thermodynamic stability of a system at absolute zero, or below critical temperature.

4.2 HOW TO WRITE A HAMILTONIAN OF THE SYSTEM

The above conclusion of Bogoliubov for a uniform system is quite clear. In applying it to a non-uniform system, declared by Hamiltonian,

$$H=\frac{\hbar^2}{2m}\int \nabla\psi^+(\bar{x})\nabla\psi(\bar{x})d\bar{x}+\frac{1}{2}\int U(\bar{x},\bar{y})\psi^+(\bar{x})\psi^+(\bar{y})\psi(\bar{x})\psi(\bar{y})d\bar{x}d\bar{y} \tag{4.23}$$

where, $U(\bar{x},\bar{y})$ is the pair potential function.

Then include the idea of representing every field ψ that is integrated over as a sum of two terms, Popov[17]. One term will be called the slowly oscillating field ψ_0 and the other the rapidly oscillating field ψ_1

$$\psi = \psi_0 + \psi_1 \tag{4.24}$$

Therefore, the slow part and the rapid can be separated by cutting off momentum k_0,

$$\psi_0(\bar{x},t) = \frac{1}{\sqrt{\beta V}} \sum_{\omega,|\bar{q}|<|\bar{k}_0|} a(\bar{q},\omega) e^{i(\omega t + \bar{q}\cdot\bar{x})} \tag{4.25}$$

$$\psi_1(\bar{x},t) = \frac{1}{\sqrt{\beta V}} \sum_{\omega,|\bar{k}|\geq|\bar{k}_0|} b(\bar{k},\omega) e^{i(\omega t + \bar{k}\cdot\bar{x})} \tag{4.26}$$

and the Hamiltonian in terms of the functional of ψ_0 and ψ_1 can be expressed as,

$$\begin{aligned}
H = \frac{\hbar^2}{2m}\int &[\psi_0^+(\bar{x},t)\psi_0(\bar{x},t) + \psi_1^+(\bar{x},t)\psi_1(\bar{x},t) + \psi_0^+(\bar{x},t)\psi_1(\bar{x},t) + \psi_1^+(\bar{x},t)\psi_0(\bar{x},t)]d\bar{x} \\
&+ \frac{1}{2}\int d\bar{x}d\bar{x}'U(\bar{x},\bar{x}')\ \{\psi_0^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_0(\bar{x},t) \\
&+ 2\psi_1^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_0(\bar{x},t) + 2\psi_0^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_1(\bar{x},t) \\
&+ 2\psi_1^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_1(\bar{x},t) + 2\psi_1^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_1(\bar{x}',t)\psi_0(\bar{x},t) \\
&+ \psi_0^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_1(\bar{x}',t)\psi_1(\bar{x},t) + \psi_1^+(\bar{x},t)\psi_1^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_0(\bar{x},t) \\
&+ 2\psi_0^+(\bar{x},t)\psi_1^+(\bar{x}',t)\psi_1(\bar{x}',t)\psi_1(\bar{x},t) + 2\psi_1^+(\bar{x},t)\psi_1^+(\bar{x}',t)\psi_1(\bar{x}',t)\psi_0(\bar{x},t) \\
&+ \psi_1^+(\bar{x},t)\psi_1^+(\bar{x}',t)\psi_1(\bar{x}',t)\psi_1(\bar{x},t)\}
\end{aligned} \tag{4.27}$$

Earlier it was stated that the Hamiltonian concept consists of two parts:Bogoliubov Hamiltonian part H_B [15], and an additional part introduced in this study, from here on called H_A. So,

$$H = H_B + H_A \tag{4.28}$$

where,

$$H_B = \frac{\hbar^2}{2m}\int\left[\psi_0^+(\bar{x},t)\psi_0(\bar{x},t)+\psi_1^+(\bar{x},t)\psi_1(\bar{x},t)\right]d\bar{x}$$
$$+\frac{1}{2}\int d\bar{x}d\bar{x}'U(\bar{x},\bar{x}')\,\{\psi_0^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_0(\bar{x},t)$$
$$+2\psi_1^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_0(\bar{x},t)+2\psi_0^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_1(\bar{x},t)$$
$$+2\psi_1^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_1(\bar{x},t)+2\psi_1^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_1(\bar{x}',t)\psi_0(\bar{x},t)\} \quad (4.29)$$

and,

$$H_A = \frac{1}{2}\int d\bar{x}d\bar{x}'U(\bar{x},\bar{x}')\{\psi_0^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_1(\bar{x}',t)\psi_1(\bar{x},t)$$
$$+\psi_1^+(\bar{x},t)\psi_1^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_0(\bar{x},t)$$
$$+2\psi_0^+(\bar{x},t)\psi_1^+(\bar{x}',t)\psi_1(\bar{x}',t)\psi_1(\bar{x},t)$$
$$+2\psi_1^+(\bar{x},t)\psi_1^+(\bar{x}',t)\psi_1(\bar{x}',t)\psi_0(\bar{x},t)$$
$$+\psi_1^+(\bar{x},t)\psi_1^+(\bar{x}',t)\psi_1(\bar{x}',t)\psi_1(\bar{x},t)\} \quad (4.30)$$

From the fact that this system is lying at extra low temperature limits and has a tendency to the excited state it is possible to neglect the last three terms in equation (4.30). Therefore, for this purpose, the new Hamiltonian is,

$$H = H_B + H_A \quad (4.31)$$

where, the approximated H_A is,

$$H_A = \frac{1}{2}\int d\bar{x}d\bar{x}'U(\bar{x},\bar{x}')\{\psi_0^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_1(\bar{x}',t)\psi_1(\bar{x},t)$$
$$+\psi_1^+(\bar{x},t)\psi_1^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_0(\bar{x},t)\} \quad (4.32)$$

4.3 HOW TO CALCULATE AN ACTION

Considering an action functional for the system it is found that,

$$\tilde{S} = \int_0^\beta dt \int \psi^+(\bar{x},t)\partial_t\psi(\bar{x},t)d\bar{x} - \int_0^\beta H dt \tag{4.33}$$

where, H is a generalized Hamiltonian. Looking at the Hamiltonian from the previous section, the action can be written as,

$$\tilde{S} = \int_0^\beta dt \int d\bar{x}\, [\psi_0^+(\bar{x},t)\partial_t\psi_0(\bar{x},t) + \psi_1^+(\bar{x},t)\partial_t\psi_1(\bar{x},t) + \psi_0^+(\bar{x},t)\partial_t\psi_1(\bar{x},t)$$
$$+ \psi_1^+(\bar{x},t)\partial_t\psi_0(\bar{x},t)] - \int_0^\beta H dt \tag{4.34}$$

From this approximation, it is true that this system belongs to the very low temperature regime with a small k regime, allowing for neglection of the terms which have power of ψ_1 and/or ψ_1^+ large than 2. If the last three terms in the equation (4.27) are neglected, then,

$$\tilde{S} = \int d\bar{x} \int_0^\beta [\psi_0^+(\bar{x},t)\partial_t\psi_0(\bar{x},t) + \psi_1^+(\bar{x},t)\partial_t\psi_1(\bar{x},t) + \psi_0^+(\bar{x},t)\partial_t\psi_1(\bar{x},t) + \psi_1^+(\bar{x},t)\partial_t\psi_0(\bar{x},t)]dt$$
$$-\frac{\hbar^2}{2m}\int d\bar{x} \int_0^\beta [\psi_0^+(\bar{x},t)\psi_0(\bar{x},t) + \psi_1^+(\bar{x},t)\psi_1(\bar{x},t) + \psi_0^+(\bar{x},t)\psi_1(\bar{x},t) + \psi_1^+(\bar{x},t)\psi_0(\bar{x},t)]dt$$
$$-\frac{1}{2}\int d\bar{x}d\bar{x}' U(\bar{x},\bar{x}') \int_0^\beta \{\psi_0^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_0(\bar{x},t)$$
$$+ 2\psi_1^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_0(\bar{x},t) + 2\psi_0^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_1(\bar{x},t)$$
$$+ 2\psi_1^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_1(\bar{x},t) + 2\psi_1^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_1(\bar{x}',t)\psi_0(\bar{x},t)$$
$$+ \psi_0^+(\bar{x},t)\psi_0^+(\bar{x}',t)\psi_1(\bar{x}',t)\psi_1(\bar{x},t) + \psi_1^+(\bar{x},t)\psi_1^+(\bar{x}',t)\psi_0(\bar{x}',t)\psi_0(\bar{x},t)\}dt \tag{4.35}$$

The above defining in equation (4.25) and equation (4.26) required that $\psi(\bar{x},t)$ to be a periodic function of variable t. Substitution of ψ_0 and ψ_1 from equation (4.25) and equation (4.26) into equation (4.35) give a new form of $\tilde{S}$ in momentum representation. The method of transformation has shown below but we would like to show the transformation of some terms, i.e. a term from kinetic part and a term from interaction part.

(1) The transformation method for any term from kinetic part:

$$\int dt\int d\bar{x}\left(\psi_0^+(\bar{x})\partial_t\psi_0(\bar{x})-\frac{\hbar^2}{2m}\partial_x\psi_0^+(\bar{x})\partial_x\psi_0(\bar{x})\right)$$
$$=\frac{1}{\beta V}\int dt\int\left[\sum_{q,q'}e^{i(q-q').x}a_{q'}^{*}\partial_t a_q+\sum_{q,q'}i\omega e^{i(q-q').x}a_{q'}^{*}a_q\right]dx$$
$$=\frac{1}{\beta V}\int dt\left[\sum_q a_q^{*}\partial_t a_q+i\omega\sum_q a_q^{*}a_q\right] \tag{4.36}$$

where, a_q and b_k are coefficient operators of the slow part and the rapid part, respectively. The above equation has used the definition $\psi_{1,0}(\bar{x})=\psi_{1,0}(\bar{x},t)$.

(2) The transformation method for any term from interaction part.

$$\int dt \int d\bar{x}d\bar{y}\,\psi_1^+(\bar{x})\psi_0^+(\bar{y})\psi_0(\bar{y})\psi_1(\bar{x})g(\bar{x},\bar{y}) = \frac{1}{(\beta V)^2}\int dt \int d\bar{x}d\bar{y}\left\{\sum_{k_1} b_{k_1}^+ e^{-i(\bar{k}_1\bar{x}+\omega t)}\sum_{q_2} a_{q_2}^+ e^{-i(\bar{q}_2\bar{y}+\omega t)}\right.$$

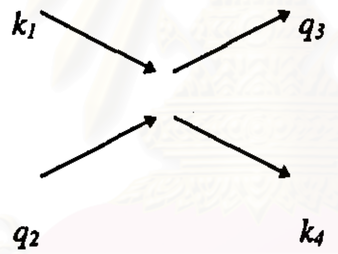
$$\left.\times\sum_{q_3} a_{q_3} e^{i(\bar{q}_3\bar{y}+\omega t)}\sum_{k_4} b_{k_4} e^{i(\bar{k}_4\bar{x}+\omega t)}\sum_{k_5,k_6} g(\bar{k}_5,\bar{k}_6)e^{i\bar{k}_5\bar{x}-i\bar{k}_6\bar{y}}\right\}$$

$$=\frac{1}{(\beta V)^2}\int dt\int d\bar{x}d\bar{y}\left\{\sum_{k_1,q_2,q_3,k_4} b_{k_1}^+ a_{q_2}^+ a_{q_3} b_{k_4} e^{i(\bar{k}_4-\bar{k}_1+\bar{k}_5)\bar{x}+i(\bar{q}_3-\bar{q}_2-\bar{k}_6)\bar{y}}\sum_{k_5,k_6} g(\bar{k}_5,\bar{k}_6)\right\}$$

$$=\frac{1}{(\beta V)^2}\int dt\left\{\sum_{k_1,q_2,q_3,k_4} b_{k_1}^+ a_{q_2}^+ a_{q_3} b_{k_4}\,\delta(\bar{k}_4-\bar{k}_1+\bar{k}_5)\delta(\bar{q}_3-\bar{q}_2-\bar{k}_6)\sum_{k_5,k_6} g(\bar{k}_5,\bar{k}_6)\right\} \qquad (4.37)$$

Two selection rules from the conservation of momentum are considered:

$$k_5 + k_4 - k_1 = 0 \quad \text{and} \quad k_6 - q_3 + q_2 = 0.$$



Let $\sum_{k_5,k_6} g(\bar{k}_5,\bar{k}_6)\delta(\bar{k}_4-\bar{k}_1+\bar{k}_5)\delta(\bar{q}_3-\bar{q}_2-\bar{k}_6) = \langle \bar{q}_3\bar{q}_2 |g| \bar{k}_1\bar{k}_4\rangle$ (4.38)

Finally,

$$\int dt\int d\bar{x}d\bar{y}\,\psi_1^+(\bar{x})\psi_0^+(\bar{y})\psi_0(\bar{y})\psi(\bar{x})_1 g(\bar{x},\bar{y}) = \frac{1}{(\beta V)^2}\int dt \sum_{k_1,q_2,q_3,k_4} b_{k_1}^+ a_{q_2}^+ a_{q_3} b_{k_4}\langle \bar{q}_3\bar{q}_2 |g| \bar{k}_1\bar{k}_4\rangle$$

(4.39)

For the others terms the method of transformation is similar to the above method.

Finally, the functional $\tilde{S}$ becomes,

$$\tilde{S} = \frac{1}{\beta V}\int dt\left\{\sum_q\left[a_q^{*}\partial_t a_q - \frac{q^2}{2m}a_q^{*}a_q\right]\right\} + \frac{1}{\beta V}\int dt\left\{\sum_k\left[b_k^{*}\partial_t b_k - \frac{k^2}{2m}b_k^{*}b_k\right]\right\}$$
$$-\frac{1}{2}\frac{1}{(\beta V)^2}\int dt\left\{\sum_q a^{+}_{q_1}a^{+}_{q_2}a_{q_3}a_{q_4}\langle\vec{q}_1\vec{q}_2|g|\vec{q}_3\vec{q}_4\rangle + 2\sum_{k,q} b^{+}_{k_1}a^{+}_{q_2}a_{q_2}a_{q_3}\langle\vec{k}_1\vec{q}_2|g|\vec{q}_3\vec{q}_4\rangle\right.$$
$$+2\sum_{k,q} a^{+}_{q_1}a^{+}_{q_2}a_{q_3}b_{k_4}\langle\vec{q}_1\vec{q}_2|g|\vec{q}_3\vec{k}_4\rangle + \sum_{k,q} a^{+}_{q_1}a^{+}_{q_2}b_{k_3}b_{k_4}\langle\vec{q}_1\vec{q}_2|g|\vec{k}_3\vec{k}_4\rangle$$
$$+\sum_{k,q} b^{+}_{k_1}b^{+}_{k_2}a_{q_3}a_{q_4}\langle\vec{k}_1\vec{k}_2|g|\vec{q}_3\vec{q}_4\rangle + 2\sum_{k,q} b^{+}_{k_1}a^{+}_{q_2}b_{k_3}a_{q_4}\langle\vec{k}_1\vec{q}_2|g|\vec{k}_3\vec{q}_4\rangle$$
$$\left.+2\sum_{k,q} b^{+}_{k_1}a^{+}_{q_2}a_{q_3}b_{k_4}\langle\vec{k}_1\vec{q}_2|g|\vec{q}_3\vec{k}_4\rangle\right\} \tag{4.40}$$

and the functional H is

$$H = \frac{1}{\beta V}\left[\frac{q^2}{2m}\sum_q a_q^{*}a_q + \frac{k^2}{2m}\sum_k b_k^{*}b_k\right]$$
$$+\frac{1}{2}\frac{1}{(\beta V)^2}\int dt\left\{\sum_q a^{+}_{q_1}a^{+}_{q_2}a_{q_3}a_{q_4}\langle\vec{q}_1\vec{q}_2|g|\vec{q}_3\vec{q}_4\rangle + 2\sum_{k,q} b^{+}_{k_1}a^{+}_{q_2}a_{q_2}a_{q_3}\langle\vec{k}_1\vec{q}_2|g|\vec{q}_3\vec{q}_4\rangle\right.$$
$$+2\sum_{k,q} a^{+}_{q_1}a^{+}_{q_2}a_{q_3}b_{k_4}\langle\vec{q}_1\vec{q}_2|g|\vec{q}_3\vec{k}_4\rangle + \sum_{k,q} a^{+}_{q_1}a^{+}_{q_2}b_{k_3}b_{k_4}\langle\vec{q}_1\vec{q}_2|g|\vec{k}_3\vec{k}_4\rangle$$
$$+\sum_{k,q} b^{+}_{k_1}b^{+}_{k_2}a_{q_3}a_{q_4}\langle\vec{k}_1\vec{k}_2|g|\vec{q}_3\vec{q}_4\rangle + 2\sum_{k,q} b^{+}_{k_1}a^{+}_{q_2}b_{k_3}a_{q_4}\langle\vec{k}_1\vec{q}_2|g|\vec{k}_3\vec{q}_4\rangle$$
$$\left.+2\sum_{k,q} b^{+}_{k_1}a^{+}_{q_2}a_{q_3}b_{k_4}\langle\vec{k}_1\vec{q}_2|g|\vec{q}_3\vec{k}_4\rangle\right\} \tag{4.41}$$

4.4 WHAT A BROKEN INVARIANCE OF HAMILTONIAN IS[20]

A new variable, the named number of particles, was introduced:

$$n = \sum_q a_q^{+}a_q + \sum_k b_k^{+}b_k \tag{4.42}$$

From the Heisenberg equation of motion, a conservation property of any quantities by the meaning of commutative with Hamiltonian, H, can be observed.

Consider a commutation of H and each term in n. Yarunin [18, 19] has shown that $\left[H,\sum_k b_k^+ b_k\right] \neq 0$ is not conserved, but the semi-classical integral of motion [25] of n is conserved. This is,

$$\begin{aligned} \frac{dn}{dt} &= \frac{d}{dt}\left(\sum_q a_q^* a_q + \sum_k b_k^+ b_k\right) \\ &= \left\{H,\sum_q a_q^* a_q\right\} + i\left[H,\sum_k b_k^+ b_k\right] = 0 \end{aligned} \tag{4.43}$$

where, $\{H,A\} = i\left(\frac{\partial H}{\partial a}\frac{\partial A}{\partial a^*} - \frac{\partial H}{\partial a^*}\frac{\partial A}{\partial a}\right)$

which leads to the canonical distribution with constraint which is supplied by this integral of motion. Applying the above idea to the revised Hamiltonian it is found that $\left[H_A,\sum_k b_k^+ b_k\right] \neq 0$ and then $\left[H,\sum_k b_k^+ b_k\right] \neq 0$. From this point of view it was found that the term H_A also breaks the translational symmetry[20], but conserved the semi-classical number of particles in the same way. Now, consider a semi-classical number of particles integral of motion.

$$\frac{dn}{dt} = \left\{H,\sum_q a_q^* a_q\right\} + i\left[H,\sum_k b_k^+ b_k\right]$$

With n from equation (4.42) and H from equation (4.41) it was found that

$$\left\{H,\sum_q a_q^* a_q\right\} = i\left[\sum_{k,q} b_k^+ b_{-k}^+ a_q a_{-q} g(\bar{k},\bar{q}) - \sum_{k,q} b_k b_{-k} a_q^* a_{-q}^* g(\bar{k},\bar{q})\right]$$
$$+\frac{i}{2}\left[\sum_{k,q}\left(b_q^+ a_q^* a_q a_q - b_k a_q^* a_q^* a_q\right) g(\bar{k},\bar{q})\right] \qquad (4.44)$$

$$\left[H,\sum_q b_k^* b_k\right] = -\left[\sum_{k,q} b_k^+ b_{-k}^+ a_q a_{-q} g(\bar{k},\bar{q}) - \sum_{k,q} b_k b_{-k} a_q^* a_{-q}^* g(\bar{k},\bar{q})\right]$$
$$-\frac{1}{2}\left[\sum_{k,q}\left(b_q^+ a_q^* a_q a_q - b_k a_q^* a_q^* a_q\right) g(\bar{k},\bar{q})\right] \qquad (4.45)$$

Then
$$\frac{dn}{dt} = \left\{H,\sum_q a_q^* a_q\right\} + i\left[H,\sum_k b_k^+ b_k\right] = 0, \qquad (4.46)$$

and
$$\left\{H_A,\sum_q a_q^* a_q\right\} + i\left[H_A,\sum_k b_k^+ b_k\right] = 0. \qquad (4.47)$$

in the same way.

What has been explained from the above is the number of particles conserved under the condition of separated particles into two types: the classical condensate type and the quantum over condensate type in equation(4.46).