PRICING MULTINAME CREDIT DERIVATIVES BY MULTICORRELATED M
ARKET FACTOR MODEL

Miss Supalak Phetcharat

จุฬาลงกรณ์มหาวิทยาลัย
CHULALONGKORN UNIVERSITY

บทคัดย่อและแฟ้มข้อมูลฉบับเต็มของวิทยานิพนธ์ตั้งแต่ปีการศึกษา 2554 ที่ให้บริการในคลังปัญญาจุฬาฯ (CUIR)
เป็นแฟ้มข้อมูลของนิสิตเจ้าของวิทยานิพนธ์ ที่ส่งผ่านทางบัณฑิตวิทยาลัย
The abstract and full text of theses from the academic year 2011 in Chulalongkorn University Intellectual Repository (CUIR)
are the thesis authors' files submitted through the University Graduate School.

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Financial Engineering
Department of Banking and Finance
Faculty of Commerce and Accountancy
Chulalongkorn University
Academic Year 2014
Copyright of Chulalongkorn University

การคิดราคาตราสารอนุพันธ์ที่อ้างอิงเครดิตหลายบริษัท
โดยแบบจำลองที่ปัจจัยเสี่ยงของตลาดมีสหสัมพันธ์กัน

นางสาวศุภลักษณ์ เพ็ชรรัตน์

จุฬาลงกรณ์มหาวิทยาลัย
CHULALONGKORN UNIVERSITY

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาวิศวกรรมการเงิน ภาควิชาการธนาคารและการเงิน
คณะพาณิชยศาสตร์และการบัญชี จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2557
ลิขสิทธิ์ของจุฬาลงกรณ์มหาวิทยาลัย

Thesis Title	PRICING MULTINAME CREDIT DERIVATIVES BY MULTICORRELATED MARKET FACTOR MODEL
By	Miss Supalak Phetcharat
Field of Study	Financial Engineering
Thesis Advisor	Sira Suchintabandid, Ph.D.

Accepted by the Faculty of Commerce and Accountancy, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

................................Dean of the Faculty of Commerce and Accountancy
(Associate Professor Pasu Decharin, Ph.D.)

THESIS COMMITTEE

....Chairman
(Anant Chiarawongse, Ph.D.)

....Thesis Advisor
(Sira Suchintabandid, Ph.D.)

....Examiner
(Thaisiri Watewai, Ph.D.)

....External Examiner
(Kridsda Nimmanunta, Ph.D.)

จุฬาลงกรณ์มหาวิทยาลัย
CHULALONGKORN UNIVERSITY

ศุภลักษณ์ เพ็ชรรัตน์ : การคิดราคาตราสารอนุพันธ์ที่อ้างอิงเครดิตหลายบริษัทโดยแบบจำลองที่ปัจจัยเสี่ยงของตลาดมีสหสัมพันธ์กัน (PRICING MULTINAME CREDIT DERIVATIVES BY MULTICORRELATED MARKET FACTOR MODEL) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: ดร.สิระ สุจินตะบัณฑิต, หน้า.

จากการศึกษาแบบจำลองความเสี่ยงทางด้านเครดิตที่ใช้สำหรับการคิดราคาตราสารอนุพันธ์ในช่วงวิกฤตการณ์เงิน ปี 2008 พบว่าแบบจำลองเหล่านั้นไม่สามารถจับเหตุการณ์ที่หลายบริษัทผิดชำระหนี้พร้อมกันและยังไม่สามารถประเมินความเสียหายทางด้านเครดิตแบบพลวัตได้ ผลก็คือแบบจำลองเหล่านั้นประเมินความเสี่ยงทางด้านเครดิตต่ำไปและยังผิดพลาดในการคิดราคาตราสารอนุพันธ์ที่มีความซับซ้อนสูง เช่น ตราสารอนุพันธ์ที่อ้างอิงเครดิตหลายบริษัท จุดมุ่งหมายของวิทยานิพนธ์นี้ก็คือต้องการเสนอแบบจำลองที่สามารถสร้างความสัมพันธ์ในการผิดชำระหนี้ได้อย่างเข้มแข็งพอเพื่อที่จะคิดราคาตราสารอนุพันธ์ที่อ้างอิงเครดิตหลายบริษัท แบบจำลองของเราที่เสนอไปจัดอยู่ในประเภทแบบจำลองความเข้ม เราได้ออกแบบอัตราการผิดชำระหนี้ของแต่ละบริษัทให้ขึ้นอยู่กับความเข้มของตัวแปรตลาด โดยความเข้มของตัวแปรตลาดสามารถแบ่งเป็นส่วนดริฟท์กระจายและส่วนกระโดด แบบจำลองนี้ต่างจากแบบจำลองอื่นๆที่มีอยู่เดิม โดยความเข้มของตัวแปรตลาดแต่ละตัวมีความสัมพันธ์กันได้ ยิ่งไปกว่านั้นเรายังให้กระบวนนับการกระโดดในความหนาแน่นของตัวแปรทางการตลาดเป็นกระบวนการแบบส่วนผสมระหว่างแกมมากับปัวซอง วิทยานิพนธ์ฉบับนี้ยังได้พัฒนาวิธีที่มีประสิทธิภาพในการนำแบบจำลองที่เสนอนี้มาคำนวณการกระจายความเสียหายของพอร์ตฟอลิโอเครดิตหลายบริษัท วิธีที่เราแนะนำได้แก่วิธีเวียนเกิดและวิธีเลียนแบบลูกโซ่มาร์คอฟ จากการทดลองพบว่าแบบจำลองนี้สามารถคิดราคาตราสารอนุพันธ์ที่อ้างอิงเครดิตได้ดีกว่าแบบจำลองการกระโดดและการกระจายแบบเดิม การบวนการที่ใช้ควบคุมการกระโดดของความเข้มของตัวแปรตลาดที่เป็นส่วนผสมระหว่างแกมมากับปัวซองนั้น มีส่วนสำคัญต่อค่าสุดปลายของการกระจายความเสียหายของพอร์ตฟอลิโอ

ภาควิชา	การธนาคารและการเงิน	ลายมือชื่อนิสิต	--
สาขาวิชา	วิศวกรรมการเงิน	ลายมือชื่อ อ.ที่ปรึกษาหลัก.	----------------------------
ปีการศึกษา	2557		

5582181226 : MAJOR FINANCIAL ENGINEERING
KEYWORDS: CREDIT DERIVATIVES / CORRELATED DEFAULTS / MARKET RISK / TAIL DEPENDENCE / INTENSITY-BASED MODEL

SUPALAK PHETCHARAT: PRICING MULTINAME CREDIT DERIVATIVES BY MULTICORRELATED MARKET FACTOR MODEL. ADVISOR: SIRA SUCHINTABANDID, Ph.D., pp.

A review of the credit risk models that were used for pricing credit derivatives and risk management during the financial crisis of 2008 shows that the models fail to capture the severe event that a lot of firms default simultaneously and measure credit losses dynamically. As a result, the models underestimate credit risk and misprice complex credit derivatives, for example, Collateralized Debt Obligations (CDOs). The aim of the study is to propose the model that has the capacity to produce strong default dependency for pricing CDOs. Our proposed model is a kind of the intensity based models. To create default correlation among the CDO's underlying firms, we construct firms' default intensity processes based on market factor intensity processes. The market factors are modeled as the jump-diffusion distribution that has a drift-diffusion component and a jump component. Unlike any existing models, our model corporates in correlated market factor intensity processes. In addition, we use the Gamma-Poisson mixture process as the counting process of jumps in market factor intensities. Another objective of this research is to develop efficient methods which are used to implement our correlated market factor model for computing the portfolio loss distribution. The methods that we suggest are a recursive method and a Mimicking Markov chain method. The empirical results show that our model prices CDO tranches better than the traditional jump-diffusion model. The correlations between the market factor intensities are economically interpretable. Gamma-Poisson mixture processes governing arrival of jumps in intensities have an immense impact to the tails of the portfolio loss distribution.

Department:	Banking and Finance	Student's Signature	....
Field of Study:	Financial Engineering	Advisor's Signature	....
Academic Year:	2014		

จุฬาลงกรณ์มหาวิทยาลัย
CHULALONGKORN UNIVERSITY

ACKNOWLEDGEMENTS

First of all, I would like to express my deepest gratitude to my thesis advisor, Dr. Sira Suchintabandid, for his advice and valuable knowledge. I would not have completed this thesis without his help and guidance. In addition, he gave me some great suggestions of other things outside of the research, which I really appreciate. I am very thankful for thesis committee: Dr. Anant Chiarawongse, Dr. Thaisiri Watewai, and Dr. Kridsda Nimmanunta for their suggestions, comments, and feedback. I am also grateful for all teachers who taught me at a Financial Engineering program.

จุฬาลงกรณ์มหาวิทยาลัย
Chulalongkorn University

CONTENTS

Page

CHAPTER I
INTRODUCTION

In this chapter, we first introduce credit derivatives and existing credit risk models that are used to price credit derivatives. Then we address problems from past literature and our motivation to propose our model. After that, we mention the objective of the study and the contributions of our work.

1.1 Problem Review

Credit derivatives are financial instruments used by market participants such as banks and hedge funds for risk management and trading of credit risk. Derivatives can be distinguished by the number of underlying reference credits being referenced. In single-name credit derivatives, the product relates to only one underlying asset. In multi-name credit derivatives, there are multiple credit references. The well-known single-name products and multi-name products are Credit Default Swaps (CDS) and Collateralized Debt Obligations (CDOs) respectively. CDOs have underlying reference entities such as bonds and CDSs. CDO tranches are classified by the different levels of the portfolio loss. However there are unobservable dependencies between defaults in a portfolio. Undiversified risk has been known in another name as systematic risk. Normally, they mitigate systematic risk in credit portfolios such as CDOs by hedging. In order to hedge risk exposure in a portfolio, there is the need to know its pricing mechanism.

For pricing credit derivatives, the model is required to characterize the loss distribution of the underlying assets. One example of credit risk models is the standard Gaussian copula. The dependence structure of the defaults can be modelled through the marginal default distributions and their default correlation. Although it is easy to implement, the model has no capacity to measure dynamically dependent defaults. In financial crisis of 2008, the Gaussian copula has been blamed for mispricing CDOs. Due to the fact that the Gaussian copula cannot explain strongly joint defaults in a portfolio, it underestimates the actual risk. Many of those who use the Gaussian copula as if it were a reliable model suffered a lot from the great loss. As a result, a number of literature have proposed their model to solve the Gaussian copula's problems. Another interesting kind of credit risk models which has been accepted worldwide is the intensity based model.

Intensity-based models have been successfully used to price single-name credit derivatives such as CDSs. The portfolio loss of intensity-based models is concerned by underlying firms' default times and the loss due to credit event. The default time of each firm in a portfolio is determined by its arrival rate of defaults which follows a stochastic process. One example of basic intensity-based models is the drift-diffusion model. Nevertheless, the drift-diffusion model has an issue with weak default correlation in multi-name credit derivatives. There are numerous models that have been invented to defeat this issue. For example, Mortensen [1] uses the affine-jump diffusion model. His model is accomplished enough to capture the default dependency in both homogenous and heterogeneous portfolios. Peng and Kou [2] propose the Conditional Survival (CS) model. The CS model uses cumulative intensity processes to construct market factors. The market factors that are shared among firm intensity processes are also allowed to have jumps in terms of cumulative intensities. Because of its flexibility and mathematical favor, we choose to develop the intensity-based model.

Our proposed model is an extension of Duffie and Garleanu [3]'s notable multi-issuer default model. In the Duffie and Garleanu [3]'s model, the defaults are concerned as the first jumps of their cox processes with stochastic intensity processes. The individual firm intensity consists of market factors and an idiosyncratic factor. In particular, each market factor independently responds to different type of default risks such as regional risk and global risk, causing the dependence between default events. There are other literature motivated by the Duffie and Garleanu [3]'s model, for example, Mortensen [1] and Peng and Kou [2]. Unlike any of them, our market factors are allowed to be correlated.

The aim of this thesis is to propose the model that has the ability to capture a strong default correlation in multi-name credit derivatives such as CDOs. In doing so, we model the default intensity process of each firm to have systematic or market factors shared among underlying firms. Market factors are formed of a continuous component and a jump component. The continuous component is assumed to follow the Ornstein Ublenbeck (OU) process that is decomposed into a drift term and a Brownian-Motion-driven diffusion term. Most importantly, market factors are correlated through their Brownian motions. With positive correlation between market factors, it is expected to increase default dependency in a portfolio. If market factors are negatively dependent, systematic risk is supposed to be reduced.

We also indicate market factors to incorporate jumps. The Gamma-Poisson mixture distribution is used to model jump times, since this distribution can represent the default contagion phenomena. Specifically, any two events modeled by a Gamma-Poisson mixture process are time interdependent, which means that occurrence of an event triggers an increase of the probability of other events occurring. The feature of the Gamma-Poisson mixture distribution is that the mean is not necessary to be equal to the variance. In addition, the Gamma-Poisson mixture processes governing arrival of jumps present more shapes in the tail of the portfolio loss distribution than would be done by the Poisson process.

Even though market factors are being part of firms' default intensities, firms can be exposed to market risk with different levels. The sensitivity of a firm's default risk to each market factor can be measured by the magnitude of its market factor loadings. Furthermore, individual firm has its particular risk represented by the idiosyncratic factor in firm's default intensity.

As another objective of this work, we show how to implement our model using two alternative methods: a recursive method and a Mimicking Markov Chain simulation method. Owing to the fact that market factors are dependent, the default state of each individual firm is unable to be described by the marginal default probability of each market factor separately. Instead, it must be done by given the path of a vector of all market factor processes in the system. This is the huge issue for applying the correlated market factor model. Therefore we provide the Laplace transform function to describe the characteristics of dependent processes. The Laplace transform function is used to solve for explicit solutions required in implementation mechanisms of a recursive method and a Mimicking Markov Chain simulation method.

To calculate the portfolio loss, we first define the conditional loss distribution given the path of market processes and then estimate the unconditional ones. We use the recursive method that is applied from Andersen [4]. The conditional loss distribution given the path of a vector-valued market-factor process can be written in a recursive form. The portfolio (unconditional) loss distribution is found by taking the expectation of the conditional loss distribution. The difficulty of solving the unconditional loss distribution from the recursive form depends on the number of underlying firms in the portfolio. For a large heterogeneous portfolio, it is hard to find a simple closed-form solution. Moreover, the computation of the portfolio loss has to involve with numerous mathematical operations on numbers in the large range that is from very small to very large number. Therefore, the number of firms in a portfolio should be limited.

However, if the portfolio is homogeneous, the loss of a portfolio is binomial distributed. We can derive the conditional mass function of the portfolio loss given the path of a vector of market-factor processes in a simple way. By taking the expectation, the unconditional one is done in a closed form. The closed-form solutions of some unconditional terms are obtained by only concerning the required inputs and assigning zeros to irrelevant inputs of the Laplace transform function which we have derived.

The Mimicking Markov Chain simulation method is adopted from Giesecke et al. [5]. Instead of using usual firms' intensities to indicate default times, the arrival time of the next default is determined by the functions of firm transition rates based on the mimicking Markov chain. The individual firm's transition rate function has the meaning of the expectation of its intensity conditioning on a vector of firms' states in the portfolio. However, Giesecke et al. [5] don't present how to use their method with the model that has correlated market factors.

We choose to apply the Mimicking Markov chain method because it provides the time and the firm that defaults without any time discretization. For that reason, we apply their simulation scheme with our model. To implement the model, we need to mimic our own continuous-time Markov chain to structure consequences of firms' states in a portfolio. Like Giesecke et al. [5], state arrivals of the mimicking Markov chain depend on the transition rate functions of firms in a portfolio. Now market factor intensities are replaced by their conditional intensity given firms' state vector embedded in individual firm's transition rate function. It is even more difficult to find the solution for the expectation of each market factor's intensity conditioning on the vector of firms especially when market factors are not independent. Fortunately, the problem is resolved by the help of the Laplace transform function that we have mentioned.

It is not easy to calibrate the model for pricing multi-name credit derivatives since default dependencies among underlying reference firms cannot be apparently observed on market data. However, tranches of the index are sensitive to systematic risk. Especially pricing the senior tranche and the super senior tranche requires strong default dependencies. Thus, market factors' parameters are estimated from the spreads of CDO tranches, whereas each firm's specific parameters such as market factor loadings are calibrated to its market quoted CDS.

The rest of the thesis is organized as follows: Chapter 2 is a background of the intensity-based model, representing CDO pricing and reviewing our inspired existing models. In Chapter 3, we discuss our proposed model: 'Multicorrelated Market Factor Model'. In Chapter 4, we show a step by step guide to implement our model using suggested methods such as a recursive method and a Mimicking Markov chain method. Chapter 5 shows numerical results and compares our model with existing models. Chapter 6 concludes the results and suggests further work. In Appendix, there are closed form solutions of the exponential-affine characteristic function of our proposed model, CDO and CDS framework, and the thinning scheme algorithm.

CHULALONGKORN UNIVERSITY

1.2 Contributions and Study Objectives

In this thesis, the contributions of the work is to propose the model incorporated correlated market factors that produces strong default correlation in multi-name portfolios such as CDOs. The objectives of the study includes:

I. Studying improvement in accuracy of pricing CDOs from a normal multi-market factor model to the model which has market factors correlated across Brownian motions.

II. Studying performance development in capturing joint defaults from the model that has jumps in market-factor intensities driven by Poisson processes to the model that has Gamma-Poisson mixture processes characterizing jumps' frequency in market factors.

III. Developing efficient methods to compute a portfolio loss distribution and exactly simulate default times for our proposed model

CHAPTER II
BACKGROUND

It is known that value of credit derivatives is derived from the loss due to defaults of assets being referenced. The prices of the multi-name credit derivatives or CDO tranches are significantly dependent on the default correlations among names. Consequently, we measure the performance of our proposed model in producing default dependency through fitting the model to index tranche spreads of CDOs.

In order to understand pricing of CDOs and the credit risk modelling, this chapter gives a grasp on CDO pricing, the basic concepts of the credit risk models, and the review of existing models. We will show our proposed model later in the next chapter.

2.1 CDO Pricing

Suppose that a portfolio has n firms. The portfolio loss process is defined as

$$L_t = \sum_{i=1}^{n} (1 - R_i)\Theta_i N^i(t)\,, t \geq 0,$$

where R_i is the deterministic recovery rate, Θ_i is the notional principle Θ_i, N^i is the indicator process that presents the status (0=survive,1=default) of the underlying firm i. When the firm defaults at time τ^i, the indicator process N_i jumps from 0 to 1, illustrated by

$$N^i(t) = 1_{\tau^i \leq t}, i = 1, \dots, n.$$

The tranches of a CDO are classified by the level of the portfolio loss. The loss process of the tranche for the attachment point K_1 and the detachment point K_2 is given by

$$U^{[K_1,K_2]}(t) = (L_t - K_1)^+ - (L_t - K_2)^+, t \geq 0.$$

As shown in the equation above, the tranche loss process $U^{[K_1,K_2]}$ is underlying on the portfolio loss process L. To calculate any coupon premiums or the loss payment of the $[K_1, K_2]$ tranche, we need to model the portfolio loss process. The credit risk model is used to describe the probability of loss $P(L_t = l)$ at any time t and construct the default correlations among underlying firms. For more information of CDO and CDS framework, it can be found in Appendix B.

2.2 Basic Concepts for The Intensity-Based Models

In our research, we emphasize on the intensity-based models. We use a Bottom-up approach to model the portfolio loss. The Bottom-up approach specifies each firm i's stochastic intensity process λ_i to drive its indicator process N_i. The advantage of the intensity-based model using the Bottom-up approach is that it is flexible to modify dependencies among indicator processes $N^1, N^2, \ldots, N^n$ through their default intensity processes $\lambda_1,\ \lambda_2 \ldots,\ \lambda_n$. For the Bottom-up approach, the loss of the portfolio can be obtained by combining all underlying firms' losses determined by their indicator processes $N^1, \ldots, N^n$. To enlarge an intuition about the portfolio loss distribution based on the Bottom-up approach, we write

$$E[(L_t - K_1)^+] = \int_0^\infty (l - K_1)^+ P(L_t = l)\, dl$$

$$= \sum_{B \in \{0\ 1\}^n} \max\left(\sum_{i=1}^{n} (1 - R_i)\Theta_i B^i - K_1\,, 0\right) P(N(t) = B).$$

where the portfolio indicator process $N = (N^1, \ldots, N^n)$ runs over $\{0\ 1\}^n$. According to the equation above, there is the need of the method used to implement the model to compute the loss distributions (calculate $P(L_t = l)$ or $P(N(t) = B)$). One example of simple methods is the simulation of default times $\tau^1, \ldots, \tau^n$to determine their indicator processes $N_1, \ldots, N_n$.

The default times τ^i could be considered as

$$\tau^i = inf\left\{t \geq 0\colon \int_0^t \lambda_i(s)ds \geq \widetilde{\epsilon_i}\right\}\ , i = 1, \ldots, n\ ,$$

where λ_i is the intensity of the ith firm , $\widetilde{\epsilon_1}$ is an independent standard exponential random variable. Any standard exponential random variable $\widetilde{\epsilon_i}$ is simulated by generating a uniform random variable $U_i \in [0,1]$ and calculating $\widetilde{\epsilon_i} = \log(1 - U_i)$. To give a clear picture of the default time τ_i, we solve it by generating a path of $\int_0^t \lambda_i(s)ds$ and an exponential random $\widetilde{\epsilon_i}$. Then we set the firm i's default time τ^i equal to the minimum time $t \geq 0$ that makes $\int_0^t \lambda_i(s)ds \geq \widetilde{\epsilon_i}$.

However, this method is not practiced in our research. The firm i's default intensity λ_i is not a deterministic function but a stochastic process, and is also correlated to intensities of other firms. Moreover, simulated default times are bias because we need to discretize time to compute $\int_0^t \lambda_i(s)ds$. Hence we suggest another method that can simulate default times exactly in Chapter 4.

2.3 Review of Existing Models

There are many existing models that have been proposed for credit risk pricing. Their main contribution is usually to propose the model than has capacity to generate fat-tailed loss distribution, long-tailed loss distribution, or asymmetric loss distribution, which cannot be explained by the standard Gaussian Copula model.

In this research, we study the intensity-based models. As stated before, this kind of the models can adjust default correlation in a portfolio via referenced default intensity processes. Generally, the processes that are generally used to drive default intensities are affine-jump diffusions which are the combination between a continuous process and a jump process. The basic affine-jump process X that has the continuous component following the Cox-Ingersoll-Ross (CIR) process that incorporates with a jump process with parameters $(k, \theta, \sigma, \mu, \ell)$ solves

$$dX(t) = k\big(\theta - X(t)\big)dt + \sigma\sqrt{X(t)}dW(t) + dJ(t), t \geq 0$$

where k is the speed of adjustment, θ is the long-term mean, σ is volatility, W is a Brownian motion, and J is the jump process that has μ as the mean of exponential-distributed jump sizes and ℓ as the Poisson arrival rate of jumps. It is widely known that the CIR process has the boundary condition $2k\theta \geq \sigma^2$. As long as the boundary rule is not broken, simulated intensities are always positive.

2.3.1 Duffie and Garleanu [3]'s Multi-Issuer Default Model

In order to describe the structure of default dependency in underlying firm, Duffie and Garleanu [3] model the default intensity process of any firm i to have systematic factors such sectorial risk factors and a global risk factor shared among other firms.

Consider a n-firm portfolio. There are S sectors which each firm particularly belongs to. The ith firm's intensity process λ_i is adapted to the filtration F generated by the firm default processes, idiosyncratic risk factors X_i, $1 \leq i \leq n$, sectorial risk factors $Y_{c(i)}$, $c(i) \in \{1, \dots, S\}$ and a global risk factor Z, where X_i, $Y_{c(i)}$ and Z are supposed to be independent affine-jump processes sharing the same parameters k, σ and μ, having different long-term mean $\theta_i, \theta_{c(i)}, \theta_z$ and jump arrival rate $\ell_i, \ell_{c(i)}, \ell_z$ respectively. Then λ^i is a basic affine-jump process with parameters $(k, \theta, \sigma, \mu, \ell)$, defined as

$$\lambda_i(t) = X_i(t) + Y_{c(i)}(t) + Z(t),$$

where $\theta = \theta_i + \theta_{c(i)} + \theta_z$ and $\ell = \ell_i + \ell_{c(i)} + \ell_z$.

Duffie and Garleanu [3] assume that a portfolio is homogeneous, that is all firms' intensity processes in the portfolio have the same model parameters.

2.3.2 Mortensen [1]'s Multi-Name Intensity Model

He modifies Duffie and Garleanu [3]'s work to handle heterogeneous portfolios. Let F denote the filtration generated by the firm default processes, idiosyncratic risk factors $X_i, 1 \leq i \leq n$, and a market risk factor Y. X_i and a_iY are supposed to be independent affine-jump processes with parameters $\left(k, \theta_i, \sqrt{a_i}\sigma, a_i\mu, \ell\right)$ and $\left(k, a_i\theta_Y, \sqrt{a_i}\sigma, a_i\mu, \ell\right)$ respectively, where $\ell = \ell_i + \ell_y$.

The ith firm's intensity process λ_i takes the form

$$\lambda_i(t) = a_iY(t) + X_i(t),$$

where the parameter a_i refers to the sensitivity of firm i to the market factor Y. It is implied that λ_i is an affine-jump process with parameters $\left(k, a_i\theta_Y + \theta_i, \sqrt{a_i}\sigma, a_i\mu, \ell\right)$.

In his paper, numerical results show that his jump-diffusion model based on only one market factor fits all index tranches well and outperforms the pure diffusion model, the Gaussian copula, the RPL Gaussian copula, and the Double-t copula. He assumes that CDO portfolios are heterogeneous and he observes that his model can price tranches of index from Markit iTraxx Europe Investment grade family in the case of homogeneous portfolios.

2.3.3 Peng and Kou [2]' Conditional Survival Model

Peng and Kou [2] propose the new Conditional Survival (CS) Model to produce default clustering. Peng and Kou [2] consider that Duffie and Garleanu [3]'s the multi-issuer default model cannot produce strong default correlation in spite of the fact that there are jumps or even simultaneous jumps in market factors shared among firms. Peng and Kou [2] illustrate that for the model of Duffie and Garleanu [3], once the jump in intensity of market factor occur, it just smoothly increases the cumulative intensity of market factor. As a result, the probability that the firm defaults is higher, though, several firms might not default simultaneously. Consequently, Peng and Kou [2] propose the CS model that has dynamics of idiosyncratic factor X_i, and market factors $M_1, \dots, M_J$ in terms of cumulative intensity processes (e.g.$X_i(t) = \int_0^t x_i(s)ds$).

Consider a portfolio of n firms, Λ_i is the cumulative intensity process of firm i, whose default time in this case is defined as

$$\tau^i = inf\{t \geq 0: \Lambda_i(t) \geq \widetilde{\epsilon_i}\}, 1 \leq i \leq n,$$

where $\widetilde{\epsilon_i}$ is an independent exponential random variable with mean 1.

The ith firm's cumulative intensity process Λ_i is adapted to the filtration F generated by the firm default processes, cumulative market factors M_j, $1 \leq j \leq J$, cumulative idiosyncratic factors $X_i, 1 \leq i \leq n$, specified as

$$\Lambda_i(t) = \sum_{j=1}^{J} a_{i,j}M_j(t) + X_i(t), 1 \leq i \leq n, t \geq 0,$$

where the factor loading $a_{i,j}$ represents the sensitivity of firm i to market factor j. The market factors $M_1, \dots, M_J$ are allowed to be full of jump processes themselves, not being part of intensity processes. Interestingly, Peng and Kou [2] don't define particular distributions for idiosyncratic factors because their CS model can relate underlying firm' conditional survival probability to the unconditional survival probability that is extracted from market data of that firm's CDS spread without the need to simulate intensities of the idiosyncratic factors $X_1, X_2, \dots, X_n$.

In numerical results of Peng and Kou [2], they use Polya processes and the integral of CIR processes to model market-factor cumulative intensities. The market factor's Polya process M_j can be viewed as the Poisson counting process that has a Gamma random variable as arrival rate of jumps. Peng and Kou [2] state that the jumps govern by the Polya process are positively increasingly correlated and then result in generating a strong degree of default dependency. In addition, they use integral of CIR processes to provide dynamic and describe dependency structure of defaults under normal situation.

In their paper, their model with three market factors could fit tranche spreads of the iTraxx Europe 5-year Index on both March 14, 2008 and September 16, 2008 really well. They choose CDO spreads on those dates in order to show that their model is efficient even in the financial crisis. Especially, 16^{th} September 2008 is the day after the collapse of Fannie Mae and Freddy Mac, and Lehman Brother. However, there is a bit of trivia about their numerical results. Some part of their parameters of the cumulative CIR process that is used to drive one market factor is $(k, \theta, \sigma) \approx (0.0526, 0.1, 1.6837)$. The boundary condition $2k\theta \geq \sigma^2$ of the CIR process is violated. Although they simulate intensities by generating Chi-square random variable and then summing them up to be the discrete integral. It is unpractically feasible because negative intensities are not allowed for the CIR process.

จุฬาลงกรณ์มหาวิทยาลัย
Chulalongkorn University

CHAPTER III
OUR PROPOSED MODEL

As mentioned in the introduction, our proposed model is primarily adjusted from the multi-issuer default model introduced by Duffie and Garleanu [3]. It also gets inspired from Mortensen [1]'s multi-name intensity model and Peng and Kou [2]'s conditional survival model. All models that we have referred are classified as the intensity-based models. The appeals of intensity-based models are their past success and flexibility. The intensity-based model is seen to be easily applied or extend such as adding more distributions. What makes our proposed model different from the others is the market factors that are allowed to be correlated. We are looking for parameters that increase the probability of joint default events or provide advantages to efficiently fitting particular CDO index tranche spreads. The results from Peng and Kou [2] imply that one market factor is not adequate to price CDOs under crisis situation. It is worth a try if making market factors correlated gains benefits.

We will deliberately discuss more details about our proposed model and make comparisons with other models in the end of this chapter.

3.1 Muticorrelated Market Factor Model

Suppose that there are n underlying reference firms in a portfolio. As mentioned before, we define τ^i as the time that the ith firm defaults in the portfolio, which is determined by the first time that its cox process jumps from 0 to 1 with the default intensity process λ_i. We define firm i's default intensity to have systematic or market factors shared among firms and its idiosyncratic factor. More specifically, the default correlations are modeled through market factors. The idiosyncratic factor represent individual firm's particular risk. Denote m as the number of market factors in the system, $X_1, X_2, \dots, X_m$ as market factors, and Y_i as the idiosyncratic factor of firm i.

The ith firm's intensity process is specified as

$$\lambda_i(t) = \sum_{j=1}^{m} \beta_{i,j} X_j(t) + Y_i(t), 1 \leq i \leq n, \tag{1}$$

where $\beta_{i,j}$ is the market factor loading representing the sensitivity of the ith firm to market factor $j, 1 \leq j \leq m$.

Let's start with the systematic part. The jth market factor has dynamics

$$dX_j(t) = k_j\left(\theta_j - X_j(t)\right)dt + \sigma_j dW_j(t) + dZ^j(t), \tag{2}$$

where k_j is rate of mean-reversion, θ_j is long-term mean, σ_j is the volatility, $W_j(t)$ is a standard Brownian motion, and Z^j is jump process.

For the Brownian motions $W_1(t), W_2(t), \ldots, W_m(t)$, they are assumed to be correlated such that

$$dW_v(t)dW_j(t) = \rho_{vj}dt, \qquad 1 \leq v \leq m. \tag{3}$$

Brownian motions are allowed to be correlated among market factors because they provide dynamic dependence among market-factor processes. Moreover, these correlation parameters are meaningful. If market factors are negative correlated, the portfolio is more diversified. Conversely, the model that has positive correlation between market factors produces stronger default dependency.

In addition, the jump processes $Z^1, Z^2, \ldots, Z^m$ are independently distributed. The Gamma-Poisson mixture processes are used to model jumps' frequency of the jump processes $Z^1, Z^2, \ldots, Z^m$ whose jump sizes are exponential distributed. For the sake of clarity, we define $Z^j, 1 \leq j \leq m$ as

$$Z^j(t) = \sum_{n=1}^{\Pi^j(t)} \Upsilon_n^j, \tag{4}$$

where $\Pi^j(t)$ is a counting Poisson process that has a arrival rate Λ^j modeled by Gamma distribution with the shape parameter α^j and the scaled parameter $\mathcal{B}^j$, and the jump sizes $\Upsilon_1^j, \Upsilon_2^j$ ... are exponentially distributed random variables with the mean μ^j.

We choose Gamma-Poisson mixture processes to model jumps because jump times are interdependent, which causes serial correlated defaults. The degree of serial correlation of any Gamma-Poisson mixture counting process Π^j can be measured by

$$\operatorname{cov}\left(\Pi^j(t), \Pi^j(t+h) - \Pi^j(t)\right) = \alpha^j {\mathcal{B}^j}^2 ht.$$

Unlike the Gamma-Poisson mixture process, the Poisson distribution has no capacity to produce jumps that are serial correlated. In addition, the Poisson process restricts that the variance are equal to the mean, but the variance of the Gamma-Poisson mixture process can be selected arbitrarily. As a result, the Gamma-Poisson mixture process has the ability to produce more shapes in the tails of the portfolio loss distribution.

For the idiosyncratic factors $Y_1, \ldots, Y_n$, we consider them as errors, which is similar to Peng and Kou [2]. We will discuss this argument more in Chapter 5.

As can be seen in the equation (1), the factor loadings $\beta_{i,j}$ of market factor $j, 1 \leq j \leq m$ are varied across firm $i, 1 \leq i \leq n$. It implies that we cannot rely on factor loadings only to create default correlation. Moreover there would be $n \times m$ factor loadings needed to be calibrated against just 5-6 tranches of the index if factor loadings were assumed to be the source of systematic risk. Schönbucher [6] mentions that "The number of parameters needed to describe the dependence structure of the defaults in the model should be limited, in particular it should not grow exponentially in the number of obligors" (p. 289). Thus the factor loadings are used to fit CDS curves of individual firm like Mortensen [1] and Peng and Kou [2]. The unobservable parameters, for example, the correlations of Brownian motions and other parameters of market factors, are used to fit the CDO tranche spreads.

3.2 Comparing with Existing Models

We compare our proposed model with the models that have been reviewed in this paper.

1. Our proposed model has the market factors that are allowed to be correlated, which doesn't actually present in other literature. Peng and Kou [2] mention that their model supports this idea but they don't show how to model them and use it to price correlated products.
2. There is a jump process incorporated in every market factor. The arrival of jumps are governed by the Gamma-Poisson mixture process. Unlike our model, Peng and Kou [2] use this kind of distribution called Polya distribution to model jumps in terms of cumulative intensities. We assume that there is not much different between jump cumulative intensities and the intensities that have jump processes being part of them. If it is true, our model fits CDO tranche spreads well as Peng and Kou [2]'s CS model. Our model has ability to produce contagious defaults and is more dynamic due to random jump sizes.
3. There are closed-form solutions such as survival (or default) probabilities that are derived from the Laplace transform function. Consequently, we can exactly compute the portfolio loss distribution whether by using a straightforwardly recursive method or a mimicking Markov chain method for simulating default times without time discretization.

จุฬาลงกรณ์มหาวิทยาลัย
Chulalongkorn University

CHAPTER IV
SUGGESTED METHODS
IN COMPUTING LOSS DISTRIBUTION

To compute a portfolio loss distribution, we must know how to define the default times or the process counting defaulted firms in a portfolio. It is known that each firm's default time is considered as the first time that its cox process or indicator process with a stochastic intensity jumps from 0 to 1. There are several ways to estimate default times such as Monte Carlo simulation and Semi-analytical transform techniques. The Monte Carlo simulation is easily used to simulate any random variables of any stochastic processes and discretize time to approximate expected values. However, simulation errors are large and it is very time consuming to compute cumulate intensities by time discretization. In this paper, two recommended methods are as follow:

1. A recursive method. Define the process that counts the number of defaulted firms in a portfolio as
$$\mathbb{N}(t) = \sum_{i=1}^{n} N^i(t)\,, t \geq 0.$$
The method is used to recursively calculate the probability $\mathrm{P}(\mathbb{N}(t) = l)$ based on the portfolio indicator process $N = (N^1, \dots, N^n)$ to compute the portfolio loss distribution. There is no default time identified. Hence, we need to assume that the defaults occur between coupon payment dates for pricing spreads of a CDO tranche.
2. A Mimicking Markov Chain method. The edge of this method over a recursive method is that the defaulted firms and default times are acknowledged. The Mimicking Markov Chain method uses transition rate functions to determine default times in a portfolio. Underlying firms' transition rate functions are straightforwardly computed using closed forms or numerical methods. There is no need to simulate default intensities and discretize time to estimate cumulative intensities. Therefore, it is useful for the process that has unknown distributions, for example, a cumulative terms of CIR process.

Nevertheless, we can't instantly use those suggested methods to implement our proposed model because market factors are correlated. To use those method, we have more works to do. We provide the Laplace transform function that can describe the structure of processes that are dependent. The Laplace transform function is used to find the solutions for suggested methods. In this chapter, we first present how to implement our proposed model using a recursive method and a Mimicking Markov chain method, and then finally show how to derive the Laplace transform function.

4.1 Recursive Method

Suppose that underlying firms have the same recovery rate R and notional principal Θ. The portfolio loss process satisfies

$$L_t = (1-R)\Theta\sum_{i=1}^{n} N^i(t) = (1-R)\Theta\mathbb{N}(t), t \geq 0$$

To compute the portfolio loss distribution, we specify

$$P(\mathbb{N}(t) = l) = E[P(\mathbb{N}(t) = l|(X(s))_{s\leq t})],$$

where $(X(s))_{s\leq t}$, $X = (X_1, \dots, X_m)$ is the path of a vector of market factor processes that are correlated. We have to take the expectation on the conditional probability given the path of a vector of correlated-market-factor processes $(X(s))_{s\leq t}$ because all firms' default intensity processes have market factors shared among them and most importantly market factors are correlated.

Before proceeding to the next step, let us introduce $P^u(\mathbb{N}(t) = v|(X(s))_{s\leq t})$ as the probability that there are v defaulted firms from u firms that consists of $u^{th}, (u-1)^{th}, \dots, 2^{nd}, 1^{st}$ firms given the path of a vector of correlated-market-factor processes$(X(s))_{s\leq t}$. Let us define the conditional survival probability of firm i given the path of a vector of correlated-market-factor processes $(X(s))_{s\leq t}$ as

$$P\left(\tau^i > t|(X(s))_{s\leq t}\right) = \exp\left(-\sum_{j=1}^{m}\beta_{i,j}\int_0^t X_j(s)ds\right)E\left[\exp\left(-\int_0^t Y_i(s)ds\right)\right]. \quad (5)$$

To compute the conditional loss distribution $P(\mathbb{N}(t) = l|(X(s))_{s\leq t})$, we set

$$P(\mathbb{N}(t) = l|(X(s))_{s\leq t}) = P^n(\mathbb{N}(t) = l|(X(s))_{s\leq t}), \quad 0 \leq l \leq n. \quad (6)$$

According to Andersen [4], we can solve $P^n(\mathrm{N}_t = l|(X(s))_{s\leq t})$, $0 \leq l \leq n$, by following the steps below recursively.

If $u = 0$, $P^0(\mathbb{N}(t) = v|(X(s))_{s\leq t}) = 1$.
Else if $u = v$,

$$P^v(\mathbb{N}(t) = v|(X(s))_{s\leq t}) = P^{v-1}(\mathbb{N}(t) = v-1|(X(s))_{s\leq t})[1 - P(\tau^v > t|(X(s))_{s\leq t})].$$

Else if $v = 0$,

$$P^u(\mathbb{N}(t) = 0|(X(s))_{s\leq t}) = P^{u-1}(\mathbb{N}(t) = 0|(X(s))_{s\leq t})P(\tau^u > t|(X(s))_{s\leq t}).$$

Else if $0 < v < u$,

$$P^u(\mathbb{N}(t) = v|(X(s))_{s\leq t}) = P^{u-1}(\mathbb{N}(t) = v-1|(X(s))_{s\leq t})[1 - P(\tau^u > t|(X(s))_{s\leq t})] + P^{u-1}(\mathbb{N}(t) = v|(X(s))_{s\leq t})P(\tau^u > t|(X(s))_{s\leq t}).$$

However, the computation of the unconditional mass function of the loss becomes intensive when the number of assets in a portfolio is large. For example, CDX IG NA and Itraxx Europe have 125 firms in their portfolios. Assuming that the underlying reference firms are homogeneous, we let the portfolio loss follows Binomial distribution, which is specified as

$$P(\mathbb{N}(t) = l|(X(s))_{s\leq t}) = \binom{n}{l}\left(1 - P(\tau > t|(X(s))_{s\leq t})\right)^{l}\left(P(\tau > t|(X(s))_{s\leq t})\right)^{n-l}. \quad (7)$$

Modified the Euler-Maclaurin sums that is represented in Papageorgiou [7], the conditional loss distribution above can be rewritten as

$$P(\mathbb{N}(t) = l|(X(s))_{s\leq t}) = \binom{n}{l}\sum_{i=0}^{l}\binom{l}{i}(-1)^{l-i}\,\mathrm{e}^{-(n-i)\sum_{j=1}^{m}\beta_j\int_0^t X_j(s)ds}E\left[\mathrm{e}^{-\int_0^t Y(s)ds}\right]^{n-i} \quad (8)$$

Note that entire firms use the same factor loadings $\beta_1, \beta_2, \dots, \beta_m$ due to the assumption of the homogeneous portfolio. By taking expectation of the equation (8), its unconditional loss distribution becomes

$$P(\mathbb{N}(t) = l) = \binom{n}{l}\sum_{i=0}^{l}\binom{l}{i}(-1)^{l-i}\,E\left[\mathrm{e}^{-(n-i)\sum_{j=1}^{m}\beta_j\int_0^t X_j(s)ds}\right]E\left[\mathrm{e}^{-\int_0^t Y(s)ds}\right]^{n-i}. \quad (9)$$

The Laplace transform function is used to solve the closed-form solutions presented in this paper. We will discuss the Laplace transform function and how to derive it deeply in the end of this chapter. However, to get a grasp on how to apply the Laplace transform function, we represent

$$E\left[\exp\left(-(n-i)\sum_{j=1}^{m}\beta_j\int_0^t X_j(s)ds\right)\right] = \phi^x(t,(n-i)(\beta_1,\dots,\beta_m),0_m,X(0)),$$

where 0_m is a m-zero vector and the Laplace transform function ϕ^x is defined as

$$\phi^x(T,u,z,X(0)) = E\left[\exp\left(-\sum_{j=1}^{m}u_j\int_0^T X_j(s)ds - \sum_{j=1}^{m}z_jX_j(T)\right)\right].$$

In addition, the idiosyncratic factor is concerned as error, then we gain

$$E\left[\mathrm{e}^{-\int_0^t Y(s)ds}\right] = 1.$$

4.2 Mimicking Markov Chain Method

Giesecke et al. [5] develop the simulation approach that is exact and efficient for a vector process. They construct the mimicking Markov chain $M = (M^1, \dots, M^n) \in \{0,1\}^n$, which has the same property as the portfolio default indicator process $N = (N^1, \dots, N^n) \in \{0,1\}^n$, in its own filtration $\mathbb{G} = (G_t)_{t\geq 0}$ generated by M. The filtration $\mathbb{G}$ contains all information for each t that is represented by the σ-algebra G_t. The mimicking Markov chain M is determined by the transition rate function $h(\cdot, M)$ instead of intensity process λ.

The transition rate function $h^i(t,B)$ is the expectation of $\lambda_i(t)I(\tau^i > t)$ conditioning on the portfolio indicator $N(t) = B$, which $B = (B^1, \dots, B^n) \in \{0,1\}^n$, defined as

$$h^i(t,B) = E\big(\lambda_i(t)I(\tau^i > t)\big|N(t) = B\big)\,.$$

For our proposed model, the function of transition rate $h^i(t,B)$ can be rewritten as

$$h^i(t,B) = \left(1 - B^i\right)\left(\sum_{j=1}^{m} \beta_{i,j} E\big(X_j(t)\big|N(t) = B\big) + E\big(Y_i(t)\big|\tau^i > t\big)\right). \tag{10}$$

However, when market factors are correlated we have to perform many steps to solve $E\big(X_j(t)\big|N(t) = B\big)$. By Bayes' theorem and the law of iterated expectations, we obtain

$$E\big(X_j(t)|N(t) = B\big) = \frac{E\Big(X_j(t)I(N(t) = B)\Big)}{P(N_t = B)} = \frac{E\Big(X_j(t)P(N(t) = B|(X(s))_{s\leq t})\Big)}{E\big(P(N(t) = B|(X(s))_{s\leq t})\big)}.$$

We will show how to find the explicit solutions of $E\big(P(N(t) = B|(X(s))_{s\leq t})\big)$ and $E\Big(X_j(t)P(N(t) = B|(X(s))_{s\leq t})\Big)$. First of all the conditional probability at time t that the portfolio default indicator process $N(t) = B$ given the path of a vector of correlated-market-factor processes $(X(s))_{s\leq t}$ is given by

$$P(N(t) = B|(X(s))_{s\leq t}) = \prod_{i=1}^{n}\left[B^i - (2B^i - 1)P(\tau^i > t|(X(s))_{s\leq t}\right].$$

Substituting (5) into the equation above, we have

$$P(N(t) = B|(X(s))_{s\leq t}) = \prod_{i=1}^{n}\left[B^j - \left(2B^j - 1\right)\mathrm{e}^{\left(-\sum_{j=1}^{m}\beta_{i,j}\int_0^t X_j(s)ds\right)} E\left[\mathrm{e}^{\left(-\int_0^t Y_i(s)ds\right)}\right]\right]. \tag{11}$$

The right hand side of the equation (11) can be in the expansion of 2^n terms. Before getting lost in the trees, we will give identity to each term in the expansion of the equation (11), which is used to explain the explicit solution of the transition rate function $h^i(t,B)$.

Let us denote by A the array of 2^n elements, each of them is assigned to a bit vector of length n mapping to its based-2 index (e.g. $\mathrm{A}^0 = (0,0,\dots,0,0), \mathrm{A}^1 =$

$(0,0,\ldots,0,1)$). Each element k of the array A corresponds to term k in the expansion formula (11). Let $\eta_k(t)$ be the coefficient of the k-th term of the expansion of the equation (11), defined as

$$\eta_k(t) = \prod_{i=1}^{n}\left[\left(1 - \mathrm{A}^{\mathrm{k}}(i)\right)B^i + \mathrm{A}^{\mathrm{k}}(i)(2B^i - 1)E\left[\exp\left(-\int_0^t Y_i(s)ds\right)\right]\right].$$

Let us introduce $b_{k,j}$ as the factor loading for the k-th term to market factor j, given by

$$b_{k,j} = \sum\nolimits_{i=1}^{n} \mathrm{A}^{\mathrm{k}}(i)(2B^i - 1)\beta_{i,j}.$$

Using the coefficients $\eta_k(t)$ and $b_{k,j}$ for $k,\ 0 \le k \le 2^{n-1}$, we can now represent the expansion of equation (11) as

$$P(N(t) = B|(X(s))_{s\le t}) = \sum_{k=0}^{2^n-1} \eta_k(t)\exp\left(-\sum_{j=1}^{m} b_{k,j}\int_0^t X_j(s)ds\right).$$

Taking expectation on both sides of the equation above leaves

$$\begin{aligned} E[P(N(t) = B|(X(s))_{s\le t})] &= \sum_{k=0}^{2^n-1} \eta_k(t)\,\mathrm{E}\left[\exp\left(-\sum_{j=1}^{m} b_{k,j}\int_0^t X_j(s)ds\right)\right] \\ &= \sum_{k=0}^{2^n-1} \eta_k(t)\,\phi^x\big(t,(b_{k,1},\ldots,b_{k,m}),0_m,X(0)\big). \end{aligned} \tag{12}$$

Note that 0_m is denoted as a m-zero vector. As seen before, the Laplace transform function is given by

$$\phi^x(T,u,z,X(0)) = E\left[\exp\left(-\sum_{j=1}^{m} u_j\int_0^T X_j(s)ds - \sum_{j=1}^{m} z_j X_j(T)\right)\right].$$

Similar to Giesecke et al.[5], the explicit solution of the iterated expectation $E\big(X_j(t)P(N(t) = B)|(X(s))_{s\le t}\big)$ can be solved by taking the derivative of the equation (12) with respect to $-z_j$, and then substituting $z = 0_m$

$$\begin{aligned} &E\big(X_j(t)P(N(t) = B)|(X(s))_{s\le t}\big) \\ &\qquad = -\sum_{k=0}^{2^n-1} \eta_k(t)\frac{\partial\phi^x}{\partial z_j}\big(t,(b_{k,1},\ldots,b_{k,m}),z,X(0)\big)|_{z=0_m}. \end{aligned} \tag{13}$$

It is unquestionable that the idiosyncratic factors $Y_1,\ldots,Y_n$ are independent distributed. We can write

$$E\big(Y_i(t)|\tau^i > t\big) = \frac{E\left(Y_i(t)P\left(\tau^i > t\right)\right)}{P(\tau^i > t)} = -\frac{\frac{\partial\phi^y}{\partial z_i}(t,1,z_i,Y_i(0))|_{z_i=0}}{\partial\phi^y(t,1,0,Y_i(0))} \tag{14}$$

where the Laplace transform function of an idiosyncratic factor is given by

$$\phi^y(T,u_i,z_i,Y_i(0)) = E\left[\exp\left(-u_i\int_0^T Y_i(s)ds - z_iY_i(T)\right)\right].$$

Substituting (12), (13), and (14) in the equation (10), we obtain

$$h^i(t,B) = -\frac{\frac{\partial\phi^y}{\partial z_i}(t,1,z_i,Y_i(0))|_{z_i=0}}{\partial\phi^y(t,1,0,Y_i(0))} - \sum_{j=1}^{m}\beta_{i,j}\frac{\sum_{k=0}^{2^n-1}\eta_k(t)\frac{\partial\phi^x}{\partial z_j}\left(t,(b_{k,1},\dots,b_{k,m}),z,X(0)\right)|_{z=0_m}}{\sum_{.k=0}^{2^n-1}\eta_k(t)\,\phi^x\left(t,(b_{k,1},\dots,b_{k,m}),0_m,X(0)\right)}.$$

In the filtration $\mathbb{G}$, we denote T_{k+1} as the time that $M(T_k)$ changes states to $M(T_{k+1})$. Like Giesecke et al. [5], the process of the portfolio transition rate function $H(t,k)$ is given by

$$H(t,k) = \sum_{i=1}^{n} h^i(t,M(T_k)),\ T_k \le t < T_{k+1}, k = 1,2,\dots$$

Commonly, the brute-force simulation is a simple method used to generate the sequence of default times $T_1, T_2, \ldots$.. The default time is T_k determined by

$$T_k = inf\left\{t \ge 0\colon \int_0^t H(s,k)ds \ge \widetilde{\epsilon_{\mathrm{k}}}\right\},$$

where $\widetilde{\epsilon_{\mathrm{k}}}$ is an independent exponential random variable with mean 1. Then the firm that defaults is acknowledged with probability $h^i(t,M(T_k))/H(t,k), i = 1,2,\dots,n$. If the ith firm defaults, the next mimicking Markov chain $M(T_{k+1})$ is the updated version of the previous chain with $M^i(T_k) = 1$. The computation stops when the default time of all firms in a portfolio are specified. By doing so, there is error from discretizing time for integration and it is computationally expensive. To avoid discretization error, we can use the thinning scheme.

4.2.1 Thinning Scheme

In Giesecke et al. [5], the thinning algorithm is applied to simulate the mimicking chain M by generating the firm defaults' identities and their default times. The advantage of this scheme is that there is no need for time scaling and discrete-time integration.

First, we determine the appropriate value of the number of intervals $\mathcal{M}$ for the intensity $H(t,k)$, next create a partition of the given interval $[0,T]$ such that $0 \le L_0 < L_1 <\ldots< L_{\mathcal{M}} = T$ to obtain a subinterval $[L_i, L_{i-1}]$ and then find the majorizing function $H^*(i,k)$ such that

$$H^*(i,k) = \sup\{H(s,k)\colon L_{i-1} \le s < L_i\}, \tag{15}$$

where $i = 1,\dots,\mathcal{M}$. After the exponentially distributed arrival time $x, x \in (L_{i-1}, L_i)$ with intensity rate $H^*(i,k)$ is generated, x is accepted with probability $H(x,k)/H^*(i,k)$.

Laplace Transform Function

We provide the Laplace transform function that is applied in the recursive method and the Mimicking Markov Chain method. Usually, the Laplace transform is used to explain the characteristic of any independent process such as an idiosyncratic factor, is defined as

$$\phi^{y}(T,u_i,z_i,Y_i(0)) = E\left[\exp\left(-u_i\int_0^T Y_i(s)ds - z_iY_i(T)\right)\right]. \tag{16}$$

Similar to Giesecke et al. [5], the Laplace transform of the idiosyncratic factor Y_i only relates to the distribution of its own idiosyncratic factor Y_i. In this research, we ignore the idiosyncratic factors $Y_1, \dots, Y_n$ because their values are assumingly small as errors analogous to Peng and Kou [2]. We concentrate on create default correlation through market factors.

Nonetheless, the distribution of market factor $X_1(t), X_2(t), \dots, X_m(t)$ cannot be transformed into their own characteristic functions because market factors are not independent. Alternatively, we describe the characteristics of the distributions of $X_1(t), X_2(t), \dots, X_m(t)$ through the sum $\sum_{j=1}^{m} X_j(t)$. We define the Laplace transform of a vector-valued market-factor process associated with their integrals as

$$\phi^{x}(T,u,z,X(0)) = E\left[\exp\left(-\sum_{j=1}^{m} u_j\int_0^T X_j(s)ds - \sum_{j=1}^{m} z_jX_j(T)\right)\right]. \tag{17}$$

Before solving the Laplace transform above, let us introduce $f(t,u,z,\Lambda,X)$ as the exponentially-affine characteristic function conditioned on the vector of jump arrival rates $\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_m)$, for all input $(t,u,z,\Lambda,x) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^m \times [0,\infty]^m \times \mathbb{R}^m$ where the constant vector $u = (u_1, \dots, u_m)$, the constant vector $z = (z_1, \dots, z_m)$, the vector of initial values at time t of market factors $x = (x_1, \dots, x_m)$.

For a given value of Λ, let $\mathrm{F} = (\mathcal{F}_t)_{t\geq 0}$ be the filtration generated by market factors $X_1, X_2, \dots, X_m$, which contains $\mathcal{F}_t := \sigma\{\bigcup_{j=1}^{m} X_j(s), 0 \leq s \leq t\}$. P is a risk neutral measure. By the Feynman-Kac approach, the characteristic function $f(t,u,z,\Lambda,x)$ has the stochastic representation

$$f(t,u,z,\Lambda,x) = \mathrm{E}_t^x\left[\exp\left(-\sum_{j=1}^{m} u_j\int_t^T X_j(s)ds - \sum_{j=1}^{m} z_jX_j(T)\right) \mid \Lambda\right] \tag{18}$$

According the notation $\mathrm{E}_t^x[\cdot \mid \Lambda]$, it implies that the expectation is taken conditional on time t information or the t-time filtration $\mathcal{F}_t$ with $X(t) = x$, and the vector of jump arrival rates Λ.

A key idea in solving the Laplace transform function (17) is to make the connection to the characteristic function $f(t,u,z,\Lambda,X(t))$ that is

$$\phi^{x}(T,u,z,X(0)) = \mathrm{E}[f(0,u,z,\Lambda,X(0))].$$

Our proposed model has jump arrival rates following Gamma distributions, not constant variables. Hence we first find the solution of the function $f(t,u,z,\Lambda,X(t))$ that treats jump counting processes as Poisson processes conditioned on the arrival rate of jumps Λ. Then we find the expected value of the function $f(t,u,z,\Lambda,X(t))$ unconditioned the jump arrival rates Λ considering that the jump counting processes are actually Gamma-Poisson mixture processes.

Now, let's start off by solving the characteristic function $f(t,u,z,\Lambda,X(t))$. To understand the equation (18) better, we rewrite the characteristic function f and multiply both sides by $e^{-\sum_{j=1}^{m} u_j \int_0^t X_j(s)ds}$, then have

$$e^{-\sum_{j=1}^{m} u_j \int_0^t X_j(s)ds} f\big(t,u,z,\Lambda,X(t)\big) = \mathrm{E}_t^x\left[e^{-\sum_{j=1}^{m} u_j \int_0^T X_j(s)ds} f(T,u,z,\Lambda,X(T)) \mid \Lambda\right].$$

Thus $e^{-\sum_{j=1}^{m} u_j \int_0^t X_j(s)ds} f\big(t,u,z,\Lambda,X(t)\big)$ is a martingale.

Let us denote by Q^j the vector of length m which has that the jth element is one and the rest of elements are zeros, $\Pi_j(t)$ a counting Poisson process of a jump process at time t with Gamma distributed intensity Λ^j with the shape parameter α^j and the scaled parameter $\mathcal{B}^j$. According to multivariable Ito's formula for jump processes, we have

$$\begin{aligned} &f(t,u,z,\Lambda,X(t)) \\ &\quad= f(0,u,z,\Lambda,X(0)) + \int_0^t f_t(s,u,z,\Lambda,X(s))ds \\ &\quad+ \sum_{j=1}^{m}\int_0^t f_{x_j}(s,u,z,\Lambda,X(s))\, d\left(k_j\left(\theta_j - X_j(s)\right)ds + \sigma_j dW_j(s)\right) \\ &\quad+ \frac{1}{2}\sum_{j=1}^{m}\sum_{v=1}^{m}\int_0^t f_{x_j x_v}(s,u,z,\Lambda,X(s))\,\rho_{jv}\sigma_j\sigma_v ds \\ &\quad+ \sum_{J=1}^{m}\int_0^t\int_0^\infty \big[f\big(s,u,z,\Lambda,X(s)+\varepsilon^j Q^j\big) \\ &\quad- f(s,u,z,\Lambda,X(s))\big]g\big(\varepsilon^j;\mu^j\big)d\varepsilon^j d\Pi^j(s). \end{aligned}$$

Note that the jump sizes $\varepsilon^1,\varepsilon^2,\dots,\varepsilon^m$ are generated from the probability density function of an exponential distribution $g\big(\varepsilon^j;\mu^j\big) = \frac{1}{\mu^j}\exp(-\frac{\varepsilon^j}{\mu^j})$ with their respective mean $\mu^j, 1 \le j \le m$.

In order to obtain the martingale part of the equality, we subtract the $\sum_{j=1}^{m}\int_0^\infty \big[f\big(s,u,z,\Lambda,X(s)+\varepsilon^j Q^j\big) - f(s,u,z,\Lambda,X(s))\big]g\big(\varepsilon^j;\mu^j\big)d\varepsilon^j\, d\big(\Lambda^j s\big)$ term from $d\Pi^j(s)$ and add it back to the corresponding ds term. We take the form

$$
\begin{aligned}
f(t,u,z,\Lambda,X(t)) \\
&= f(0,u,z,\Lambda,X(0)) \\
&+ \int_0^t \Bigg[f_t(s,u,z,\Lambda,X(s)) + \sum_{j=1}^{m} f_{x_j}(s,u,z,\Lambda,X(s))k_j\left(\theta_j - X_j(s)\right) \\
&+ \frac{1}{2}\sum_{j=1}^{m}\sum_{v=1}^{m}\int_0^t f_{x_j x_v}(s,u,z,\Lambda,X(s))\,\rho_{jv}\sigma_j\sigma_v \\
&+ \sum_{j=1}^{m}\Lambda^j \int_0^\infty \big[f\big(s,u,z,\Lambda,X(s)+\varepsilon^j Q^j\big) \\
&- f(s,u,z,\Lambda,X(s))\big]g\big(\varepsilon^j;\mu^j\big)d\varepsilon^j \Bigg] ds \\
&+ \sum_{j=1}^{m}\Bigg[\int_0^t f_{x_j}(s,u,z,\Lambda,X(s))\,d\left(\sigma_j dW_j(s)\right) \\
&+ \int_0^t\int_0^\infty \big[f\big(s,u,z,\Lambda,X(s)+\varepsilon^j Q^j\big) \\
&- f(s,u,z,\Lambda,X(s))\big]g\big(\varepsilon^j;\mu^j\big)d\varepsilon^j\, d\big(\Pi^j(s)-\Lambda^j s\big)\Bigg],
\end{aligned}
$$

Hence, the stochastic differential equation of the characteristic function f becomes

$$
\begin{aligned}
df = \Bigg[& f_t(t,u,z,\Lambda,X(t)) + \sum_{i=1}^{m} f_{x_i}(t,u,z,\Lambda,X(t))k_j\left(\theta_j - X_j(t)\right) \\
&+ \frac{1}{2}\sum_{j=1}^{m}\sum_{v=1}^{m} f_{x_j x_v}(t,u,z,\Lambda,X(t))\rho_{jv}\sigma_j\sigma_v \\
&+ \sum_{j=1}^{m}\Lambda^j \int_0^\infty \big[f\big(t,u,z,\Lambda,X(t)+\varepsilon^j Q^j\big) \\
&- f(t,u,z,\Lambda,X(t))\big]g\big(\varepsilon^j;\mu^j\big)d\varepsilon^j \Bigg] dt \\
&+ \sum_{j=1}^{m}\Bigg[f_{x_j}(t,u,z,\Lambda,X(t))\sigma_j dW_j(t) \\
&+ \int_0^\infty \big[f\big(t,u,z,\Lambda,X(t)+\varepsilon^j Q^j\big) \\
&- f(t,u,z,\Lambda,X(t))\big]g\big(\varepsilon^j;\mu^j\big)d\varepsilon^j\, d\big(\Pi^j(t)-\Lambda^j t\big)\Bigg].
\end{aligned}
$$

By using the product rule, we obtain

$$d\left(e^{-\sum_{j=1}^{m} u_j \int_0^t X_j(s)ds} f\right)$$

$$= f de^{-\sum_{j=1}^{m} u_j \int_0^t X_j(s)ds} + e^{-\sum_{j=1}^{m} u_j \int_0^t X_j(s)ds} df$$
$$+ de^{-\sum_{j=1}^{m} u_j \int_0^t X_j(s)ds} df.$$

Because $e^{-\sum_{j=1}^{m} u_j \int_0^t X_j(s)ds} f$ is a martingale, the dt term must be zero. The partial differential equation of $e^{-\sum_{j=1}^{m} u_j \int_0^t X_j(s)ds} f$ is

$$0 = -\left(\sum_{j=1}^{m} u_j X_j(t)\right) f + f_t + \sum_{j=1}^{m} f_{x_j} k_j \left(\theta_j - X_j(t)\right) + \frac{1}{2}\sum_{j=1}^{m}\sum_{v=1}^{m} f_{x_j x_v} \rho_{jv}\sigma_j\sigma_v$$
$$+ \sum_{j=1}^{m} \Lambda^j \int_0^\infty \left[f\left(s,u,z,\Lambda,X(t)+\varepsilon^j Q^j\right) - f(s,u,z,\Lambda,X(t))\right] g\left(\varepsilon^j;\mu^j\right) d\varepsilon^j.$$

We want to have the function $f(t,u,z,\Lambda,X(t))$ in the affine form

$$f(t,u,z,\Lambda,x) = e^{a(T-t,u,z)+\sum_{j=1}^{m} b^j(T-t,u_j,z_j)X_j(t)+\sum_{j=1}^{m} c^j(T-t,u_j,z_j)\Lambda^j} \tag{19}$$

where $a: [0,T]\times\mathbb{R}^{\mathrm{m}}\times\mathbb{R}^{\mathrm{m}} \to \mathbb{R}$, $b^j: [0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, and $c^j: [0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ with the initial conditions $a(0,u,z) = 0$, $b^j(0,u_j,z_j) = -z_j, c^j(0,u_j,z_j) = 0$, for $1 \le j \le m$. In doing so, the affine-form function $f(t,u,z,\Lambda,X(t))$ (18) must satisfy the partial differential equation

$$0 = -\left(\sum_{j=1}^{m} u_j X_j(t)\right) - \frac{\partial a(T-t,u,z)}{\partial(T-t)} - \sum_{j=1}^{m} \frac{\partial b^j(T-t,u_j,z_j)}{\partial(T-t)} X_j(t)$$
$$- \sum_{j=1}^{m} \frac{\partial c^j(T-t,u_j,z_j)}{\partial(T-t)} \Lambda^j + \sum_{j=1}^{m} b^j(T-t,u_j,z_j) k_j \left(\theta_j - X_j(t)\right)$$
$$+ \frac{1}{2}\sum_{j=1}^{m}\sum_{v=1}^{m} \rho_{jv}\sigma_j\sigma_v b^j(T-t,u_j,z_j) b^v(T-t,u_v,z_v)$$
$$+ \sum_{j=1}^{m} \Lambda^j \int_0^\infty \left[e^{b^j(T-t,u_j,z_j)\varepsilon^j} - 1\right] g\left(\varepsilon^j;\mu^j\right) d\varepsilon^j.$$

Accordingly, the ordinary differential equations are

$$\frac{\partial a(s,u,z)}{\partial s} = \sum_{j=1}^{m} k_j\, \theta_j b^j(s,u_j,z_j) + \frac{1}{2}\sum_{j=1}^{m}\sum_{v=1}^{m} \rho_{jv}\sigma_j\sigma_v b^j(s,u_j,z_j) b^v(s,u_v,z_v),$$

$$\frac{\partial b^j(s,u_j,z_j)}{\partial s} = -u_j - k_j b^j(s,u_j,z_j),$$

$$\frac{\partial c^j(s,u_j,z_j)}{\partial s} = \int_0^\infty \left[e^{b^j(s,u_j,z_j)\varepsilon^j} - 1\right] g\left(\varepsilon^j;\mu^j\right) d\varepsilon^j.$$

The solutions of them can be found in Appendix A.

Here is a quick summary of the explicit solution of the Laplace transform function (17)

$$\begin{aligned}\phi^x(T,u,z,X(0)) &= \mathrm{E}[f(0,u,z,\Lambda,X(0))]\\ &= e^{a(T,u,z)+\sum_{j=1}^m b^j(T,u_j,z_j)X_j(0)}\prod_{j=1}^m E\left[e^{c^j(T,u_j,z_j)\Lambda^j}\right].\\ &= e^{a(T,u,z)+\sum_{j=1}^m b^j(T,u_j,z_j)X_j(0)}\prod_{j=1}^m\left(1-c^j(T,u_j,z_j)\mathcal{B}^j\right)^{-\alpha^j}\end{aligned}$$

The moment generating function of the gamma distribution is given by

$$E\left[e^{c\Lambda^j}\right]=\int_0^\infty e^{c\Lambda^j}\ g\left(\Lambda^j;\alpha^j,\mathcal{B}^j\right)d\Lambda^j=\left(1-c\mathcal{B}^j\right)^{-\alpha^j},$$

where the probability density function $g\left(\Lambda^j;\alpha^j,\mathcal{B}^j\right)=\frac{\Lambda^{j^{\alpha^j-1}}e^{-\frac{\Lambda^j}{\mathcal{B}^j}}}{\Gamma(\alpha^j)\mathcal{B}^{j^{\alpha}}}$.

According to the stochastic differential equation (2), it is possible to have negative market factors. However negative intensities are not allowed to happen in practice. To make it clear, we use the equation (2) for deriving the closed-form solution of the Laplace transform that we have stated in the introduction. In reality, market factors are assumed to be always positive in such a way that

$$X_j^+(t)=\max(X_j(t),0)\ \text{ at every time } t\geq 0, 1\leq j\leq m.$$

To be more accurate, the ith firm's intensity process can be rewritten as

$$\lambda_i(t)=\sum_{j=1}^m\beta_{i,j}\,X_j^+(t)+Y_i(t), 1\leq i\leq n.$$

We need some constraints to prevent negative intensities in order to have the solutions we derive from the Laplace transform close to the real outcomes. In doing so, we select values of the any market factor j's parameters that satisfy the two constraints. The first constraint is as follows:

$$\begin{aligned}1\geq \mathrm{E}\left[\exp\left(-\int_0^{s_1}X_j(v)dv\right)\right]\geq \mathrm{E}\left[\exp\left(-\int_0^{s_2}X_j(v)dv\right)\right]\geq\ldots\\ \geq \mathrm{E}\left[\exp\left(-\int_0^{t}X_j(v)dv\right)\right]\geq 0\end{aligned}$$

where $0\leq s_1\leq s_2\leq\ldots\leq t$.

The approach above also applies to the firm i's idiosyncratic factor i or any positive intensity process. In this thesis, the important thing to keep in mind is that market factors are correlated. It must include the second constraint:

$$0\leq E\left[\exp\left(-\sum_{j=1}^m u_j\int_0^T X_j(s)ds-\sum_{j=1}^m z_jX_j(T)\right)\right]\leq 1,\text{for any } z_j,u_j\in\mathbb{R}^+.$$

Unlike the first constraint, $E\left[\exp\left(-\sum_{j=1}^{m} u_j \int_0^T X_j(s)ds - \sum_{j=1}^{m} z_j X_j(T)\right)\right]$ (a.k.a the Laplace transform we have derived) is not necessary to be decreasing with respect to time when parameters $u_i > 1, z_i > 1, 1 \leq i \leq m$ plugged in the equation.

จุฬาลงกรณ์มหาวิทยาลัย
CHULALONGKORN UNIVERSITY

CHAPTER V
NUMERICAL RESULTS

In this chapter, we show how to calibrate our model to multi-name portfolios such as CDOs. The data and tools that are used for computation are stated later. Then we display numerical results of our model used to price the CDO index tranches. We also analyze the portfolio loss distributions that are implied from quoted market index tranches using our proposed model and make a comparison of performance with some models we have studied.

Our model is able to apply to both heterogeneous and homogeneous portfolios. We demonstrate them as follows:

- For the homogeneous portfolio, every firm in a portfolio is assumed to have the same weight, notional principle and recovery rate. Restating the equation (1)

$$\lambda_i(t) = \sum_{j=1}^{m} \beta_{i,j} X_j(t) + Y_i(t), 1 \leq i \leq n.$$

We set the market factor loading $\beta_{i,j}$ of any firm i to the market factor j to be 1 and ignore the idiosyncratic factor Y. Each market factor has 7 parameters. We only use market factors' parameters for fitting index tranches. More specifically, we find the calibrated parameters of market factors such as the rate of mean-reversion k, the volatility σ, the shape parameter α, the scaled parameter $\mathcal{B}$, the mean of jump size μ, and the correlation parameter ρ that reduces the Root Mean Square Error (RMSE) the most. The RMSE is given by

$$\text{RMSE} = \sqrt{\frac{1}{K}\sum_{k=1}^{K}\left(\frac{s_k - \frac{\left(s_k^a + s_k^b\right)}{2}}{s_k^a - s_k^b}\right)^2}$$

where $\mathrm{s_k}$ is the k-th credit index tranche spread of the model and $\mathrm{s_k^a}$, $\mathrm{s_k^b}$ are the ask and bid price of index tranche spread from the market, and K is the number of index ranches. The long-term mean θ and the initial value of the market intensity $X(0)$ are used to fit the CDO index spread. The CDO index spread can be view as the weighted mean of underlying CDS spreads, computed as the running spread of the 0-100% index tranche.

Due to the fact that the model has many parameters, there are many local solutions of parameter values. It is hard to find the optimal solutions by using numerical methods alone. The optimal solutions are majorly led by initial value parameters. Thus we first initialize values of parameters and then use a numerical method such as Multivariate Newton's Method to adjust them.

To initialize the model's values of $m \times 7$ parameters, we perform as follows:

1. It begins with setting based parameter values.
2. We estimate the parameters of a jump component of the market factor (with large value of the scaled parameter of Gamma $\mathcal{B}$, the small value of Gamma's shape parameter $\mathcal{B}$) that has the ability to create the high serial correlation of defaults. Increasing serial default correlation through jump parameters particularly results in increasing the senior and super senior tranche spreads, which doesn't cause exaggeratedly upshifting in the prices of other tranches. Those jump process's parameters have an immense impact on the extreme tail risk and therefore are adjusted to match quoted market spreads of the senior tranche and especially the super senior tranche.
3. There are another kind of jump processes in market factors (with the small value of the scaled parameter of Gamma $\mathcal{B}$, the large value of Gamma's shape parameter α) that can produce heavier and longer tail of the portfolio loss distribution with low serial default correlation. The parameters of that market factor's jump process are used to fit the equity tranche and the mezzanine tranches. If the mean of jump size μ are too large, all index tranches are more likely to be exceedingly overestimated. Hence the calibrated jump size's mean usually has a small value. Moreover, this jump process of the market factor doesn't affect the spread of the super senior tranche.
4. The rest of market factors' parameters are used to fit the index tranches more accurately. Changing values of parameters of market factors such as the rates of mean-reversion has a considerable effect on many index tranches. As a result, we might have to recalibrate market factors' jump process's parameters. We also use the correlation parameters for specifically adjusting the equity tranche and the mezzanine 1 tranche.
5. We continue revising market factors' parameters according to step 2-4 until the RMSE is acceptable. Each step of initialization has the reasons which will be illustrated later in the rest of this chapter.

- For the heterogeneous portfolio, our proposed model can be calibrated by adopting Peng and Kou [2]'s calibration algorithm. Under the heterogeneous portfolio assumption, the model is calibrated to all index tranche market quotes, the CDO index spread including its referenced firms' CDS spread all together. According to Peng and Kou [2]'s calibration algorithm, market factors' parameters are considered as free parameters used to fit index tranches. The idiosyncratic factors are concerned as small error terms while market factor loadings can be estimated by using their optimization problem. Nonetheless, there are numerous parameters such as market factor loadings (about $m \times 7 \times n$ parameters, m is the number of market factors in the system, n is the number of referenced firms in the index) to be gauged. Papageorgiou [7] argues that by doing so the model is over parameterized. Papageorgiou [7] uses the name grouping method to decrease the number of idiosyncratic factors and the results are more satisfied. However, we don't show the results from the case of heterogeneous portfolios because the empirical results under the homogeneous portfolio assumption are satisfyingly acceptable.

5.1 Data and Tools

CDX NA IG and Itraxx Europe index are chosen to be calibrated on models. We consider those indices that mature in 5 years. CDX NA IG and Itraxx Europe are relied on 125 CDS indices of investment grade firms in North American and Europe respectively. Every series of CDX and Itraxx is released on every March and September. The recently issued index that has a 5-year maturity is regarded as the on-the-run index. The on-the-run index is actively traded and then has no arbitrage. On the contrary, the off-the-run index is older and passively traded. There are the equity tranche, the mezzanine tranche, the senior tranche and the super senior tranche classified by the level of the portfolio loss, ranked from the riskiest to the safest. From series 1 to series 11, the equity tranche has different mechanism of pricing from the other tranches, paying an upfront cash with a 500-bps fixed running spread whereas the other tranches pay running spreads only. After the subprime crisis 2007-2008, other tranches have upfront fees with fixed running spreads. For instance, all index tranches of are quoted on upfront fees with a 100-bps fixed running spread for CDX NA IG series 12 to 14.

As stated earlier in the introduction, we assume credit derivatives portfolios to be homogeneous. The prices of the CDS index and the index tranches that we choose are from Mortensen [1], Peng and Kou [2], Choi [8] and Bloomberg. The selected indexes from those literature are mostly on-the-run indexes. A recovery rate of any underlying reference firm is set to be 40%. For interest rates, we found that they can be assumed to be constant or bootstrapped from swap rates. The characteristics of interest rates don't directly have an impact on the performance of the model in capturing the events of joint defaults. The important things are the model of default intensities and the method that is used to implement the model for computing the portfolio loss distribution.

Papageorgiou [7] suggests that in order to reduce errors from calculating the large portfolio loss distribution which is based on the binomial distribution, the number of underlying reference firms should not be exceeded 30. However, we solve the big number issue by using the GNU MPFR library. It is an open source and supports C/C++ programming language for computations of high precision numbers. The computation time depends on the digit precision whose numbers are set. When using the MPFR library, our program could compute spreads of the 125-firm CDO tranches faster than compared to simulation methods.

5.2 Results

We price credit index tranche using our proposed model and then make a comparison between our proposed models themselves under following circumstances:

- With and without correlation between market factors.
- Two correlated market factors and three independent market factors.

We set up the case studies above to investigate performance improvement of the correlated market factor model. The model is also compared with other models that are as follows:

- Mortensen [1]'s Multi-Name Intensity Model. We want to verify that Gamma-Poisson mixture processes outperform Poisson processes for modeling jumps in intensity processes.
- Peng and Kou [2]'s Conditional Survival Model. We want to verify that our model that has jump processes being components of market factors is capable of joint default events, fat tails and tail dependence. There is no need to model jump processes as cumulative intensities like Peng and Kou [2]'s Conditional Survival model accordingly. If correct, our proposed model could price CDOs as well as theirs.

จุฬาลงกรณ์มหาวิทยาลัย
CHULALONGKORN UNIVERSITY

Test 1

To examine the sensitivity of the portfolio loss distribution to market factors' correlation, we establish the models that have the same parameter values of market factors with different correlation parameters. As shown in Figure 1, the correlation parameters are easy to understand. The more positive correlation between market factors the model has, the more dispersion of the portfolio loss distribution is. In contrast, the models with negative correlation parameters generate higher peak and thinner-tail distributions.

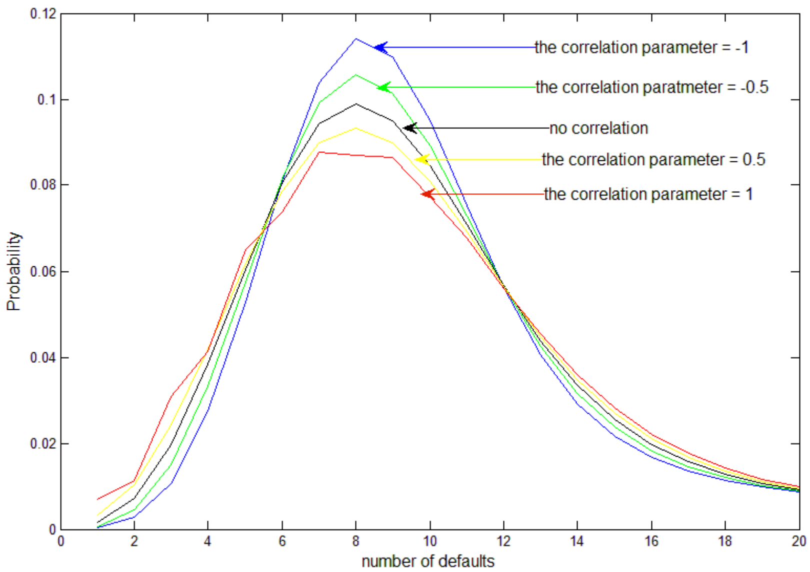

Figure 1: The 5-year loss distribution of the 125-firm portfolio from three two-market-factor models. The models with the correlation parameters -1 (perfectly negative), -0.5, without correlation, and the correlation parameter -0.25, 1 (perfectly positive) are corresponded to the blue line, the green line, the black line, the yellow line, and the red line respectively. All models have the sets of the first market factor' parameters $(k_1 = 2, \theta_1 = 0.005, X_1(0) = 0.001, \sigma_1 = 0.0085, u_1 = 0.5, \alpha_1 = 0.05, \mathcal{B}_1 = 1)$ and $(k_2 = 1, \theta_2 = 0.0085, X_2(0) = 0.0005, \sigma_2 = 0.0055, u_2 = 0.05, \alpha_2 = 1, \mathcal{B}_2 = 0.08)$ for the second market factor. The recovery rate is 40% and the interest rate is 5%.

Let's begin with calibrating our proposed model to CDX NA IG S2 5Y and CDX NA IG S5 5Y on August 23, 2004 and December 5, 2005 respectively. CDX NA IG S2 5Y was launched on March 23, 2004 and matured on June 22, 2009 with quarterly coupons. CDX NA IG S4 5Y was launched on September 21, 2005 and matured on December 20, 2010. The interest rate and recovery rate are assumed to be 5% and 35% respectively. The equity tranche has an upfront fee with a running spread of 500 bps while other tranches are quoted on running spreads without upfront fees. We choose those index series in order to compare our results to Mortensen [1]'s. Most importantly, we analyze the impact of the correlation between market factors on index tranche spreads and study performance in pricing of three-independent-market-factor model. The calibrated parameters are represented in Table 1.

Date (Series)	Part	Para-meter	2-factor model		3-independent-factor model		
			Factor 1	Factor 2	Factor 1	Factor 2	Factor 3
23/4/2004 (CDX S2)	Cont.	k	0.1000	1.3000	0.1000	1.3000	2.0000
		θ	0.0040	0.0042	0.0030	0.0035	0.0006
		x_0	0.0004	0.0006	0.0003	0.0005	0.0002
		σ	0.0008	0.0006	0.0006	0.0005	0.0004
	Jump	u	8.0000	0.0720	6.0000	0.0600	0.1000
		α	0.0010	10.000	0.0010	10.000	1.0000
		B	25.000	0.0100	25.000	0.0100	0.0300
	Correlation ρ		-1.0000				
5/12/2005 (CDX S5)	Cont.	k	0.6000	2.0000	0.6000	2.0000	1.5000
		θ	0.0030	0.0040	0.0030	0.0035	0.0004
		x_0	0.0020	0.0008	0.0020	0.0007	0.0002
		σ	0.0010	0.0048	0.0010	0.0042	0.0005
	Jump	u	1.0000	0.0168	1.0000	0.0147	0.0005
		α	0.0016	0.0700	0.0016	0.0700	3.0000
		B	2.0000	2.5000	2.9000	2.5000	0.2000
	Correlation ρ		-1.0000				

Table 1: Estimated parameters of our models on August 23, 2004 for CDX NA IG S2 5Y and December 5, 2005 for CDX NA IG S4 5Y.

As can be seen in Table 2 and Table 3, the 2-correlated-market-factor model prices CDO tranche spread slightly better than the 2-uncorrelated-market-factor model. Although the correlation parameter is -1 or 1, the model still has two market factors if there are different jump processes. The results implies that the negative correlation parameter causes in decreasing the mezzanine 1 (3-7%) tranche spread while increasing the equity (0-3%) tranche spread. Even though the changes in the spreads of the equity (0-3%) tranche and the first mezzanine (3-7%) tranche are visually seen small in Table 3, they are considerably large when concerned with bid-ask spread. Admittedly, it is feasible that adjusting some parameters can reproduce the outcome of the model that has correlated market factors. However several index tranches get affected while changing some values of parameters as the rates of mean-reversion. Contrary to other parameters, the correlation parameter can be varied and has a particular influence on the equity tranche and the first mezzanine tranche without considerably affecting other index tranches.

Tranches %	Source					
	Market	Bid/Ask	Mortensen	2-factor model		3-idp-factor model
				Correlated	Uncorrelated	
0-3	40.0%	2.0%	46.9%	39.4%	39.32%	40.11%
3-7	312.5	15.0	340.2	318.3	319.29	349.83
7-10	122.5	7.0	119.7	118.4	118.52	120.94
10-15	42.5	7.0	61.9	45.0	45.0	42.61
15-30	12.5	3.0	14.3	12.7	12.7	12.08
RMSE			2.1	0.37	0.40	1.12

Table 2: Comparison of the results of our models and the old results of Mortensen [1],and the CDX NA IG S2 5Y index tranche spreads on August 23, 2004. The equity (0-3%) tranche pays an upfront cash with 500-bps running spread. The other tranches are quoted on running coupons.

Tranches%	Source					
	Market	Bid/Ask	Mortensen	2-factor model		3-idp-factor model
				Correlated	Uncorrelated	
0-3	41.1%	0.8%	43.2%	40.96%	40.59%	41.07%
3-7	117.5	6.8	125.9	122.02	127.42	121.04
7-10	32.9	5.3	30.6	33.27	33.40	29.36
10-15	15.8	3.0	21.3	15.85	15.88	14.53
15-30	7.9	1.0	8.8	7.33	7.33	8.01
RMSE			1.58	0.40	0.76	0.43

Table 3: Comparison of the results of our models and the old results of Mortensen [1], and the CDX NA IG S2 5Y index tranche spreads on December 5, 2005. The equity (0-3%) tranche pays an upfront cash with 500-bps running spread. The other tranches are quoted on running coupons.

Table 2 displays that the RMSE of the 3-uncorrelated-market-factor model is greater than the 2-factor model's. Conversely, Table 3 shows that the prices of CDO tranches from the 3-indepenent-market-factor model are more accurate than the results of the 2-independent-market-factor model. We have to calibrate a set of parameters for one more market factor but obtain similar results. Consequently, the correlation parameter is more satisfied in the sense of convenience. The reason why the model that has three independent market factors is counterintuitively underachieving is that the two-correlated-market-factor model has already created the necessary shapes of the portfolio loss distribution. It is really hard to compete the model that has a small value of RMSE.

Moreover, it is shown that our model can solve Mortensen [1]'s problem about overpricing equity (0-3%) tranche, the mezzanine 1 (3-7%) tranche, and the mezzanine 3 (10-15%) tranche in Table 2. Likewise, Table 3 shows that our model fits the mezzanine 1 (3-7%) tranche better that the jump-diffusion model of [4]'s. The

RMSEs from August 23, 2004 and December 5, 2005 of our proposed model are smaller.

Note that the results of Mortensen [1] based on the assumption that the index's underlying reference firms are heterogeneous.' jump-diffusion model has a market factor of 6 free parameters used to fit the index tranche and $7 \times n$ idiosyncratic parameters including the market factor loading calibrated to underlying CDSs. Thus there are $6 + 7 \times n$ estimated parameters. Mortensen [1] states that for iTraxx Europe the homogenous and heterogeneous portfolios yield similar outcomes, which doesn't apply to CDX NA. He said that the heterogeneous portfolio assumption is more attainable for CDX NA. In spite of the assumption of homogenous portfolios, our model could price the CDX index's tranche spreads with smaller RMSEs.

Test 2

Following Peng and Kou [2]'s footsteps, we calibrate our models to ITraxx Europe S8 5Y on March 14, 2008 and iTraxx Europe S9 5Y on September 16, 2008, which are good examples to demonstrate market during financial crisis of 2008. The iTraxx Europe S8 5Y was released on September 20, 2007 and matured on December 20, 2012. The iTraxx Europe S9 5Y was released on March 20, 2008, and matured on June 20, 2013. Both index series have 124 underlying firms rated as investment grade. There are numerous firms shared between those iTraxx series. Every firm is assumed to have a 40% recovery rate. The interest rates are extracted from Euro swap rates.

To make sure that our proposed model literally outperforms the Mortensen [1]'s jump-diffusion model, we use Mortensen [1]'s jump-diffusion model calibrated against those series under the same homogenous portfolio assumption. In this case, there are 6 parameters to be gauged, which all of them are treated as free parameters for matching spreads of the index tranches.

For the jump-diffusion model, the values of parameters we obtain from calibration to iTraxx Europe S8 5Y on March 14, 2008 are $k = 0.02$, $\theta = 0.14$, $\lambda_0 = 0.003$, $\sigma = 0.0721$, $\mu = 0.7$, and $\ell = 0.025$. Whereas the estimated of iTraxx Europe S9 5Y on September 16, 2008 are $k = 0.02$, $\theta = 0.13$, $\lambda_0 = 0.003$, $\sigma = 0.0721$, $\mu = 0.7$, and $\ell = 0.02$. For our proposed model, the estimated parameters are shown in Table 4.

Date(Series)	k	θ	x_0	σ	u	α	B	ρ
14/3/2008 (iTraxx S8)	0.1	0.00750	0.00048	0.00060	30.0000	0.00800	550.000	-1
	2	0.00440	0.00040	0.00400	0.12000	24.0000	0.00500	
16/9/2008 (iTraxx S9)	0.1	0.00250	0.00015	0.00050	1.00000	0.01000	37.0000	1
	2	0.00500	0.00050	0.00500	0.10500	7.00000	0.02000	

Table 4: Estimated parameters of the two-correlated-market-factor model on March 14, 2008 for iTraxx Europe S8 5Y and September 16, 2008 for iTraxx Europe S9 5Y

As noted before, there are differently correlated market factors in the model if market factors have distinctive jump processes. The series 8 and series 9 of iTraxx Europe have a lot of underlying firms in common so their calibrated parameters have closely values. The correlation parameter $\rho = -1$ is used to diverse the portfolio of iTraxx Europe S8 on March 14, 2008. For the iTraxx Europe S9 on September 16, 2008, it is interpretable that the correlation parameter $\rho = 1$ is used to disperse the portfolio loss distribution by increasing the equity tranche spread and decreasing the mezzanine tranche spread.

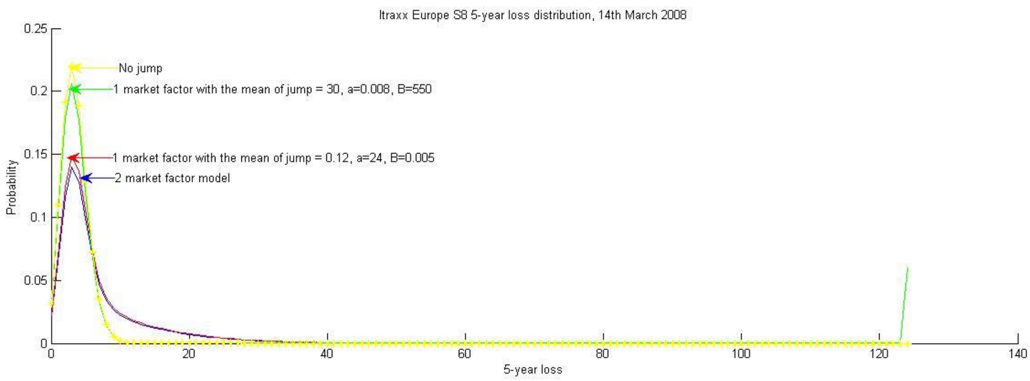

Figure 2: The implied 5-year loss distribution for iTraxx Europe S8 5Y on March 14, 2008 of the two-factor models with the different jump processes.

With two market factors, our proposed model is flexible to fit index tranche spreads. The Gamma-Poisson mixture distribution offers two interesting shapes of the portfolio loss distribution. As can be seen in Figure 2, the green line demonstrates the intensity-based model incorporating with the jump process with the mean of exponential distributed jump sizes $\mu = 30$, and the arrival rate of jumps driven by Gamma-Poisson mixture process with the shape parameter $\alpha = 0.008$ and the scaled parameter $\mathcal{B} = 500$. This kind of jump processes has high serial correlation and generates steeply increased probability of loss at the end of tail, sharpest peak and thinnest right tail of the probability of loss around 0-20 firms. The jump process with low serial correlation has the jump size's mean $\mu = 0.12$, and arrival of jump driven by Gamma-Poisson mixture process with the shape parameter $\alpha = 24$ and the scaled parameter $\mathcal{B} = 0.005$, represented by the red line. Between losses of 0-20 firm, the lower serial-correlated jump process creates a fatter right tail of the portfolio loss distribution. Corresponded by the blue line, the combination between those two jump processes produces the lowest peak of the portfolio loss distribution but has the capacity to generate both characteristics of fat-tailed distributions. The yellow line displays what would be expected from the bivariate drift-diffusion without jump processes. As shown in Figure 3, the default times that are driven by the model that has correlated market factors with jump processes are more clustering when compared with the model that has dependent market factors without jump processes.

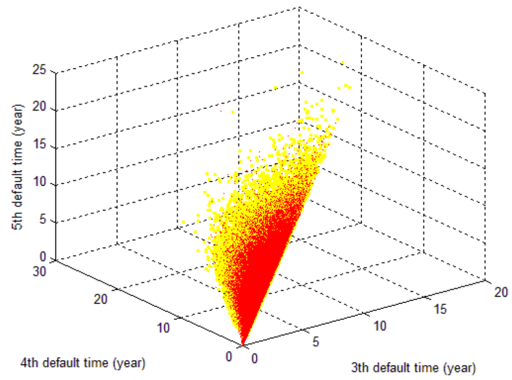

Figure 3: The scatter plot of 50000 scenarios of the times of third τ^3, fourth τ^4, and fifth τ^5 defaults in a 125-firm portfolio that are simulated by using a Mimicking Markov chain method. The values of parameters are from calibration on March 14 2008 of iTraxx Europe series 8, which are shown in Table 4. The red maker and the yellow maker respectively correspond to the model with and without jump processes. The means of the default times τ^3, τ^4, τ^5 which are generated by the model with jump processes are 3.5246, 4.2879 and 4.9934 years respectively. For the model without jumps in intensities, we obtain the averages of the default times τ^3, τ^4, τ^5 which respectively are 4.2648, 5.3537, and 6.4013 years.

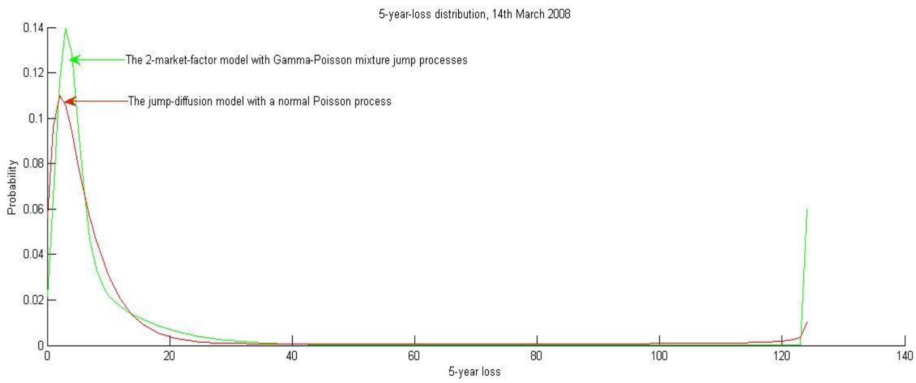

Figure 4: The implied 5-year loss distribution of the two-correlated-market-factor model and the jump-diffusion model from the Itraxx Europe S8 5Y index on March 14, 2008.

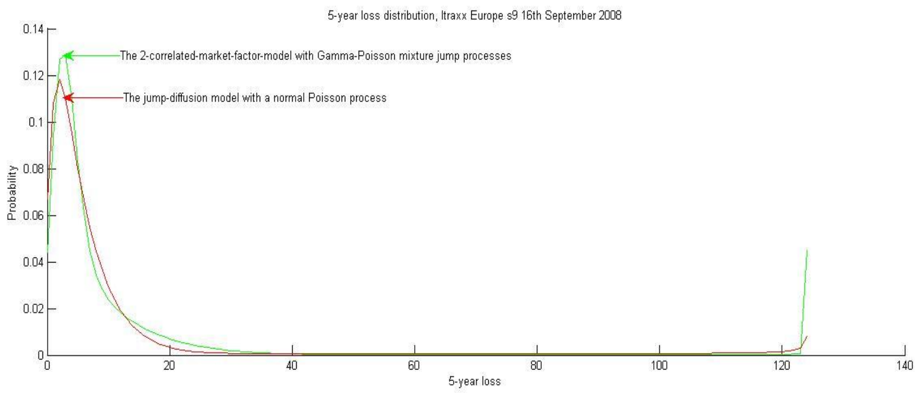

Figure 5: The implied 5-year loss distribution of the two-correlated-market-factor model and the jump-diffusion model from the itraxx Europe S9 5Y index on September 16, 2008

In Figure 4 and Figure 5, the 5-year loss distribution of the two-correlated-market-factor model has a higher peak and fatter right tail of 5-year loss and is more leptokurtic than the jump-diffusion model's loss distribution. The jump process with high exponential distributed jump size's mean and the jump arrival rate distributed as Gamma that has low value of the shape parameter α and high value of the scaled parameter $\mathcal{B}$ has high serial correlation. As a result, it particularly helps to fit the senior and super senior tranche spread. It is shown by dramatically upward trend of the green line at the end of the right tail of 5-year loss distribution in Figure 4 and Figure 5. With large mean of jump size and jump's arrival rate of the Poisson process, the model overestimates all CDO tranche spreads. No matter how many times we've tried to manipulate parameters values, the Poisson process could not reach the same level and curvature as the Gamma-Poisson mixture process has done in the tail. It is because the Poisson process has no ability to create serial correlation.

Tranches %	Source				
	Market	Bid/Ask	Jump-diffusion model (Mortensen)	Peng&Kou	Our model
0-3	51.4%	1.6%	50.07%	50.48%	51.9%
3-6	649.0	24.3	668.65	691.14	658.9
6-9	401.1	24.5	331.03	395.47	374.5
9-12	255.3	19.8	237.00	261.23	249.5
12-22	143.4	11.8	192.47	168.62	166.6
22-100	69.9	2.9	59.88	66.96	68.0
RMSE			2.57	1.24	0.98

Table 5: Comparison of our empirical results of the jump-diffusion model and the 2-correlated factor model, and the old results of Peng and Kou [2], and the iTraxx Europe S8 5Y index tranche spreads on March 14, 2008. The equity (0-3%) tranche pays an upfront cash with 500-bps running spread. The other tranches are quoted on running coupons.

Tranches %	Source				
	Market	Bid/Ask	Jump-diffusion model (Mortensen)	Peng&Kou	Our model
0-3	45.98%	1.18%	46.40%	46.10%	46.93%
3-6	618.25	14	588.52	630.49	627.94
6-9	374.50	12.59	276.39	347.43	335.19
9-12	215.16	10.55	191.96	217.43	211.39
12-22	102.17	5.33	154.18	131.53	134.84
22-100	58.81	2.58	47.99	52.18	53.78
RMSE			5.52	2.66	2.95

Table 6: Comparison of our empirical results from the jump-diffusion model and the 2-correlated factor model, the old results of Peng and Kou [2], and the iTraxx Europe S9 5Y index tranche spreads on September 16, 2008. The equity (0-3%) tranche pays an upfront cash with 500-bps running spread. The other tranches are quoted on running coupons.

After analyzing the implied portfolio loss distributions of the jump-diffusion model and our proposed model, we now examine performance of those models for pricing. Table 5 exhibits that the jump-diffusion model underprices the mezzanine 2 (6-9%) tranche and the super senior (22-100%) tranche, overpricing the senior (12-22%) tranche. The results of the jump-diffusion model in Table 6 show that the mezzanine 1 (3-6%) tranche, the mezzanine 2 (6-9%) tranche, and the super senior (22-100%) tranche are underestimated, while the senior (12-22%) tranche is overestimated significantly. In Table 5 and Table 6, our proposed model underprices the second mezzanine (6-9%) tranche and overprices the senior (12-22%) tranche exceeding their bid-ask spreads. However compared to the jump-diffusion model, our proposed has smaller RMSEs and perform better in pricing after all. It is unarguable that our proposed model outperforms the jump-diffusion process because it has more free parameters to fit tranche spreads. Nevertheless, those free parameters are meaningless if they are not able to introduce different traits of distribution. Fortunately, Gamma-Poisson mixture processes can produce either high or low serial correlation of jumps in market factors.

We also compare our proposed model with Peng and Kou [2]'s Conditional Survival (CS) Model. For clarification, we use the former results of Peng and Kou [2] to compare with our empirical results. Peng and Kou [2] assume that the portfolio is heterogeneous. They use the model that has three market factors which are in terms of cumulative intensities following the integral CIR process with parameters $(k_1, x_1(0), \sigma_1)$, two Polya processes (a.k.a Gamma-Poisson mixture processes) with parameters $(\alpha_1, \mathcal{B}_2)$ and $(\alpha_3, \mathcal{B}_3)$ to fit index tranches. While market factor loadings are calibrated to referenced CDSs of n underlying firms. Hence there are $7 + 3 \times n$ estimated parameters.

Peng and Kou [2]'s CS model and our proposed model obviously perform better than the jump-diffusion model with small RMSEs. Furthermore our proposed model has verified that it could imitate the past numerical results of Peng and Kou [2]. Thus why their model is successfully used to price CDO tranches is because of characteristics of Gamma-Poisson jump counting processes. When there is the large jump in cumulative term of intensities, infinities are likely to be boundless. Theoretically, it is unsatisfied to allow infinite intensities to happen.

จุฬาลงกรณ์มหาวิทยาลัย
Chulalongkorn University

Date (Series)	Source	0-3%	3-7%	7-10%	10-15%	15-30%	RMSE	0-100%
19/7/2007 (CDX S7) *Bloomberg	Market	29.50%	90.25	19.93	9.65	3.41		43.15
	Bid/Ask	0.23%	2.00	1.42	1.19	0.87		
	Model	29.75%	91.95	19.70	7.69	4.28	1.05	41.08
14/3/2008 (CDX S9)	Market	67.92%	836.9	462.22	265.56	129.45		182
	Bid/Ask	0.56%	9.21	9.07	9.11	4.97		
	Choi	67.77%	843.35	440.26	288.30	125.56	1.63	167.54
	Model	68.02%	840.96	444.45	279.10	129.49	1.12	160.23
20/7/2008 (CDX S10)	Market	51.55%	447.32	240.75	124.01	66.66		115
	Bid/Ask	0.78%	7.37	6.5	4.75	3.37		
	Choi	51.92%	459.27	226.01	139.51	51.56	2.78	102.64
	Model	51.84%	455.65	232.31	132.85	67.17	1.14	102.23
16/9/2008 (CDX S7) *Bloomberg	Market	82.31%	1202.27	518.16	222.31	117.9		232.25
	Bid/Ask	4.401%	42	27.929	13.606	7.69		
	Model	77.32%	1278.54	473.87	233.53	112.72	1.28	232.41
16/10/2008 (CDX S11)	Market	71.50%	1297	676.67	209.34	66.50		173
	B/A	1.5%	50	26.67	11.33	10		
	Choi	77.84%	1386.39	588.96	269.74	53.74	3.52	162.14
	Model	71.61%	1341.91	584.59	237.99	68.56	1.95	156.13

Table 7: The fitting results of the two-correlated-market-factor on market tranche spreads for the CDX NA IG 5Y indexes between 2007-2008 and the past results of Choi [8]. The equity (0-3%) tranche pays an upfront cash with 500-bps running spread. The other tranches are quoted on running coupons.

Additionally, our proposed model could price CDOs and its CDS index really well with small RMSEs as compared to the results of Choi [8]. Choi [8] uses the equity-credit intensity-Based model which is analogous to Mortensen [1]'s jump-diffusion model. However there is trivial difference from Mortensen [1] that the CDO tranche spreads and the CDS spreads are also dependent on the stock market index such as the S&P 500 index. The link between the CDO and the S&P 500 index is formed through correlated Brownian motions of the jump-diffusion model' market factor and the stock price model that has a stochastic variance process. According to Choi [8]'s backward problem, the jump-diffusion model's parameters are first calibrated to market prices of the CDO and CDSs. Then relevant market factor's calibrated parameters are plugged into Black-Scholes model in order to extract implied volatilities from S&P 500 options. Consequently, the CDO tranche spreads and CDS spreads obtained by using this approach are not affected by changes in the values of the Black-Scholes model's parameters. Our empirical results from our model and the past results from Choi [8] are shown in Table 7 and 8. Note that Choi [8] assumes that the CDO portfolios are heterogeneous. The CDS index spread from Choi [8] is the weighted average of all underlying CDS spreads.

Date (Series)	Source	0-3%	3-7%	7-10%	10-15%	15-30%	RMSE	0-100%
8/7/2009 (CDX S12)	Market	64.00%	34.89%	16.73%	6.80%	-0.83%		139.00
	Bid/Ask	0.52%	0.53%	0.63%	0.48%	0.25%		
	Choi	66.67%	29.97%	15.73%	7.61%	-0.94%	4.86	130.06
	Model	63.93%	33.34%	18.19%	8.31%	-0.82%	2.18	127.70
8/9/2009 (CDX S12)	Market	62.38%	27.31%	10.88%	5.21%	-1.84%		121.00
	Bid/Ask	0.50%	0.50%	0.50%	0.50%	0.29%		
	Choi	61.48%	20.77%	7.75%	1.27%	-3.45%	7.83	108.33
	Model	62.37%	25.84%	12.70%	5.10%	-1.61%	2.12	107.35
8/12/2009 (CDX S13)	Market	53.42%	22.54%	8.57%	1.75%	-2.44%		98.00
	Bid/Ask	1.00%	0.78%	0.62%	0.50%	0.40%		
	Choi	54.74%	19.45%	9.05%	3.88%	-1.26%	2.99	95.11
	Model	53.49%	20.52%	8.99%	2.53%	-2.66%	1.40	84.36
8/3/2010 (CDX S13)	Market	53.81%	19.75%	7.38%	0.88%	-2.60%		89.00
	Bid/Ask	1.00%	1.00%	1.13%	0.75%	0.50%		
	Choi	54.70%	17.92%	7.30%	2.19%	-2.32%	1.22	86.65
	Model	53.80%	19.16%	7.75%	1.67%	-2.91%	0.63	82.74
8/6/2010 (CDX S9)	Market	52.95%	14.95%	-1.61%	0.81%	-1.48%		141.00
	Bid/Ask	0.52%	0.45%	0.43%	0.28%	0.09%		
	Choi	52.27%	8.38%	-3.62%	0.96%	-1.98%	7.32	127.57
	Model	53.42%	13.06%	-2.62%	0.96%	-1.47%	2.47	147.78
8/9/2010 (CDX S9)	Market	47.98%	7.23%	-5.78%	1.53%	-1.71%		118.00
	Bid/Ask	0.25%	0.25%	0.25%	0.26%	0.05%		
	Choi	47.75%	5.25%	-6.18%	-1.03%	-2.10%	5.10	110.28
	Model	47.56%	7.72%	-5.14%	0.003%	-1.78%	1.96	127.57

Table 8: The fitting results of the two-correlated-market-factor on tranche spreads of the CDX NA IG 5Y indexes between 2009-2010 and the past results of Choi [8]. The equity (0-3%) tranche, the first mezzanine (3-7%) tranche, and the second mezzanine (7-10%) tranche pay upfront cashes with 500-bps running coupons. The other tranches are quoted upfront cashes with 100-bps running coupons.

CHAPTER VI
CONCLUSION AND FURTHER WORK

The purpose of the study is to propose the model that has the ability to create strong default dependency of underlying assets for pricing CDOs. In doing so, we model the default intensity processes of firms to have systematic or market factors shared among firms. We define the market factors' processes to have two components which are a continuous process and a jump process. The continuous component follows the drift-diffusion process. The jump process has the Gamma-Poisson mixture process as the jump counting process and the jump sizes are exponential distributed. There are two distinctive properties of our proposed model. First market factors can be correlated. Second, the arrival rates of jumps in market factor intensities are driven by Gamma-Poisson Mixture process.

Unfortunately, correlation between Brownian motions is not substantial enough to construct fat-tailed distributions. However it is economically meaningful. If market factors are positively correlated to each other, this yields in more dispersion of the portfolio loss distribution. This applies to the case of negative correlation among market factors as well but resulting in the reversed consequence. It is found to be helpful to use the correlation parameter to particularly adjust spreads the equity tranche and the first mezzanine tranche.

Empirical results show that the model that has two market factors has the potential to create effective shapes of the portfolio loss distribution. The two-market-factor model has even good performance in fitting index tranches traded during the global financial crisis of 2008-2009. Incorporating three or more market factors in the model are somewhat better but not significantly improving in pricing. The time spent on calibration and computation is of course based on the number of model parameters. The market factors should be selective and have abilities to introduce different characteristics of distributions such as long tail, fat tail, and high serial correlation.

As the objective of the study, we compare our proposed model with the existing credit risk models. Mortensen [1] uses Poisson processes to model jumps in default intensities. The numerical results show that the Gamma-Poisson mixture processes outperform the Poisson processes. Compared to the Poisson process, the Gamma-Poisson mixture process has the ability to generate more shapes of the portfolio loss distribution and even high serial default correlation. For example, the jump process, which has independently exponential distributed jump sizes with the large mean and Gamma distributed arrival rates with a small value of the shape parameter and a large value of the scaled parameter, causes uprising in probability of loss at the end of tail. It is used to fit the senior tranche and super senior tranche spreads easily and doesn't have any impact on other tranches.

Peng and Kou [2] proposed the CS model that has market factors as cumulative intensities to generate default clustering. Unlike us, they use Polya (Gamma-Poisson mixture) processes to model jumps in term of cumulative intensities. Peng and Kou [2] states that jumps in intensities don't result in producing simultaneous defaults. However our model that has the Gamma-Poisson mixture process modeling jumps' frequency can price all index tranche spreads with small RMSEs like theirs. The results imply that the serial correlation in defaults that is generated by the Gamma-Poisson process of our model is strong enough. In addition, Peng and Kou [2]'s Conditional Survival Model is counterintuitive. Default intensities can be infinity and untraceable if intensities are allowed to have jumps in cumulative terms.

In conclusion, our model is efficient and dynamic enough to price all tranches of CDOs and its CDS index. We also show the way to implement our model using suggested methods: a Recursive method and Mimicking Markov chain method.

For further work, we plan to use our model applied to risk measure such as value at risk and expected shortfall. We also want to improve the calibration algorithm that is adequately fast and accessible to price sname CDS spread

จุฬาลงกรณ์มหาวิทยาลัย
Chulalongkorn University

REFERENCES

1. Mortensen, A., *Semi-analytical valuation of basket credit derivatives in intensity-based models.* Journal of Derivatives, 2006.
2. Peng, X. and S. Kou, *Default clustering and valuation of collateralized debt obligations*. 2009, Columbia University.
3. Duffie, D. and N. Garleanu, *Risk and Valuation of Collateralized Debt Obligations.* Financial Analysts Journal, 2001.
4. Andersen, L., J. Sidenius, and S. Basu, *All your hedges in one basket*, in *Risk magazine*. 2003.
5. Giesecke, K., et al., *Exact and efficient simulation of correlated default.* Siam Journal, 2010. **1**: p. 868–896.
6. Schönbucher, P.J., *Credit Derivatives Pricing Models: Models, Pricing and Implementation*. 2003. 396.
7. Papageorgiou, E., *Single-name and Multi-name Credit Derivatives: Pricing and Calibration Using Multiscale Asymptotic Methods*, in *Department of Operations Research and Financial Engineering*. 2007, Princeton University.
8. Choi, E., *An Examination of the Systematic Risks in the Multi-Name Credit and Equity Markets*, in *Department of Operations Research and Financial Engineering*. 2012, Princeton University. p. 144.

จุฬาลงกรณ์มหาวิทยาลัย
CHULALONGKORN UNIVERSITY

APPENDIX

A The Exponentially-Affine Characteristic Function

The exponentially-affine characteristic function $f(t,u,z,\Lambda,X)$ has jumps distributed as Poisson with arrival rates $\Lambda=\{\Lambda_i,\ 1\le i\le J\}$, for all $(t,u,z,\Lambda,X)\in[0,T]\times\mathbb{R}^{\mathrm{m}}\times\mathbb{R}^{\mathrm{m}}\times[0,\infty]^m\times\mathbb{R}^m$ is written as

$$f(t,u,z,\Lambda,X_t)=\exp\left(a(T-t,u,z)+\sum_{i=1}^{m}\left(b^i(T-t,u_i,z_i)X_t^i+c^i(T-t,u_i,z_i)\Lambda^{\mathrm{i}}\right)\right),$$

where

$$\begin{aligned}a(t,u,z)=&-\sum_{i=1}^{m}\theta_i\left(\left(e^{-k_it}-1\right)\left(-z_i+\frac{u_i}{k_i}\right)+tu_i\right)\\&+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\rho_{ij}\sigma_i\sigma_j\left(\frac{tu_iu_j}{k_ik_j}+\frac{1}{k_i}\left(e^{-k_it}-1\right)\frac{u_j}{k_j}\left(-z_i+\frac{u_i}{k_i}\right)\right.\right.\\&+\frac{1}{k_j}\left(e^{-k_jt}-1\right)\frac{u_i}{k_i}\left(-z_j+\frac{u_j}{k_j}\right)\\&\left.\left.-\frac{\left(e^{-(k_i+k_j)t}-1\right)\left(-z_i+\frac{u_i}{k_i}\right)\left(-z_j+\frac{u_j}{k_j}\right)}{k^i+k^j}\right)\right]\end{aligned}$$

$$b^i(t,u_i,z_i)=\left(-z_i+\frac{u_i}{k_i}\right)e^{-k_it}-\frac{u_i}{k_i},$$

$$c^i(t,u_i,z_i)=\frac{1}{1+\frac{\mu^iu}{k_i}}\left(t+\frac{1}{k_i}\ln\left(\frac{1-\mu^ib^i(t,u_i,z_i)}{1+\mu^iz_i}\right)\right)-t.$$

B CDS and CDO Framework

For a CDS, there are two counterparties that enter to a contract which are a protection buyer (CDS buyer) and a protection seller (CDS seller). The premium leg corresponds the periodic payments S_t^i of the ith firm's CDS that the protection buyer has to pay until the credit event happens, defined as

$$PL_t^i = S_t^i E\left[\sum_{j=1}^{M} \exp\left(-r\left(\min\{T_j, \tau^i\} - t\right)\right) \int_{T_{j-1}}^{T_j} I\left(\tau^i > s\right) ds\right],$$

where M is the number of coupon payment dates $T_1, T_2, \dots, T_M$. When the default event occurs, the protection seller covers the loss given firm i's recovery rate R_i for the protection buyer represented by the default leg. The default leg is specified as

$$DL_t^i = E\left[(1 - R_i) \exp\left(-r\left(\tau^i - t\right)\right) I\left(\tau^i \le T\right)\right].$$

Since the premium leg and the default leg have the equal value, the CDS spread of firm i can be obtained by $S_t^i = \frac{DL_t^i}{PL_t^i}$.

Let us consider the CDO tranches with a reference credit pool of n names. The loss process of a portfolio at time t is defined as

$$L_t = \sum_{i=1}^{n} (1 - R_i)\Theta_i 1_{\tau_i \le t}.$$

$U_t^{[K_1,K_2]}$ is the process of loss of the tranche at time t for the attachment point K_1 and the detachment point K_2, defined as follows

$$U^{[K_1,K_2]}(t) = (L_t - K_1)^+ - (L_t - K_2)^+$$

Like CDS mechanism, there are protection sellers (CDO buyers) and protection buyers (CDO sellers). The default leg refers to the present value of the sum of contingent payments upon default that protection sellers (CDO buyers) must pay as agreed. The default leg at time t is specified as

$$DL_t = E\left[\int_t^T \exp\left(-r(s - t)\right) dU^{[K_1,K_2]}(s)\right].$$

The premium leg is the sum of payments that the protection seller (CDO buyer) receives from the protection buyer (CDO seller). Let $S_t^{[K_1,K_2]}$ denote the running spread of the CDO tranche, F denote a fixed upfront fee. The premium payment at time t is defined as

$$PL_t = F(K_2 - K_1) + S_t^{[K_1,K_2]} E\left[\sum_{j=1}^{M} \exp\left(-r\left(T_j - t\right)\right)\left(T_j - T_{j-1}\right) \int_{T_{j-1}}^{T_j} \frac{O_s^{[K_1,K_2]}}{T_j - T_{j-1}} ds\right],$$

where the national outstanding $O_t^{[K_1,K_2]}$ is given by

$$O_t^{[K_1,K_2]} = K_2 - K_1 - U^{[K_1,K_2]}(t).$$

If default times are unknown, we assume that default occurring between coupon dates. This assumption is used in Peng and Kou [2] and Mortensen [1].

C Thinning Scheme Algorithm

The algorithm of Thinning scheme is used to generate sequences of $(T_k, I_k)_{k=1,2,..}$ where the kth default time $T_k \leq T$ and the firm that defaults $I_k \leq n$, where n is the number of underlying firms in a portfolio. Denote the number of intervals $\mathcal{M}$, the transition rate function $h(t, B)$ where $B = (B^1, \dots, B^n) \in \{0,1\}^n$, the portfolio transition rate function $H(t, k)$, and the majoring intensity function $H^*(i, k)$. Inputs are the current interval i such that $i = \{i^*: L_{i^*-1} \leq t < L_{i^*}\}$, the firms' states vector M, the current time t, the number of the firms that have defaulted k, and the vector Q^j which has that the jth element is equal to one and the rest of elements are zero. First, we initialize t=0, k=0, $T=0_n$, $M=0_n$ and i=1. Then we proceed as follows:

1. Generate x~exponential random variable with the mean $H^*(i, k)$.
2. If $t + x < L_i$ set $t \leftarrow t + x$ and if $t > T$ or $i > \mathcal{M}$ stop, else go to step 4. Else if $t + x \geq L_i$, go to step 3.
3. Set $x \leftarrow H^*(i, k)(t + x - L_i)/H^*(i + 1, k), t \leftarrow L_i, i \leftarrow i + 1$. Go to step 2.
4. Generate $\omega_2 \sim random[0,1]$.
   If $\omega_2 \leq H(t, k)/H^*(i, k)$, set $T_k \leftarrow t$ and go to step 5. Otherwise, go to 1.
5. Define the defaulted firm J by drawing from the pool of survival firms. Each firm $j, 1 \leq j \leq n$ has the probability being selected as $\frac{h^j(t, M_{T_k})}{H(t,k)}$.
   Then set $I_k = J, M_{T_{k+1}} = M_{T_k} + Q^j$, and $k = k + 1$. Return to step 1.

VITA

Supalak Phetcharat was born on December 1, 1989. In 2012, she graduated a Bachelor of Engineering (B.Eng.) in Computer Engineering from the Faculty of Engineering, Chulalongkorn University and continued studying a master's degree in financial engineering at the Department of Banking and Finance, Faculty of Commerce and Accountancy, Chulalongkorn University.

จุฬาลงกรณ์มหาวิทยาลัย
Chulalongkorn University