ขอบเขตเบอร์รี-เอสซีนสำหรับทฤษฎีบทลิมิตกลางหลายมิติผ่านวิธีของสไตน์

นายดาวุด ทองทา

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรดุษฎีบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2554

BERRY-ESSEEN BOUNDS FOR MULTIDIMENSIONAL CENTRAL LIMIT THEOREM VIA STEIN'S METHOD

Mr. Dawud Thongtha

A Dissertation Submitted in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2011

Thesis Title	BERRY-ESSEEN BOUNDS FOR MULTIDIMENSIONAL CENTRAL LIMIT THEOREM VIA STEIN'S METHOD
By	Mr. Dawud Thongtha
Field of Study	Mathematics
Thesis Advisor	Professor Kritsana Neammanee, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Doctoral Degree

. . . . Dean of the Faculty of Science
(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

. . . . Chairman
(Associate Professor Imchit Termwuttipong, Ph.D.)

. . . . Thesis Advisor
(Professor Kritsana Neammanee, Ph.D.)

. . . . Examiner
(Assistant Professor Nattakarn Chaidee, Ph.D.)

. . . . Examiner
(Kittipat Wong, Ph.D.)

. . . . External Examiner
(Kittipong Laipaporn, Ph.D.)

ดาวุด ทองทา: ขอบเขตเบอร์รี-เอสซีนสำหรับทฤษฎีบทลิมิตกลางหลายมิติผ่านวิธีของสไตน์. (BERRY-ESSEEN BOUNDS FOR MULTIDIMENSIONAL CENTRAL LIMIT THEOREM VIA STEIN'S METHOD) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : ศ.ดร.กฤษณะ เนียมมณี, 65 หน้า.

เราให้ขอบเขตในการประมาณค่าปกติหลายตัวแปรสำหรับทฤษฎีบทเบอร์รี-เอสซีนหลายมิติภายใต้สมมุติฐานว่าเวกเตอร์สุ่มมีโมเมนต์ค่าสัมบูรณ์อันดับที่สามแต่ไม่จำเป็นต้องมีการแจกแจงเดียวกัน เราได้ขอบเขตเอกรูปบนทรงกลมปิด ครึ่งระนาบ และสี่เหลี่ยมมุมฉาก และได้ขอบเขตไม่เอกรูปบนสองเซตแรก โดยใช้ระเบียบวิธีของสไตน์ด้วยอสมการความเข้มข้น นอกจากนี้เราได้ให้ค่าคงตัวในขอบเขตเอกรูปบนเซตเหล่านั้นด้วย

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต...

สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก..................

ปีการศึกษา 2554

5073847723 : MAJOR MATHEMATICS

KEYWORDS: BERRY-ESSEEN THEOREM / UNIFORM AND NON-UNIFORM BOUNDS/ STEIN'S METHOD/ MULTIVARIATE NORMAL APPROXIMATION

DAWUD THONGTHA: BERRY-ESSEEN BOUNDS FOR MULTIDIMENSIONAL CENTRAL LIMIT THEOREM VIA STEIN'S METHOD. ADVISOR : PROF.KRITSANA NEAMMANEE, Ph.D., 65 pp.

We give bounds in multivariate normal approximation for multidimensional Berry-Esseen theorem. With the assumption that the random vectors have an absolute third moments but they may not be identically distributed. We obtain uniform bounds on a closed sphere, a half plane and a rectangular set and non-uniform bounds on the first two sets. The Stein's method using concentration inequality approach is applied. Moreover, we provide constants in uniform bounds on these sets.

Department : Mathematics and Computer Science Student's Signature ....

Field of Study : Mathematics Advisor's Signature ....

Academic Year : 2011

ACKNOWLEDGEMENTS

First and foremost, I am deeply indebted to my thesis advisor, Professor Dr. Kritsana Neammanee, for his valuable advice both in academic matters and livelihoods. His expertise and optimism have enabled to achieve the goal I almost found impossible.

I also would like to express my sincere gratitude to the rest of my thesis committee members, Associate Professor Dr.Imchit Termwuttipong, Assistant Professor Dr.Nattakarn Chaidee, Dr.Kittpat Wong and Dr.Kittipong Laipaporn, for their help. Each individual kindly advised and corrected the mistakes in this research.

My special thanks go to Assistant Professor Dr.Songkiat Sumetkijakan for his generous advice on academic matters. This research would not have been completed without the useful suggestions I received from him. I also extend my sincere thanks to all my teachers for the knowledge they provided me.

I would like to express my deepest gratitude to my beloved family members who continuously give me love, support and power to complete this doctoral degree, one of the greatest achievements in my life.

Another person I cannot fail to mention is Patcharin Thanapairin, the love of my life. Her love and encouragement tremendously helped me overcome the difficulties that I had encountered.

I also would like to thank Kamolwan Eomtongin for her companionship and willingness to help me with the English language used in this research.

Additionally, I truly appreciate the strong friendship and memorable experiences that I shared with my dear friends in Chulalongkorn University.

Finally, I wish to thank the Office of the Higher Education Commission for giving me the long term scholarship to do this doctoral degree.

CONTENTS

page

CHAPTER I

INTRODUCTION

Berry-Esseen inequality is one of the most important tools in the theory of probability. This inequality helps us to quantify the rate of the convergence in the central limit theorem. For each $n \in \mathbb{N}$, let $X_1, X_2, ..., X_n$ be independent and identically distributed random variables with zero means and $\sum_{i=1}^{n} EX_i^2 = 1$. Define

$$S_n = \sum_{i=1}^{n} X_i$$

and let Φ_1 be the standard normal distribution, i.e.,

$$\Phi_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt.$$

The Berry-Esseen inequality was stated under the assumption that $\sum_{i=1}^{n} E|X_i|^3 < \infty$. The uniform and non-uniform versions of the inequality are

$$\sup_{x \in \mathbb{R}} |P(S_n \leq x) - \Phi_1(x)| \leq C_0 \sum_{i=1}^{n} E|X_i|^3$$

and

$$|P(S_n \leq x) - \Phi_1(x)| \leq \frac{C_1}{1 + |x|^3} \sum_{i=1}^{n} E|X_i|^3,$$

respectively, where both C_0 and C_1 are positive constants. The uniform version was independently discovered by Berry [5] and Esseen [13] in 1941 and 1945, respectively, while the non-uniform version was discovered by Nagaev [17] in 1965.

Over several decades, many authors put their effort to find the rate of this convergence both uniform and non-uniform versions such as Shevtsova [23], Siganov [24], Neammanee and Thongtha [18], Chen and Shao [10, 12], Paditz [19] and Chaidee [7], etc.

For multidimensional case, let $k \in \mathbb{N}$ be fixed and $n \in \mathbb{N}$ be arbitrary, $Y_i = (Y_{i1}, Y_{i2}, ..., Y_{ik}), i = 1, 2, ..., n,$ be independent and identically distributed random vectors in $\mathbb{R}^k$ with zero means,

$$\sum_{i=1}^{n} EY_{ij}^2 = 1 \text{ for } j = 1, 2, \ldots, k \text{ and} \tag{1.1}$$

$$EY_{ij}Y_{il} = 0 \text{ for } j \neq l. \tag{1.2}$$

Define

$$W_n = \sum_{i=1}^{n} Y_i.$$

Let F_n be the distribution of W_n and Φ_k the standard Gaussian distribution in $\mathbb{R}^k$, i.e.,

$$\Phi_k(A) = \frac{1}{(2\pi)^{\frac{k}{2}}} \int_A e^{-\frac{1}{2}\sum_{i=1}^{k} x_i^2} d^k x$$

where $A \subseteq \mathbb{R}^k$ and $x = (x_1, x_2, ..., x_k) \in \mathbb{R}^k$. Under the above assumption, Bergström [4] guaranteed that F_n converges weakly to Φ_k. The uniform bound of this convergence have been repeatedly refined over subsequent decades by many researchers such as Esseen [13], Rao [21] and Bahr [2], etc. Esseen [13] assumed the finiteness of the forth moments,

$$\sum_{j=1}^{k} E|Y_{1j}|^4 < \infty,$$

and used Fourier method to find a uniform bound over the closed sphere $B_k(r) = \{x \in \mathbb{R}^k \mid x_1^2 + x_2^2 + \cdots + x_k^2 \leq r^2\}$ for $r > 0$. He proved that

$$|F_n(B_k(r)) - \Phi_k(B_k(r))| \leq \frac{C_k}{n^{\frac{k}{k+1}}}$$

where C_k is an absolute constant depending on k. Rao [21] generalized Esseen's result to any convex Borel subset C of $\mathbb{R}^k$, and his estimation is

$$|F_n(C) - \Phi_k(C)| \leq \frac{C_k}{\sqrt{n}}(\log n)^{\frac{k-1}{2(k+1)}} \tag{1.3}$$

In 1967, Bahr [2] assumed

$$E(\sum_{j=1}^{k} Y_{1j}^2)^{\frac{s}{2}} < \infty,$$

for an integer $s > k > 1$ and improved the rate of convergence in (1.3) by the inequality

$$|F_n(C) - \Phi_k(C)| \leq \frac{C_k}{\sqrt{n}}. \tag{1.4}$$

In the case that each random vector Y_i may not be identically distributed, Bhattacharya [6] assumed that for $i = 1, 2, ..., n$,

$$\sum_{j=1}^{k} E|Y_{ij}|^{3+\delta} < \infty \quad \text{for some } \delta > 0,$$

and he gave a bound of the estimation on any Borel subset of $\mathbb{R}^k$. The rate of convergence in [6] is the same as in (1.4). In 1991, Götze [14] assumed the finiteness of the third moments and used the Stein's method to find a uniform bound of this convergence. He proved that on any measurable convex set C in $\mathbb{R}^k$,

$$|F_n(C) - \Phi_k(C)| \leq C_k\gamma_3 \tag{1.5}$$

where $\gamma_3 = \sum_{i=1}^{n} E||Y_i||^3$, $||\cdot||$ is the Euclidean norm in $\mathbb{R}^k$ and

$$C_k = 124.4a_k\sqrt{k} + 10.7,$$

where $a_k = 2.04, 2.4, 2.69, 2.94$ for $k = 2, 3, 4, 5$, respectively and $a_k \leq 1.27\sqrt{k}$ for $k \geq 6$. His estimation is of order $O(n^{-\frac{1}{2}})$. In 2009, Reinert and Röllin [22] assumed the finiteness of the third moments and used the Stein's method to find uniform bounds. Their estimation is of order $O(n^{-\frac{1}{4}})$, but the result can be applied to the case that the random vectors Y_i, $i = 1, 2, \ldots, n$, need not be independent.

Bahr [1] is the first one who investigated the non-uniform bound of this estimation. By assuming the identically distributed on $Y_i's$, he gave a rate of convergence on $B_k(r)$. Under the assumption

$$E(\sum_{j=1}^{k} Y_{1j}^2)^{\frac{s}{2}} < \infty,$$

for an integer $s \geq 3$, the result is

$$|F_n(B_k(r)) - \Phi_k(B_k(r))| \leq \frac{C_k \cdot d(n)}{r^s n^{\frac{s-2}{2}}} \quad \text{for} \quad r \geq \left(\frac{5}{4}m(s-2)\log n\right)^{\frac{1}{2}} \tag{1.6}$$

where m is the largest eigenvalue of the covariance matrix of $\sqrt{n}Y_i$, $d(n)$ is bounded by one and $\lim_{n\to\infty} d(n) = 0$.

In this dissertation, we will find both uniform and non-uniform Berry-Esseen bounds without assuming that $Y_i's$ are identically distributed nor all components of Y_i are independent.

In the first part of our investigation, we obtain both uniform and non-uniform bounds on the half plane $A_k(r) = \{x \in \mathbb{R}^k \mid x_1 + x_2 + \cdots + x_k \leq r\}$ for $r \in \mathbb{R}$. We investigate the bounds by applying Berry-Esseen inequality in $\mathbb{R}$. In this part, we give our results under various assumptions on Y_{ij}: the random vaiables Y_{ij} are bounded, $E|Y_{ij}|^p < \infty$ for some $2 < p < 3$ and $E|Y_{ij}|^3 < \infty$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$. The results are as follows:

Theorem 1.1. *Let $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{ik}), i = 1, 2, \ldots, n$, be independent random vectors in $\mathbb{R}^k$ with zero means, satisfying (1.1) and (1.2). Define $W_n = \sum_{i=1}^{n} Y_i$. Let F_n be the distribution function of W_n. If $|Y_{ij}| \leq \delta_0$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$, then*

$$\sup_{r\in\mathbb{R}} |F_n(A_k(r)) - \Phi_k(A_k(r))| \leq 3.3\sqrt{k}\delta_0$$

and there exists a constant C which does not depend on δ_0 such that for every real numbers r,

$$|F_n(A_k(r)) - \Phi_k(A_k(r))| \leq \frac{Ck^2\delta_0}{(\sqrt{k})^3 + |r|^3}.$$

Theorem 1.2. *Let $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{ik}), i = 1, 2, \ldots, n$, be independent random vectors in $\mathbb{R}^k$ with zero means, satisfying (1.1) and (1.2). Define $W_n = \sum_{i=1}^{n} Y_i$. Let F_n be the distribution function of W_n. If $E|Y_{ij}|^p < \infty$ for some $2 < p < 3, i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$, then*

$$\sup_{r\in\mathbb{R}} |F_n(A_k(r)) - \Phi_k(A_k(r))| \leq 75(4)^{p-1}k^{\frac{p}{2}} \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^p$$

and there exists an absolute constant C such that for $r \in \mathbb{R}$,

$$|F_n(A_k(r)) - \Phi_k(A_k(r))| \leq \frac{C(5k)^p}{(\sqrt{k} + |r|)^p} \sum_{j=1}^{k} \sum_{i=1}^{n} E|Y_{ij}|^p.$$

Theorem 1.3. *Let $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{ik}), i = 1, 2, \ldots, n,$ be independent random vectors in $\mathbb{R}^k$ with zero means, satisfying (1.1) and (1.2). Define $W_n = \sum_{i=1}^{n} Y_i$. Let F_n be the distribution function of W_n. If $E|Y_{ij}|^3 < \infty$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$, then*

$$\sup_{r \in \mathbb{R}} |F_n(A_k(r)) - \Phi_k(A_k(r))| \leq 0.5600\sqrt{k} \sum_{j=1}^{k} \sum_{i=1}^{n} E|Y_{ij}|^3$$

and for all real numbers r,

$$|F_n(A_k(r)) - \Phi_k(A_k(r))| \leq \frac{31.935k^2}{(\sqrt{k})^3 + |r|^3} \sum_{j=1}^{k} \sum_{i=1}^{n} E|Y_{ij}|^3.$$

In the second part, we use Stein's method to find uniform bounds and give the constants C on the half plane $A_k(r)$, the closed sphere $B_k(r)$ and the rectangular set $R_k(r) = \{x \in \mathbb{R}^k \mid |x_j| \leq r_j, j = 1, 2, \ldots, k\}$ where $r = (r_1, r_2, \ldots, r_k)$ and $r_j > 0$ for all $j = 1, 2, \ldots, k$. In this part, we assume further that

$$\sum_{j=1}^{k} E|Y_{ij}|^3 < \infty \quad \text{for all} \quad i = 1, 2, ..., n.$$

Here are our results.

Theorem 1.4. *Let $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{ik}), i = 1, 2, \ldots, n,$ be independent random vectors in $\mathbb{R}^k$ with zero means and Y_{ij} are independent for all $j = 1, 2, \ldots, k$. Define $W_n = \sum_{i=1}^{n} Y_i$. Let F_n be the distribution function of W_n. Assume that $\sum_{i=1}^{n} EY_{ij}^2 = 1$ for $j = 1, 2, \ldots, k$ and $\sum_{j=1}^{k} E|Y_{ij}|^3 < \infty$ for $i = 1, 2, \ldots, n$. Then*

$$\sup_{r \in \mathbb{R}} |F_n(B_k(r)) - \Phi_k(B_k(r))| \leq C\beta_3$$

where $C = \frac{4.55}{k} + \frac{3}{k\sqrt{k}}$ and $\beta_3 = \sum_{j=1}^{k} \sum_{i=1}^{n} E|Y_{ij}|^3$.

Theorem 1.5. *Under the assumptions of Theorem 1.4, we have*

$$\sup_{r\in\mathbb{R}} |F_n(A_k(r)) - \Phi_k(A_k(r))| \leq C\beta_3$$

where $C = \dfrac{4.55}{k} + \dfrac{3}{k\sqrt{k}}$ *and* $\beta_3 = \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^3$.

Observe that the orders of the estimations in Theorem 1.4 and Theorem 1.5 are $O(n^{-\frac{1}{2}})$ which is finer than the result in [22] and the constants are smaller than the constant in (1.5).

Corollary 1.6. *Let* X_i, $i = 1, 2, \dots, n$, *be independent random variables with zero mean and* $\sum_{i=1}^{n} EX_i^2 = 1$. *Define* $W_n = \sum_{i=1}^{n} X_i$. *Let* F_n *be the distribution function of* W_n. *If* $E|X_i|^3 < \infty$ *for* $i = 1, 2, \dots, n$, *then*

$$\sup_{x\in\mathbb{R}} |F_n(x) - \Phi_1(x)| \leq 7.55 \sum_{i=1}^{n} E|X_i|^3.$$

Theorem 1.7. *Under the assumption of Theorem 1.4, we have*

$$\sup_{r\in\mathbb{R}} |F_n(R_k(r)) - \Phi_k(R_k(r))| \leq C\beta_3$$

where $C = \dfrac{4.55}{k} + \dfrac{3}{k\sqrt{k}}$ *and* $\beta_3 = \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^3$.

In the last part of our results, we use the same method as in the the second part to find a non-unifrom bound on $B_k(r)$. The result is as follows:

Theorem 1.8. *Under the assumption of Theorem 1.4, there exists a positive constant* C_k *(depends on* k*) such that*

$$|F_n(B_k(r)) - \Phi_k(B_k(r))| \leq \frac{C_k\beta_3}{1+r^3}$$

for $r > 0$, *where* $\beta_3 = \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^3$.

Note that the result in Theorem 1.8 is obtained for all positive real numbers r which is broader than the radius r in (1.6).

The contents of this dissertation are organized into five chapters. Firstly, chapter II, a premilinary part, consists of basic information in probability theory and integration on sphere. The information and propositions concerning the Stein's method are explained in chapter III. The proofs of our results are given in chapter IV, chapter V and chapter VI. In Chapter IV, we give uniform and non-uniform bounds by using Berry-Essen theorem in $\mathbb{R}$. Uniform bounds provided in Chapter V are investigated by using the Stein technique. Finally, Chapter VI, contains a proof of non-uniform bound given in Theorem 1.8.

CHAPTER II

PRELIMINARIES

In this chapter, we review some basic knowledges in probability and the idea of integration on sphere.

2.1 Basic Knowledge in Probability

In this section, we give some basic knowledges in probability which will be used in our work.

A **probability space** is a measure space $(\Omega, \mathcal{F}, P)$ for which $P(\Omega) = 1$. The measure P is called a **probability measure**. The set Ω will be referred to as a **sample space** and its elements are called **points** or **elementary events**. The elements of $\mathcal{F}$ are called **events**. For any event A, the value $P(A)$ is called the **probability of** A.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is called a **random variable** if for every Borel set B in $\mathbb{R}$, $X^{-1}(B)$ belongs to $\mathcal{F}$. We shall use the notation $P(X \in B)$ in place of $P(\{\omega \in \Omega | X(\omega) \in B\})$. In the case that $B = (-\infty, a]$ or $[a, b]$, $P(X \in B)$ is denoted by $P(X \leq a)$ or $P(a \leq X \leq b)$, respectively.

Let X be a random variable. A function $F : \mathbb{R} \to [0, 1]$ defined by

$$F(x) = P(X \leq x)$$

is called the **distribution function** of X.

A random variable X with the distribution function F is said to be a **discrete random variable** if the image of X is countable and it is called a **continuous random variable** if F can be written in the form

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

for some nonnegative integrable function f on $\mathbb{R}$. In this case, we say that f is the **probability function** of X.

Now we will give some examples of random variables.

We say that X is a **normal** random variable with parameters μ and σ^2, written as $X \sim N(\mu, \sigma^2)$, if its probability function is defined by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(x-\mu)^2\Big).$$

Moreover, if $X \sim N(0,1)$ then X is said to be a **standard normal** random variable.

We say that X is a **discrete uniform** random variable with parameter n if there exist $x_1, x_2, \ldots, x_n$ such that $P(X = x_i) = \dfrac{1}{n}$ for any $i = 1, 2, \ldots, n$, denoted by $X \sim U(n)$.

A random variable X is a **gamma** random variable with parameters α and β, written as $X \sim Gam(\alpha, \beta)$, if its probability function is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} & \text{if} \quad x \geq 0, \\ 0 & \text{if} \quad x < 0 \end{cases}$$

where $\alpha, \beta > 0$ and Γ, called the gamma function, is defined by

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy. \tag{2.1}$$

A ramdom variable X is a **chi-square** random variable with degree of freedom γ, denoted by $X \sim \chi^2(\gamma)$, if $X \sim Gam(\frac{\gamma}{2}, 2)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{F}_\alpha$ is a sub σ-algebra of $\mathcal{F}$ for each $\alpha \in \Lambda$. We say that $\{\mathcal{F}_\alpha | \alpha \in \Lambda\}$ is **independent** if and only if for $k \in \mathbb{N}$ and subset $J = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ of Λ,

$$P\left(\bigcap_{m=1}^{k} A_{\alpha_m}\right) = \prod_{m=1}^{k} P(A_{\alpha_m})$$

where $A_{\alpha_m} \in \mathcal{F}_{\alpha_m}$ for $m = 1, 2, \ldots, k$.

Let $\mathcal{E}_\alpha \subseteq \mathcal{F}$ for all $\alpha \in \Lambda$. We say that $\{\mathcal{E}_\alpha | \alpha \in \Lambda\}$ is **independent** if and only if $\{\sigma(\mathcal{E}_\alpha) | \alpha \in \Lambda\}$ is independent where $\sigma(\mathcal{E}_\alpha)$ is the smallest σ-algebra with $\mathcal{E}_\alpha \subseteq \sigma(\mathcal{E}_\alpha)$.

We say that the set of random variables $\{X_\alpha |\ \alpha \in \Lambda\}$ is **independent** if $\{\sigma(X_\alpha)|\ \alpha \in \Lambda\}$ is independent, where $\sigma(X) = \{X^{-1}(B) \mid B$ is a Borel subset of $\mathbb{R}\}$.

Theorem 2.1. *Random variables $X_1, X_2, \ldots, X_n$ are* **independent** *if for any Borel sets $B_1, B_2, \ldots, B_n$, we have*

$$P\left(\bigcap_{i=1}^{n}\{X_i \in B_i\}\right) = \prod_{i=1}^{n} P(X_i \in B_i).$$

Proposition 2.2. *If X_{ij} ; $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m_i$ are independent and $f_i : \mathbb{R}^{m_i} \to \mathbb{R}$ are measurable, then $\{f_i(X_{i1}, X_{i2}, \ldots, X_{im_i}),\ i = 1, 2, \ldots, n\}$ is independent.*

Let X be any random variable on a probability space $(\Omega, \mathcal{F}, P)$. If $\int_\Omega |X| dP < \infty$, then we define its **expected value** to be

$$E(X) = \int_\Omega X dP.$$

Proposition 2.3. *Let X be a random variable such that $E(|X|) < \infty$.*

(1) If X is a discrete random variable, then $E(X) = \sum_{x \in ImX} xP(X = x)$.

(2) If X is a continuous random variable with probability function f, then

$$E(X) = \int_{\mathbb{R}} x f(x) dx.$$

Proposition 2.4. *Let X and Y be random variables such that $E(|X|) < \infty$ and $E(|Y|) < \infty$. Then, we have the followings:*

(1) $E(aX + bY) = aE(X) + bE(Y)$ for $a, b \in \mathbb{R}$.

(2) If $X \leq Y$, then $E(X) \leq E(Y)$.

(3) $|E(X)| \leq E(|X|)$.

Let X be a random variable which $E(|X|^k) < \infty$. Then $E(|X|^k)$ is called the **k-th moment** of X about the origin and call $E[(X - E(X))^k]$ the **k-th moment** of X about the mean.

We call the second moment of X about the mean, the **variance** of X, denoted by $Var(X)$. Then

$$Var(X) = E[X - E(X)]^2.$$

We note that

(1) $Var(X) = E(X^2) - [E(X)]^2.$

(2) If $X \sim N(\mu, \sigma^2)$, then $E(X) = \mu$ and $Var(X) = \sigma^2$.

Proposition 2.5. *If $X_1, X_2, \ldots, X_n$ are independent, $E|X_i| < \infty$ and $EX_i^2 < \infty$ for $i = 1, 2, \ldots, n$, then*

(1) $E(X_1 X_2 \cdots X_n) = E(X_1)E(X_2)\cdots E(X_n)$,

(2) $Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 Var(X_i)$ *for any real numbers* $a_1, a_2, \ldots, a_n$.

The following inequalities are useful in our work.

1. **Hölder's inequality**

If X and Y are random variables such that $E(|X|^p) < \infty$, $E(|Y|^q) < \infty$ where $1 \leq p, q \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$E(|XY|) \leq [E|X|^p]^{\frac{1}{p}}[E|Y|^q]^{\frac{1}{q}}.$$

2. **Chebyshev's inequality**

For any $p > 0$ and any random variable X such that $E(|X|^p) < \infty$,

$$P(\{|X| \geq \varepsilon\}) \leq \frac{E|X|^p}{\varepsilon^p} \quad \text{for all} \quad \varepsilon > 0.$$

Let X be a finite expected value random variable on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{D}$ a sub σ-algebra of $\mathcal{F}$. Define a probability measure $P_{\mathcal{D}} : \mathcal{D} \to [0, 1]$ by

$$P_{\mathcal{D}}(E) = P(E)$$

and a sign-measure $\mathcal{Q}_X : \mathcal{D} \to \mathbb{R}$ by

$$\mathcal{Q}_X(E) = \int_E X dP \quad \text{for any } E \in \mathcal{D}.$$

Thus, $\mathcal{Q}_X$ is absolutely continuous with respect to $P_\mathcal{D}$. By Radon-Nikodym theorem, there exists a unique measurable function $E^\mathcal{D}(X)$ on $(\Omega, \mathcal{F}, P)$ such that

$$\int_E E^\mathcal{D}(X) dP_\mathcal{D} = \mathcal{Q}_X(E) = \int_E X dP \quad \text{for any } E \in \mathcal{D}.$$

We call $E^\mathcal{D}(X)$ the **conditional expectation** of X with respect to $\mathcal{D}$.

In addition, for any random variables X and Y on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$, we will denote $E^{\sigma(Y)}(X)$ by $E^Y(X)$.

Theorem 2.6. *Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$, then the followings hold for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$.*

(1) If X is random variable on $(\Omega, \mathcal{D}, P_\mathcal{D})$, then $E^\mathcal{D}(X) = X$ a.s.$[P_\mathcal{D}]$.

(2) $E^\mathcal{F}(X) = X$ a.s.$[P]$.

(3) If $\sigma(X)$ and $\mathcal{D}$ are independent, then $E^\mathcal{D}(X) = E(X)$ a.s.$[P_\mathcal{D}]$.

Theorem 2.7. *Let X and Y be random variables on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|)$ and $E(|Y|)$ are finite. Then, for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$, the followings hold.*

(1) If $X \leq Y$, then $E^\mathcal{D}(X) \leq E^\mathcal{D}(Y)$ a.s. $[P_\mathcal{D}]$.

(2) $E^\mathcal{D}(aX + bY) = aE^\mathcal{D}(X) + bE^\mathcal{D}(X)$ a.s. $[P_\mathcal{D}]$ for any $a, b \in \mathbb{R}$.

Theorem 2.8. *Let X and Y be random variables on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|XY|)$ and $E(|Y|)$ are finite and $\mathcal{D}_1, \mathcal{D}_2$ sub σ-algebras of $\mathcal{F}$. If X is a random variable with respect to $\mathcal{D}_1$, then*

(1) $E^{\mathcal{D}_1}(XY) = XE^{\mathcal{D}_1}(Y)$ a.s. $[P_{\mathcal{D}_1}]$.

(2) $E^{\mathcal{D}_2}(XY) = E^{\mathcal{D}_2}(XE^{\mathcal{D}_1}(Y))$ a.s. $[P_{\mathcal{D}_2}]$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{D}$ a sub σ-algebra of $\mathcal{F}$. For any event A on $\mathcal{F}$, we define the **conditional probability of A given $\mathcal{D}$** by

$$P(A|\mathcal{D}) = E^\mathcal{D}(I_A)$$

where I_A is defined by

$$I_A(w) = \begin{cases} 1 & \text{if } \ w \in A, \\ 0 & \text{if } \ w \notin A. \end{cases}$$

Let $k \in \mathbb{N}$ and $X_1, X_2, \ldots, X_k$ be random variables. The k-dimensional vector $\mathbf{X} = (X_1, X_2, \ldots, X_k)$ is called a **random vector** in $\mathbb{R}^k$. A function $F_{\mathbf{X}} : \mathbb{R}^k \to [0,1]$ defined by

$$F_{\mathbf{X}}(\boldsymbol{x}) = P(X_1 \leq x_1, X_2 \leq x_2, \ldots, X_k \leq x_k)$$

for all $\boldsymbol{x} = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k$, is called a **joint distribution function** of the random vector $\mathbf{X}$.

If the random variables $X_1, X_2, \ldots, X_k$ are discrete, then the random vector $\mathbf{X}$ is considered as a **discrete random vector** and its **joint probability function** is

$$P_{\mathbf{X}}(\boldsymbol{x}) = P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k).$$

If $F_{\mathbf{X}}$ can be written in the form

$$F_{\mathbf{X}}(\boldsymbol{x}) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \cdots \int_{-\infty}^{x_k} f_{\mathbf{X}}(\mathbf{t}) d^k \mathbf{t}$$

for some nonnegative integrable function $f_{\mathbf{X}}$ on $\mathbb{R}^k$, then the random vector $\mathbf{X}$ is called a **continuous random vector**. This function $f_{\mathbf{X}}$ is the **joint probability function** of $\mathbf{X}$.

The **expected value** of a random vector, denoted by $\mu_{\mathbf{X}}$, is the vector of expected values, i.e.

$$\mu_{\mathbf{X}} = (E(X_1), E(X_2), \ldots, E(X_k)).$$

The $k \times k$ matrix

$$E\{(X - \mu_{\mathbf{X}})^T (X - \mu_{\mathbf{X}})\}$$

is called a **covariance matrix** of a random vector $\mathbf{X}$, denoted by $cov(\mathbf{X})$. We note that

$$\begin{aligned} cov(\mathbf{X}) &= E(\mathbf{X}^T\mathbf{X}) - \mu_{\mathbf{X}}^T\mu_{\mathbf{X}} \\ &= \begin{bmatrix} Var(X_1) & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{21} & Var(X_2) & \cdots & \sigma_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{k1} & \sigma_{k2} & \cdots & Var(X_k) \end{bmatrix} \end{aligned}$$

where $\sigma_{ij} = E(X_i - EX_i)(X_j - EX_j)$ for $i, j = 1, 2, \ldots, k$.

An example of a random vector is a multivariate normal distribution. We say that $\mathbf{X}$ has a **multivariate normal** distribution, written as $X \sim N_k(\mu_{\mathbf{X}}, \Sigma)$ if its joint probability density function can be expressed as

$$f_{\mathbf{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{k}{2}}\sqrt{\det\Sigma}} \exp\left\{-\frac{1}{2}(\boldsymbol{x} - \mu_{\mathbf{X}})\Sigma^{-1}(\boldsymbol{x} - \mu_{\mathbf{X}})^T\right\} \text{ for } \boldsymbol{x} \in \mathbb{R}^k$$

where Σ is a covariance matrix of $\mathbf{X}$.

Proposition 2.9. *Let $\boldsymbol{X}$ be an k-dimensional random vector with $\mu_{\boldsymbol{X}} < \infty$. Then,*

(1) $E(\boldsymbol{X}\boldsymbol{a}^T + b) = \mu_{\boldsymbol{X}}\boldsymbol{a}^T + b$ *for any vector of constant $\boldsymbol{a} \in \mathbb{R}^k$ and any constant b in $\mathbb{R}$,*

(2) $E(\boldsymbol{X}A + \boldsymbol{a}) = A \cdot \mu_{\boldsymbol{X}} + \boldsymbol{a}$ *for any $k \times m$ matrix A and any vector of constant $\boldsymbol{a} \in \mathbb{R}^m$.*

Proposition 2.10. *Let $\boldsymbol{X}$ be an k-dimensional random vector with covariance matrix $cov(\boldsymbol{X})$. Then,*

(1) $cov(\boldsymbol{X}A + \boldsymbol{a}) = A \cdot [cov(\boldsymbol{X})] \cdot A^T$ *for any $k \times m$ matrix A and any vector of constant $\boldsymbol{a} \in \mathbb{R}^m$,*

(2) $cov(\boldsymbol{X})$ *is a symmetric and positive semi-definite matrix.*

2.2 Integration on Sphere

A k-dimensional sphere, briefly "k-sphere", is defined as a set of k-tuples of points $(x_1, x_2, \ldots, x_k)$ in $\mathbb{R}^k$ that are equidistant from a unique point. The unique point is called the center and a line from the center to a point on the sphere is called a radius of the sphere. The equation for an k-sphere centered at the origin is

$$x_1^2 + x_2^2 + \cdots + x_k^2 \leq r^2$$

where r is length of a radius of the sphere. A unit k-sphere is a k-sphere of unit radius which we denote its area by S_k. Let V_k be the k-dimensional volumn of a k-sphere of radius r. The formula of V_k is given by

$$V_k = \int_0^r S_k t^{k-1} dt. \tag{2.2}$$

The constant S_k, which depends on k, satisfies

$$\int_0^\infty S_k e^{-t^2} t^{k-1} dt = \int_{\mathbb{R}^k} e^{-\sum_{i=1}^k x_i^2} d^k x = \pi^{\frac{k}{2}}.$$

As a gamma function defined by (2.1), we find that

$$S_k = \frac{2\pi^{\frac{k}{2}}}{\Gamma(\frac{k}{2})}. \tag{2.3}$$

By (2.3) and the explicit form of gamma function,

$$\Gamma(n) = (n-1)!, \quad \Gamma\left(n + \frac{1}{2}\right) = \frac{\sqrt{\pi}(2n)!}{4^n n!} \quad \text{and} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \text{for all} \quad n \in \mathbb{N},$$

the area S_k can be written as $S_1 = 2$, $S_2 = 2\pi$ and for $k \geq 3$,

$$S_k = \begin{cases} \dfrac{2^{\frac{k+1}{2}} \pi^{\frac{k-1}{2}}}{((k-2)!)!} & \text{if } k \text{ is odd,} \\ \\ \dfrac{2\pi^{\frac{k}{2}}}{(\frac{k}{2}-1)!} & \text{if } k \text{ is even.} \end{cases} \tag{2.4}$$

We note that $S_1 = 2$ is the number of points in $S_1 = \{-1, 1\}$, $S_2 = 2\pi$ is the length of the circumference of the unit circle and $S_3 = 4\pi$ is the area of the unit 3-sphere.

Therefore, we can find the integration of the standard Gaussian distribution Φ_k over $B_k(r)$ by using (2.2) and (2.3). The result is

$$\begin{aligned}\Phi_k(B_k(r)) &= \frac{S_k}{(2\pi)^{\frac{k}{2}}}\int_0^r e^{-\frac{t^2}{2}}t^{k-1}dt \\ &= \frac{1}{2^{\frac{k-2}{2}}\Gamma(\frac{k}{2})}\int_0^r t^{k-1}e^{-\frac{t^2}{2}}dt \qquad (2.5)\\ &= \frac{1}{\Gamma(\frac{k}{2})}\int_0^{\frac{r^2}{2}} t^{\frac{k-2}{2}}e^{-t}dt \qquad (2.6)\end{aligned}$$

where (2.6) is obtained from integrating (2.5) by substitution. The equation (2.5) and (2.6) are useful equations for estimating $1 - \Phi_k(B_k(r))$ in Chapter III.

CHAPTER III

STEIN'S METHOD

At the beginning of ascertaining bounds of the Berry-Esseen theorem, a widely used technique is Fourier transformation. This method focuses on the characteristic function rather than the distribution function of random variables. However, this technique is quite complicated especially for the dependent case.

In 1972, Stein [25] introduced a new approach to find an explicit bound for the error in normal approximation. This technique is based on a partial differential equation instead of the Fourier transformation. The advantage of this approach is that it can be used in many situations in which dependence plays a part. This technique is called "Stein's method". The keys of this technique are the Stein's equation and its corresponding solution.

The Stein equation is considered as an equation of a partial differential operator T. The equation used in normal approximation is of the form

$$T(f)(w) = h(w) - \mathcal{N}(h), \quad w \in \mathbb{R} \tag{3.1}$$

where f is a function, h is a function called the *test function* and $\mathcal{N}(h)$ is a constant defined by

$$\mathcal{N}(h) = E(h(Z_1)), \quad Z_1 \text{ is a standard normal random variable.}$$

Thus, for a random variable W, the equation (3.1) becomes

$$T(f)(W) = h(W) - \mathcal{N}(h). \tag{3.2}$$

From (3.2), we obtain a bound of the normal approximation by estimating $T(f)(W)$ instead of $h(W) - \mathcal{N}(h)$. Therefore, the bound of the approximation depends on the solution f of the eqution (3.1).

Stein gave a bound of normal approximation by introducing the operator T in

(3.1) as follows:

$$T(f)(w) := f'(w) - wf(w) \text{ for } w \in \mathbb{R}.$$

He also gave its corresponding solution f defined by

$$f_h(w) = e^{\frac{w^2}{2}} \int_{-\infty}^{w} [h(x) - \mathcal{N}(h)] e^{-\frac{x^2}{2}} dx$$

for all real-valued measureable functions h with $\mathcal{N}(h) < \infty$.

Apart from the normal distribution, many researchers have seriously worked to find equations for other distributions such as Poisson distribution [9], gamma distribution [16], chi-square distribution [20] and hypergeometric distribution [15], etc.

In multidimensional case, many researchers gave a stein's equation for multivariate normal distribution under various assumptions on h. Götze [14] gave an equation and found a bound of the approximation when h belongs to a class of uniformly bounded measurable functions. This class includes a class of indicator functions on measurable convex sets. Barbour [3] introduced an equation to find a bound of the approximation when h belongs to a class of twice Fŕechet differentiable functions. Chatterjee and Meckes [8] gave an equation and used exchangeable pair approach to find a bound of the approximation when $h \in C^2(\mathbb{R}^k)$. Reinert and Röllin [22] used the similar approach of [8] with a different equation to give a bound of the approximation. In [22], the equation can be applied to the case that the test function h belongs to a class of indicator functions on measurable convex sets.

In this chapter, the information is organized into two sections. In section 3.1, we will introduce the Stein's equation and give its solution. The properties of the solution f needed to prove our results are given in section 3.2.

3.1 Stein's Equation

This section is devoted to introducing the Stein's equation for multidimensional vector space $\mathbb{R}^k$ and its solution. They are given in the event that the test function

h is an indicator function on Borel sets in $\mathbb{R}^k$. The result is stated in the following proposition.

Proposition 3.1. *For $k \in \mathbb{N}$ and a Borel set B in $\mathbb{R}^k$, let $h_B : \mathbb{R}^k \to \mathbb{R}$ be defined by*

$$h_B(w) = \begin{cases} 1 & \text{if} \quad w \in B, \\ 0 & \text{if} \quad w \notin B \end{cases}$$

where $w = (w_1, w_2, \ldots, w_k) \in \mathbb{R}^k$. A solution f_B of the equation

$$\sum_{i=1}^{k} f_{w_i}(w) - \sum_{i=1}^{k} w_i f_B(w) = \sqrt{k}[h_B(w) - \Phi_k(B)] \tag{3.3}$$

is

$$f_B(w) = \begin{cases} -\sqrt{2\pi} e^{\frac{1}{2}\bar{w}^2}(1 - \Phi_k(B))(1 - \Phi_1(\bar{w})) & \text{if} \quad w \in B, \bar{w} \geq 0, \\ \sqrt{2\pi} e^{\frac{1}{2}\bar{w}^2}(1 - \Phi_k(B))\Phi_1(\bar{w}) & \text{if} \quad w \in B, \bar{w} < 0, \\ \sqrt{2\pi} e^{\frac{1}{2}\bar{w}^2}\Phi_k(B)(1 - \Phi_1(\bar{w})) & \text{if} \quad w \notin B, \bar{w} \geq 0, \\ -\sqrt{2\pi} e^{\frac{1}{2}\bar{w}^2}\Phi_k(B)\Phi_1(\bar{w}) & \text{if} \quad w \notin B, \bar{w} < 0 \end{cases} \tag{3.4}$$

where $\bar{w} = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} w_i$ and f_{w_i} are the partial derivatives of f_B with respect to w_i for $i = 1, 2, \ldots, k$.

Proof. **Case 1)** Let $w \in \mathrm{Int}(B)$ and $\bar{w} \geq 0$.

$$\begin{aligned} f_{w_i}(w) &= -\sqrt{2\pi}(1 - \Phi_k(B)) \left[e^{\frac{1}{2}\bar{w}^2} \frac{\partial}{\partial w_i}(1 - \Phi_1(\bar{w})) + (1 - \Phi_1(\bar{w})) \frac{\partial}{\partial w_i} e^{\frac{1}{2}\bar{w}^2} \right] \\ &= -\sqrt{2\pi}(1 - \Phi_k(B)) \left[-\frac{1}{\sqrt{2k\pi}} + \frac{\bar{w} e^{\frac{1}{2}\bar{w}^2}(1 - \Phi_1(\bar{w}))}{\sqrt{k}} \right] \\ &= \frac{1}{\sqrt{k}}(1 - \Phi_k(B)) + \frac{\bar{w}}{\sqrt{k}} f_B(w). \end{aligned}$$

Thus

$$\begin{aligned} \sum_{i=1}^{k} f_{w_i}(w) &= \sqrt{k}(1 - \Phi_k(B)) + \sqrt{k}\bar{w} f_B(w) \\ &= \sqrt{k}[h_B(w) - \Phi_k(B)] + \sum_{i=1}^{k} w_i f_B(w). \end{aligned}$$

Case 2) Let $w \in \mathrm{Int}(B)$ and $\bar{w} < 0$.

$$\begin{aligned}
f_{w_i}(w) &= \sqrt{2\pi}(1-\Phi_k(B))\left[e^{\frac{1}{2}\bar{w}^2}\frac{\partial}{\partial w_i}\Phi_1(\bar{w}) + \Phi_1(\bar{w})\frac{\partial}{\partial w_i}e^{\frac{1}{2}\bar{w}^2}\right] \\
&= \sqrt{2\pi}(1-\Phi_k(B))\left[\frac{1}{\sqrt{2k\pi}} + \frac{\bar{w}e^{\frac{1}{2}\bar{w}^2}\Phi_1(\bar{w})}{\sqrt{k}}\right] \\
&= \frac{1}{\sqrt{k}}(1-\Phi_k(B)) + \frac{\bar{w}}{\sqrt{k}}f_B(w).
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{i=1}^{k} f_{w_i}(w) &= \sqrt{k}(1-\Phi_k(B)) + \sqrt{k}\bar{w}f_B(w) \\
&= \sqrt{k}[h_B(w) - \Phi_k(B)] + \sum_{i=1}^{k} w_i f_B(w).
\end{aligned}$$

The proof of other cases is similar to either case 1) or case 2). Note that each f_{w_i} does not exist on the boundary of B. However, we can define their partial derivatives from (3.3). If w is a point on the boundary of B, we have

$$\sum_{i=1}^{k} f_{w_i}(w) = \sum_{i=1}^{k} w_i f_B(w) + \sqrt{k}[h_B(w) - \Phi_k(B)].$$

To preserve the piecewise continuity of f_{w_i}, we define f_{w_i} by

$$f_{w_i}(w) = \frac{1}{k}\sum_{i=1}^{k} w_i f_B(w) + \frac{1}{\sqrt{k}}[h_B(w) - \Phi_k(B)], \tag{3.5}$$

for $i = 1, 2, \ldots, k$. Hence, we have the Proposition 3.1. □

Remark 3.2. *For the functions f_B and f_{w_i} defined as in Proposition 3.1, we have*

(1) f_{w_i} are equal for all $i = 1, 2, \ldots, k$.

(2) f_B and f_{w_i} are piecewise continuous for $i = 1, 2, \ldots, k$.

The first remark is obtained by differentiating all cases in (3.4) together with (3.5). The derivatives are

$$f_{w_i}(w) = \frac{1}{k}\sum_{i=1}^{k} w_i f_B(w) + \frac{1}{\sqrt{k}}[h_B(w) - \Phi_k(B)], \tag{3.6}$$

for $i = 1, 2, \ldots, k$. The second remark is immediately obtained from (3.4) and (3.6).

As previously mentioned, the keys of Stein's technique are the Stein's equation and its solution. In order to prove our theorems, we choose an equation (3.3) to form a Stein's equation for multidimensional normal approximation. In the next section, we will give some properties of f which are used to prove our results.

3.2 Properties of Solution

For $r > 0$, let f_r be the solution of Stein's equation defined in (3.4) with respect to the Borel set $B_k(r) = \{w \in \mathbb{R}^k \mid w_1^2 + w_2^2 + \cdots + w_k^2 \leq r^2\}$. In this section, we give propositions concerning the solution f_r. Proposition 3.3 and Proposition 3.5 provide bounds of the solution f_r and its partial derivatives $f_{r_{w_i}}, i = 1, 2, \ldots, k$, while Proposition 3.6 gives us bounds of a function concerning f_r. From now on, the constant C_k has different values in different places. To prove these propositions, we let

$$\bar{w} = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} w_i.$$

Proposition 3.3. *For $k \in \mathbb{N}$, $w \in \mathbb{R}^k$ and $r > 0$, we have*

(1) $|f_r(w)| \leq \dfrac{1}{|\bar{w}|}$ *for $\bar{w} \neq 0$,*

(2) $|f_r(w)| \leq 2$ *and*

(3) $|f_{r_{w_i}}(w)| \leq \dfrac{2}{\sqrt{k}}$ *for $i = 1, 2, \ldots, k$.*

Proof. To prove the proposition, we use the following inequalities.
If $\bar{w} > 0$, then

$$1 - \Phi_1(\bar{w}) \leq \frac{1}{\sqrt{2\pi}\bar{w}e^{\frac{1}{2}\bar{w}^2}} \tag{3.7}$$

and for $\bar{w} < 0$,

$$\Phi_1(\bar{w}) \leq \frac{1}{\sqrt{2\pi}|\bar{w}|e^{\frac{1}{2}\bar{w}^2}} \tag{3.8}$$

(see inequalities (25) and (26), page 23 in [26]).

1) From the above inequalities, we obtain that for $\bar{w} > 0$,

$$|f_r(w)| \le \sqrt{2\pi} e^{\frac{1}{2}\bar{w}^2}(1 - \Phi_1(\bar{w})) \le \sqrt{2\pi} e^{\frac{1}{2}\bar{w}^2} \cdot \frac{1}{\sqrt{2\pi}\bar{w} e^{\frac{1}{2}\bar{w}^2}} = \frac{1}{|\bar{w}|}. \tag{3.9}$$

Likewise, this inequality holds for $\bar{w} < 0$ when we apply (3.8) instead of (3.7) in (3.9). The inequality in this case is that

$$|f_r(w)| \le \sqrt{2\pi} e^{\frac{1}{2}\bar{w}^2}\Phi_1(\bar{w}) \le \sqrt{2\pi} e^{\frac{1}{2}\bar{w}^2} \cdot \frac{1}{\sqrt{2\pi}|\bar{w}| e^{\frac{1}{2}\bar{w}^2}} = \frac{1}{|\bar{w}|}. \tag{3.10}$$

Thus, (1) is proved. Furthermore, if $w \in B_k(r)$, by (3.4) and (3.9)–(3.10),

$$|f_r(w)| = \begin{cases} \sqrt{2\pi} e^{\frac{1}{2}\bar{w}^2}(1 - \Phi_k(B_k(r)))(1 - \Phi_1(\bar{w})) & \text{if } \bar{w} > 0, \\ \sqrt{2\pi} e^{\frac{1}{2}\bar{w}^2}(1 - \Phi_k(B_k(r)))\Phi_1(\bar{w}) & \text{if } \bar{w} < 0 \end{cases} \tag{3.11}$$

$$\le \frac{1 - \Phi_k(B_k(r))}{|\bar{w}|} \qquad \text{for } \bar{w} \ne 0. \tag{3.12}$$

2) To prove (2), we consider $\bar{w}$ in two cases. If $|\bar{w}| \ge \frac{1}{2}$, then (1) implies that

$$|f_r(w)| \le \frac{1}{|\bar{w}|} \le 2.$$

Whilst if $|\bar{w}| < \frac{1}{2}$, by (3.11),

$$\begin{aligned} |f_r(w)| &\le \begin{cases} \sqrt{2\pi} e^{\frac{1}{2}\bar{w}^2}(1 - \Phi_1(\bar{w})) & \text{if } \bar{w} > 0, \\ \sqrt{2\pi} e^{\frac{1}{2}\bar{w}^2}\Phi_1(\bar{w}) & \text{if } \bar{w} < 0 \end{cases} \\ &\le \sqrt{2\pi} e^{\frac{1}{8}}\Phi_1(0) \\ &\le 1.42. \end{aligned}$$

Therefore, we have (2).

3) By using equation (3.6) and (1), we have

$$\begin{aligned} |f_{r_{w_i}}(w)| &\le \frac{1}{k}|\sum_{i=1}^{k} w_i||f_r(w)| + \frac{1}{\sqrt{k}}[h_{B_k(r)}(w) - \Phi_k(B_k(r))] \\ &\le \frac{1}{k|\bar{w}|}|\sum_{i=1}^{k} w_i| + \frac{1}{\sqrt{k}} \\ &\le \frac{2}{\sqrt{k}}. \end{aligned}$$

Hence, (3) is proved and the proposition is completed. □

Proposition 3.4 is used to prove Proposition 3.5. This proposition gives us an inequality concerning the integration of Gaussian formula over $B_k(r)$. To prove the proposition, we use helpful equations (2.5) and (2.6) which are proposed in Chapter II.

Proposition 3.4. *For $k \in \mathbb{N}$ and $r > 0$, there exists an absolute constant C_k (depends on k only) such that*

$$1 - \Phi_k(B_k(r)) \leq \frac{C_k}{1 + r^6}.$$

Proof. To prove the proposition, it suffices to show that

$$\Phi_k(B_k(r)) \geq 1 - \frac{C_k}{1 + r^6} \tag{3.13}$$

for some absolute constant C_k. The proof of (3.13) is divided into two cases and proved by using mathematical induction. Firstly, we will show that (3.13) holds for all positive odd integers. For a basis step,

$$\begin{aligned}
\Phi_1(B_1(r)) &= \Phi_1(r) - \Phi_1(-r) \\
&= 2\Phi_1(r) - 1 \\
&= 1 - 2(1 - \Phi_1(r)) \\
&\geq 1 - \frac{2}{\sqrt{2\pi} r e^{\frac{r^2}{2}}} \\
&\geq 1 - \frac{C_k}{1 + r^6}
\end{aligned}$$

where we have used (3.7) in the first inequality. For an induction step, we assume that (3.13) holds for a positive odd integer k. Thus, by (2.5),

$$\begin{aligned}
\Phi_{k+2}(B_{k+2}(r)) &= \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k+2}{2})} \int_0^r t^k \cdot t e^{-\frac{t^2}{2}} dt \\
&= \frac{1}{2^{\frac{k}{2}}\Gamma(\frac{k+2}{2})} \left[(-r^k e^{-\frac{r^2}{2}}) + k \int_0^r t^{k-1} e^{-\frac{t^2}{2}} dt \right] \\
&\geq -\frac{C_k}{1 + r^6} + \frac{k}{2^{\frac{k}{2}}\Gamma(\frac{k+2}{2})} \int_0^r t^{k-1} e^{-\frac{t^2}{2}} dt
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{C_k}{1+r^6} + \frac{1}{2^{\frac{k-2}{2}}\Gamma(\frac{k}{2})}\int_0^r t^{k-1}e^{-\frac{t^2}{2}}dt \\
&= -\frac{C_k}{1+r^6} + \Phi_k(B_k(r)) \\
&\geq 1 - \frac{C_k}{1+r^6}
\end{aligned}
$$

where we have used the formulas:

$$
\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \text{and} \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n)!\cdot\sqrt{\pi}}{4^n\cdot n!} \quad \text{for} \quad n \in \mathbb{N}
$$

in the third equality. Hence, by mathematical induction, the inequality (3.13) is true for all positive odd integers. Next, we will show that (3.13) holds for all positive even integers. For a basis step, by (2.6) and $\Gamma(1) = 1$, we can compute directly that

$$
\Phi_2(B_2(r)) = \int_0^{\frac{r^2}{2}} e^{-t}dt = 1 - e^{-\frac{r^2}{2}} \geq 1 - \frac{C_k}{1+r^6}. \tag{3.14}
$$

For an induction step, we assume that (3.13) holds for a positive even integer k. So, by (2.6),

$$
\begin{aligned}
\Phi_{k+2}(B_{k+2}(r)) &= \frac{1}{\Gamma(\frac{k+2}{2})}\int_0^{\frac{r^2}{2}} t^{\frac{k}{2}}e^{-t}dt \\
&= \frac{1}{\Gamma(\frac{k+2}{2})}\left[-(\frac{r^2}{2})^{\frac{k}{2}}e^{-\frac{r^2}{2}} + \frac{k}{2}\int_0^{\frac{r^2}{2}} t^{\frac{k-2}{2}}e^{-t}dt\right] \\
&\geq -\frac{C_k}{1+r^6} + \frac{k}{2\Gamma(\frac{k+2}{2})}\int_0^{\frac{r^2}{2}} t^{\frac{k-2}{2}}e^{-t}dt \\
&= -\frac{C_k}{1+r^6} + \Phi_k(B_k(r)) \\
&\geq 1 - \frac{C_k}{1+r^6}
\end{aligned}
$$

where we have used the fact that

$$
\Gamma(n) = (n-1)! \quad \text{for} \quad n \in \mathbb{N}
$$

in the third equality. By mathematical induction, the inequality (3.13) is true for all positive even integers and then holds for all positive integer. Hence, the proposition is proved. □

Proposition 3.5 gives us bounds of the expectation of an absolute value of $f_r(W)$ and $f_{r_{w_i}}(W)$ for all $i = 1, 2, \ldots, k$ where $W = (W_1, W_2, \ldots, W_k)$ is a random vector in $\mathbb{R}^k$. For notational convenience, we let

$$\widetilde{W} = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} W_i.$$

Proposition 3.5. *For $k \in \mathbb{N}$, let $W = (W_1, W_2, \ldots, W_k)$ be a random vector in $\mathbb{R}^k$ such that $\sum_{i=1}^{k} EW_i^2 < \infty$. Then, there exists an absolute constant C_k (depends on k) such that for $r > 0$,*

(1) $E|f_r(W)| \leq \frac{C_k}{1+r^2}$ *and*

(2) $E|f_{r_{w_i}}(W)| \leq \frac{C_k}{1+r^2}$ *for $i = 1, 2, \ldots, k$.*

Proof. 1) Note that

$$E|f_r(W)| = E|f_r(W)|I(W \in B_k(r)) + E|f_r(W)|I(W \notin B_k(r)). \tag{3.15}$$

Firstly, we will find a bound of $E|f_r(W)|I(W \in B_k(r))$. Note that

$$\begin{aligned} E|f_r(W)|I(W \in B_k(r)) \leq\ & E|f_r(W)|I(W \in B_k(r))I\left(|\widetilde{W}| < \frac{1}{2}\right) \\ & + E|f_r(W)|I(W \in B_k(r))I\left(|\widetilde{W}| \geq \frac{1}{2}\right). \end{aligned} \tag{3.16}$$

By (3.4), (3.12) and Proposition 3.4, we have

$$E|f_r(W)|I(W \in B_k(r))I\left(|\widetilde{W}| < \frac{1}{2}\right) \leq \sqrt{2\pi}e^{\frac{1}{8}}(1 - \Phi_k(B_k(r))) \leq \frac{C_k}{1+r^6} \tag{3.17}$$

and

$$\begin{aligned} E|f_r(W)|I(W \in B_k(r))I\left(|\widetilde{W}| \geq \frac{1}{2}\right) &\leq E\left(\frac{1 - \Phi_k(B_k(r))}{|\widetilde{W}|}\right) I\left(|\widetilde{W}| \geq \frac{1}{2}\right) \\ &\leq 2(1 - \Phi_k(B_k(r))) \\ &\leq \frac{C_k}{1+r^6}. \end{aligned} \tag{3.18}$$

Thus, we can conclude from (3.16)–(3.18) that

$$E|f_r(W)|I[W \in B_k(r)] \leq \frac{C_k}{1+r^6}. \tag{3.19}$$

Next, we will estimate the second term of (3.15). By proposition 3.3(2), we obtain

$$\begin{aligned} E|f_r(W)|I[W \notin B_k(r)] &\le 2EI[W \notin B_k(r)] \\ &= 2P\left(\sum_{i=1}^{k} W_i^2 > r^2\right) \\ &\le \frac{2}{r^2}\sum_{i=1}^{k} EW_i^2 \\ &\le \frac{C_k}{1+r^2} \end{aligned} \tag{3.20}$$

where Chebyshev's inequality is used in the second inequality. By (3.15), (3.19)–(3.20), we complete the proof of (1).

2) In the same way as (3.15), we note that

$$E|f_{r_{w_i}}(W)| = E|f_{r_{w_i}}(W)|I(W \in B_k(r)) + E|f_{r_{w_i}}(W)|I(W \notin B_k(r)) \tag{3.21}$$

for $i = 1, 2, \ldots, k$. We obtain from (3.6), (3.12) and Proposition 3.4 that

$$\begin{aligned} E|f_{r_{w_i}}(W)|I[W \in B_k(r)] &\le \frac{1}{\sqrt{k}}E|\widetilde{W}f_r(W)| + \frac{1}{\sqrt{k}}(1-\Phi_k(B_k(r))) \\ &\le \frac{C_k}{\sqrt{k}}(1-\Phi_k(B_k(r))) \\ &\le \frac{C_k}{1+r^6}. \end{aligned} \tag{3.22}$$

For the second term of (3.21), By Proposition 3.3(3) and Chebyshev's inequality, we have

$$\begin{aligned} E|f_{r_{w_i}}(W)|I[W \notin B_k(r)] &\le \frac{2}{\sqrt{k}}EI[W \notin B_k(r)] \\ &\le \frac{2}{\sqrt{k}r^2}\sum_{i=1}^{k} EW_i^2 \\ &\le \frac{C_k}{1+r^2}. \end{aligned} \tag{3.23}$$

So, by (3.21)–(3.23), the proof of (2) is completed. □

In Proposition 3.6, we give bounds of a function concerning f. In this proposition, the notation $W_{i,u}$ is introduced as follows: For a random vector $W = (W_1, W_2, \ldots, W_k)$, $u \in \mathbb{R}$ and $i = 1, 2, \ldots, k$, define

$$W_{i,u} := (W_1, W_2, \ldots, W_i + u, \ldots, W_k).$$

Proposition 3.6. *For $k \in \mathbb{N}$ and a Borel set B in $\mathbb{R}^k$, let $g_i : \mathbb{R}^k \to \mathbb{R}$ be defined by*

$$g_i(w) = \frac{\partial}{\partial w_i} \sum_{j=1}^{k} w_j f_B(w)$$

for $i = 1, 2, \ldots, k$. Then

(1) $| g_i(w) | \leq \dfrac{2}{1 + |\bar{w}|^3}$.

(2) *If $B = B_k(r)$ for $r > 0$, then there exists an absolute constant C_k (depends on k) such that*

$$E|g_i(W_{i,u})| \leq \frac{C_k}{1+r^6} + \frac{C_k}{1+r^4} \sum_{m=1}^{k} EW_m^4$$

for $r \geq 4$, $|u| \leq \dfrac{r}{4}$ and $EW_m^4 < \infty$ for $m = 1, 2, \ldots, k$.

Proof. 1.) We can compute directly that

$$g_i(w) = \begin{cases} (1-\Phi_k(B))[\sqrt{2\pi}(1+\bar{w}^2)e^{\frac{1}{2}\bar{w}^2}\Phi_1(\bar{w}) + \bar{w}] & \text{if } w \in B \text{ and } \bar{w} < 0, \\ -(1-\Phi_k(B))[\sqrt{2\pi}(1+\bar{w}^2)e^{\frac{1}{2}\bar{w}^2}(1-\Phi_1(\bar{w})) - \bar{w}] & \text{if } w \in B \text{ and } \bar{w} \geq 0, \\ -\Phi_k(B)[\sqrt{2\pi}(1+\bar{w}^2)e^{\frac{1}{2}\bar{w}^2}\Phi_1(\bar{w}) + \bar{w}] & \text{if } w \notin B \text{ and } \bar{w} < 0, \\ \Phi_k(B)[\sqrt{2\pi}(1+\bar{w}^2)e^{\frac{1}{2}\bar{w}^2}(1-\Phi_1(\bar{w})) - \bar{w}] & \text{if } w \notin B \text{ and } \bar{w} \geq 0. \end{cases} \tag{3.24}$$

Note that for $x \geq 0$,

$$0 \leq \sqrt{2\pi}(1+x^2)e^{\frac{x^2}{2}}(1-\Phi_1(x)) - x \leq \frac{2}{1+x^3} \tag{3.25}$$

(see inequality (5.4) in [10]). If we replace x by $-x$, then for $x < 0$,

$$0 \leq \sqrt{2\pi}(1+x^2)e^{\frac{x^2}{2}}\Phi_1(x) + x \leq \frac{2}{1+|x|^3}. \tag{3.26}$$

The proof of 1) is completed by using the equations (3.25)–(3.26).

2) We note that

$$E|g_i(W_{i,u})| = E|g_i(W_{i,u})|I[W_{i,u} \in B_k(r)] + E|g_i(W_{i,u})|I[W_{i,u} \notin B_k(r)]. \tag{3.27}$$

By Proposition 3.4 and (3.24)–(3.26), we obtain

$$E|g_i(W_{i,u})|I[W_{i,u} \in B_k(r)] \leq 2(1 - \Phi_k(B_k(r))) \leq \frac{C}{1+r^6}. \tag{3.28}$$

From (1) and Chebyshev's inequality,

$$\begin{aligned}
E|g_i(W_{i,u})|I(W_{i,u} \notin B_k(r)) &\leq 2P\left(\sum_{\substack{m=1\\m\neq i}}^{k} W_m^2 + (W_i+u)^2 > r^2\right)\\
&\leq 2P\left(\sum_{\substack{m=1\\m\neq i}}^{k} W_m^2 + 2W_i^2 + 2u^2 > r^2\right)\\
&= 2P\left(\sum_{m=1}^{k} W_m^2 + W_i^2 > r^2 - 2u^2\right)\\
&\leq 2P\left(\sum_{m=1}^{k} W_m^2 + W_i^2 > \frac{7r^2}{8}\right)\\
&\leq \frac{C_k}{1+r^4} E\left(\sum_{m=1}^{k} W_m^2 + W_i^2\right)^2\\
&\leq \frac{C_k}{1+r^4} E\left(\sum_{m=1}^{k} W_m^4 + W_i^4\right)\\
&\leq \frac{C_k}{1+r^4} \sum_{m=1}^{k} EW_m^4
\end{aligned} \tag{3.29}$$

where we used the fact that

$$(a_1 + a_2 + \cdots + a_k)^2 \leq k(a_1^2 + a_2^2 + \cdots + a_k^2) \tag{3.30}$$

in the second and the fifth inequality. By (3.27)–(3.29), we have (2) and hence the proposition. □

Remark 3.7. *Each function g_i defined in Proposition 3.6 is piecewise continuous.*

This remark is obtained by the definition of g_i and Remark 3.2(2).

CHAPTER IV

BOUNDS ON NORMAL APPROXIMATION ON A HALF PLANE IN $\mathbb{R}^k$

For $n \in \mathbb{N}$, let $X_i, i = 1, 2, \ldots, n$, be independent and identically distributed random variables with zero mean and $\sum_{i=1}^{n} EX_i^2 = 1$. Define

$$S_n = \sum_{i=1}^{n} X_i$$

and Φ_1 the standard normal distribution in $\mathbb{R}$. Suppose that $E|X_i|^3 < \infty$ for $i = 1, 2, \ldots, n$. The uniform and non-uniform versions of the Berry-Esseen inequality are

$$\sup_{x \in \mathbb{R}} |P(S_n \le x) - \Phi_1(x)| \le C_0 \sum_{i=1}^{n} E|X_i|^3$$

and

$$|P(S_n \le x) - \Phi_1(x)| \le \frac{C_1}{1 + |x|^3} \sum_{i=1}^{n} E|X_i|^3,$$

respectively, where C_0 and C_1 are positive constants. Without assuming that $X_i's$ are identically distributed, the best constant C_0 and C_1 were given by Shevtsova [23] and Paditz [19], respectively. The statements are as follow:

Theorem 4.1. *([23]) Let $X_i, i = 1, 2, \ldots, n$, be independent random variables such that $EX_i = 0$ and $E|X_i|^3 < \infty$. Assume that $\sum_{i=1}^{n} EX_i^2 = 1$. Then*

$$\sup_{x \in \mathbb{R}} |P(S_n \le x) - \Phi_1(x)| \le 0.5600 \sum_{i=1}^{n} E|X_i|^3.$$

Theorem 4.2. *([19]) Under the assumption of theorem 4.1, we have*

$$|P(S_n \le x) - \Phi_1(x)| \le \frac{31.935}{1 + |x|^3} \sum_{i=1}^{n} E|X_i|^3$$

for all real numbers x.

In 2001, Chen and Shao [10] relaxed the condition to the finiteness of the second moments and gave uniform and non-uniform versions of the inequality. The constant of the non-uniform version was investigated by Neammanee and Thongtha [18] in 2007. Here are the results.

Theorem 4.3. *([10]) Let $X_i, i = 1, 2, \ldots, n$, be independent random variables such that $EX_i = 0$ and $\sum_{i=1}^{n} EX_i^2 = 1$. Then*

$$\sup_{x\in\mathbb{R}} |P(S_n \le x) - \Phi_1(x)| \le 4.1 \sum_{i=1}^{n} \{E|X_i|^2 I(|X_i| > 1) + E|X_i|^3 I(|X_i| \le 1)\}$$

and for all real numbers x, there exists an absolute constant C such that

$$|P(S_n \le x) - \Phi_1(x)| \le C \sum_{i=1}^{n} \left\{ \frac{E|X_i|^2 I(|X_i| > 1+|x|)}{1+|x|^2} + \frac{E|X_i|^3 I(|X_i| \le 1+|x|)}{1+|x|^3} \right\}.$$

Theorem 4.4. *([18])Under the assumptions of Theorem 4.3, we have*

$$|P(S_n \le x) - \Phi_1(x)| \le C \sum_{i=1}^{n} \left\{ \frac{E|X_i|^2 I(|X_i| > 1+|x|)}{1+|x|^2} + \frac{E|X_i|^3 I(|X_i| \le 1+|x|)}{1+|x|^3} \right\}$$

for all real numbers x where

$$C = \begin{cases} 13.11 & \text{if } 0 \le |x| < 1.3, \\ 28.54 & \text{if } 1.3 \le |x| < 2, \\ 46.32 & \text{if } 2 \le |x| < 3, \\ 61.40 & \text{if } 3 \le |x| < 7.98, \\ 40.12 & \text{if } 7.98 \le |x| < 14, \\ 39.39 & \text{if } |x| \ge 14. \end{cases}$$

In the case that each X_i is bounded, the uniform and non-uniform versions were given in [12] and [7], respectively.

Theorem 4.5. *([12]) Let $X_i, i = 1, 2, \ldots, n$, be independent random variables such that $EX_i = 0$, $\sum_{i=1}^{n} EX_i^2 = 1$ and $|X_i| \le \delta_0$, then*

$$\sup_{x\in\mathbb{R}} |P(S_n \le x) - \Phi_1(x)| \le 3.3\delta_0.$$

Theorem 4.6. *([7]) Under the assumptions of Theorem 4.5, there exists a constant C not depends on δ_0 such that for every real numbers x,*

$$|P(S_n \leq x) - \Phi_1(x)| \leq \frac{C\delta_0}{1+|x|^3}.$$

In 2004, Chen and Shao [11] introduced four assumptions on local dependence and gave bounds of normal approximation under the assumptions. These conditions are circumstances in which dependence involved and the Stein's method can be applied to these situations.

Let $\mathcal{J}$ be a finite index set of cardianality n, and let $\{X_i, i \in \mathcal{J}\}$ be a random field with zero means and finite variances. For $A \subset \mathcal{J}$, let X_A denote $\{X_i, i \in A\}$,$A^c = \{j \in \mathcal{J} : j \notin A\}$ and $|A|$ the cardinality of A. The situations are proposed as follows:

$(LD1)$ For each $i \in \mathcal{J}$ there exists $A_i \subset \mathcal{J}$ such that X_i and $X_{A_i^c}$ are independent.

$(LD2)$ For each $i \in \mathcal{J}$ there exists $A_i \subset B_i \subset \mathcal{J}$ such that X_i is independent of $X_{A_i^c}$ and X_{A_i} is independent of $X_{B_i^c}$.

$(LD3)$ For each $i \in \mathcal{J}$ there exists $A_i \subset B_i \subset C_i \subset \mathcal{J}$ such that X_i is independent of $X_{A_i^c}$, X_{A_i} is independent of $X_{B_i^c}$ and X_{B_i} is independent of $X_{C_i^c}$.

$(LD4^*)$ For each $i \in \mathcal{J}$ there exists $A_i \subset B_i \subset B_i^* \subset C_i^* \subset D_i^* \subset \mathcal{J}$ such that X_i is independent of $X_{A_i^c}$, X_{A_i} is independent of $X_{B_i^c}$ and then X_{A_i} is independent of $\{X_{A_j}, j \in B_i^{*c}\}, \{X_{A_l}, l \in B_i^*\}$ is independent of $\{X_{A_j}, j \in C_i^{*c}\}$ and $\{X_{A_l}, l \in C_i^*\}$ is independent of $\{X_{A_j}, j \in D_i^{*c}\}$.

Remark 4.7. $(LD4^*) \Rightarrow (LD3) \Rightarrow (LD2) \Rightarrow (LD1)$.

The followings are the uniform Berry-Esseen bound under $(LD3)$ and non-uniform bound under $(LD4^*)$ stated in [11].

Theorem 4.8. *Let $2 < p \leq 3$. Assume that $(LD3)$ is satisfied with*

$$\max(|N(C_i)|, |\{j : i \in C_j\}|) \leq \kappa$$

where $N(C_i) = \{j \in \mathcal{J} : C_i \cap B_j \neq \varnothing\}$. Then

$$\sup_{x\in\mathbb{R}} |P(S_n \leq x) - \Phi_1(x)| \leq 75\kappa^{p-1} \sum_{j\in\mathcal{J}} E|X_i|^p.$$

Theorem 4.9. *Assume that* $E|X_i|^p < \infty$ *for* $2 < p \leq 3$ *and that* $(LD4^*)$ *is satisfied. Let* $\kappa = \max_{i\in\mathcal{J}} \max(|D_i^*|, |\{j : i \in D_j^*\}|)$. *Then*

$$|P(S_n \leq x) - \Phi_1(x)| \leq \frac{C\kappa^p}{(1+|x|)^p} \sum_{j\in\mathcal{J}} E|X_i|^p$$

where C *is an absolute constant.*

Let $n, k \in \mathbb{N}$ and $Y_i = (Y_{i1}, Y_{i2}, ..., Y_{ik}), i = 1, 2, ..., n$ be independent random vectors in $\mathbb{R}^k$ with zero means,

$$\sum_{i=1}^{n} EY_{ij}^2 = 1 \text{ for } j = 1, 2, \dots, k \quad \text{and} \tag{4.1}$$

$$EY_{ij}Y_{il} = 0 \text{ for } j \neq l. \tag{4.2}$$

Define

$$W_n = \sum_{i=1}^{n} Y_i.$$

Let F_n be the distribution of W_n and Φ_k the standard Gaussian distribution in $\mathbb{R}^k$. In this chapter, we will use Berry-Eesseen bounds in $\mathbb{R}$ to find bounds on multivariate normal approximation on the set

$$A_k(r) = \left\{(w_1, w_2, \dots, w_k) \in \mathbb{R}^k \mid \sum_{i=1}^{k} w_i \leq r\right\} \quad \text{for} \quad r \in \mathbb{R}.$$

We give our results on various assumptions: each random variable Y_{ij} is bounded, $E|Y_{ij}|^3 < \infty$ and $E|Y_{ij}|^p < \infty$ for some $2 < p < 3$. Our estimations are stated in the following theorems.

Theorem 4.10. *If* $|Y_{ij}| \leq \delta_0$ *for* $i = 1, 2, \dots, n$ *and* $j = 1, 2, \dots, k$, *then*

$$\sup_{r\in\mathbb{R}} |F_n(A_k(r)) - \Phi_k(A_k(r))| \leq 3.3\sqrt{k}\delta_0$$

and there exists a constant C *not depends on* δ_0 *such that for every real numbers* r,

$$|F_n(A_k(r)) - \Phi_k(A_k(r))| \leq \frac{Ck^2\delta_0}{(\sqrt{k})^3 + |r|^3}.$$

Theorem 4.11. *If* $E|Y_{ij}|^p < \infty$ *for some* $2 < p < 3, i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, k$, *then*

$$\sup_{r\in\mathbb{R}} |F_n(A_k(r)) - \Phi_k(A_k(r))| \leq 75(4)^{p-1}k^{\frac{p}{2}} \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^p$$

and there exists an abs0lute constant C *such that for all real numbers* r,

$$|F_n(A_k(r)) - \Phi_k(A_k(r))| \leq \frac{C(5k)^p}{(\sqrt{k} + |r|)^p} \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^p.$$

Theorem 4.12. *If* $E|Y_{ij}|^3 < \infty$ *for* $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, k$, *then*

$$\sup_{r\in\mathbb{R}} |F_n(A_k(r)) - \Phi_k(A_k(r))| \leq 0.5600\sqrt{k} \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^3$$

and for all real numbers r,

$$|F_n(A_k(r)) - \Phi_k(A_k(r))| \leq \frac{31.935k^2}{(\sqrt{k})^3 + |r|^3} \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^3.$$

The proof of our main theorems are given in section 4.2. In the next section, we will give a proprosition which is used to prove the theorems.

4.1 Auxiliary Results

The first auxiliary result gives us that the random field $\{Y_{i,j} \mid i = 1, 2, \ldots, n, j = 1, 2, \ldots, k\}$ according to the conditions (4.1) and (4.2) satisfies $(LD4^*)$. This result is used to prove Theorem 4.11.

Proposition 4.13. *For* $k, n \in \mathbb{N}$, *let* $Y_i = (Y_{i1}, Y_{i2}, ..., Y_{ik}), i = 1, 2, ..., n$ *be independent random vectors in* $\mathbb{R}^k$ *with zero mean. If each* Y_i *assents to the conditions* (1.1) *and* (1.2). *Then* $\{Y_{ij} \mid i = 1, 2, \ldots, n, j = 1, 2, \ldots, k\}$ *satisfies* $(LD4^*)$.

Proof. This proposition is completed by setting $A_{ij} \subset B_{ij} \subset B^*_{ij} \subset C^*_{ij} \subset D^*_{ij}$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$ as follows:

$$A_{ij} = \{il \mid l = 1, 2, \ldots, k\} \text{ for } i = 1, 2, \ldots, n,$$

$$B_{ij} = \{il, (i+1)l \mid l = 1, 2, \ldots, k\} \text{ for } i = 1, 2, \ldots, n-1 \text{ and } B_{nj} = B_{(n-1)j},$$

$$B^*_{ij} = C_{ij} = \{il, , (i+1)l, (i+2)l \mid l = 1, 2, \ldots, k\} \text{ for } i = 1, 2, \ldots, n-2 \text{ and}$$
$$B^*_{(n-m)j} = C_{(n-m)j} = B_{(n-2)j} \text{ for } m = 1, 2,$$
$$C^*_{ij} = \{il, (i+1)l, \ldots, (i+3)l \mid l = 1, 2, \ldots, k\} \text{ for } i = 1, 2, \ldots, n-3 \text{ and}$$
$$C^*_{(n-m)j} = C^*_{(n-3)j} \; m = 1, 2, 3,$$
$$D^*_{ij} = \{il, (i+1)l, \ldots, (i+4)l \mid l = 1, 2, \ldots, k\} \text{ for } i = 1, 2, \ldots, n-4 \text{ and}$$
$$D^*_{(n-m)j} = D^*_{(n-4)j} \; m = 1, 2, 3, 4.$$

So, we have the proposition. □

From the sets defined in the above proposition, we can compute directly that for each $i = 1, 2, \ldots, n$,

$$\max(|N(C_i)|, |\{j : i \in C_j\}| \leq 4 \tag{4.3}$$

and

$$\max_{1 \leq i \leq n} \max(|D^*_i|, |\{j : i \in D^*_j\}|) \leq 5 \tag{4.4}$$

where $N(C_i)$ is defined in Theorem 4.8.

In order to prove the main theorems, we use the Berry-Esseen Theorems in $\mathbb{R}$ in which the limit distribution is Φ_1. However, the limit distribution in our theorems is the standard Gaussian distribution Φ_k in $\mathbb{R}^k$. In the following proposition, we give a relation between Φ_1 and Φ_k.

Proposition 4.14. *For $k \in \mathbb{N}$ and $r \in \mathbb{R}$, we have*

$$\Phi_k(A_k(r)) = \Phi_1\left(\frac{r}{\sqrt{k}}\right).$$

Proof. To prove the proposition, let $w = (w_1, w_2, \ldots, w_k) \in A_k(r)$ and $B = \{b_1, b_2, \ldots, b_k\}$ be an orthonormal basis for $\mathbb{R}^k$ with $b_1 = \frac{1}{\sqrt{k}}(1, 1, \ldots, 1)$. The existence of B is guaranteed by the Gram-Schmidt process. Set

$$t_1 = \langle b_1, w\rangle \text{ and } t_i = \langle b_i, w\rangle \text{ for } i = 2, 3, \ldots, k.$$

Then

$$t_1 = \frac{1}{\sqrt{k}}\sum_{i=1}^{k} w_i \leq \frac{r}{\sqrt{k}}, -\infty < t_i < \infty, \text{ for } i = 2, 3, \ldots, k, \text{ and}$$
$$\sum_{i=1}^{k} \langle b_i, w\rangle\, b_i = w = \sum_{i=1}^{k} \langle e_i, w\rangle\, e_i$$

where $\{e_1, e_2, \ldots, e_k\}$ is the usual orthonormal basis for $\mathbb{R}^k$. We obtain that

$$\sum_{i=1}^{k} w_i^2 = ||\sum_{i=1}^{k} w_i e_i||^2 = ||\sum_{i=1}^{k} \langle e_i, w\rangle\, e_i||^2 = ||\sum_{i=1}^{k} \langle b_i, w\rangle\, b_i||^2 = ||\sum_{i=1}^{k} t_i b_i||^2$$
$$= \sum_{i=1}^{k} t_i^2. \tag{4.5}$$

Let J be the Jacobian matrix,

$$J = \begin{bmatrix} \frac{\partial w_1}{\partial t_1} & \frac{\partial w_2}{\partial t_1} & \cdots & \frac{\partial w_k}{\partial t_1} \\ \frac{\partial w_1}{\partial t_2} & \frac{\partial w_2}{\partial t_2} & \cdots & \frac{\partial w_k}{\partial t_2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial w_1}{\partial t_k} & \frac{\partial w_2}{\partial t_k} & \cdots & \frac{\partial w_k}{\partial t_k} \end{bmatrix}.$$

Thus $|\det(J)| = 1$. Then, by (4.5),

$$\begin{aligned} \Phi_k(A_k(r)) &= \frac{1}{(2\pi)^{\frac{k}{2}}} \int_{A_k(r)} e^{-\frac{1}{2}\sum_{i=1}^{k} w_i^2} d^k w \\ &= \frac{1}{(2\pi)^{\frac{k}{2}}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{\frac{r}{\sqrt{k}}} e^{-\frac{1}{2}\sum_{i=1}^{k} t_i^2} |\det J| dt_1 dt_2 \cdots dt_k \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{r}{\sqrt{k}}} e^{-t^2} dt \\ &= \Phi_1(\frac{r}{\sqrt{k}}). \end{aligned}$$

Hence, the proposition is proved. □

4.2 Proof of Main Results

We are now ready to prove our results in this section. Theorem 4.10 is proved by applying Theorem 4.5 and Theorem 4.6. Theorem 4.8 and Theorem 4.9 are applied in the proof of Theorem 4.11. Likewise, the bounds in Theorem 4.12 are obtained by applying Theorem 4.1 and Theorem 4.2.

Proof of Theorem 4.10

Proof. For each $i = 1, 2, \ldots, n$ and , $j = 1, 2, \ldots, k$, we define

$$W_{jn} = \sum_{i=1}^{n} Y_{ij} \quad \text{and} \quad T_i = \sum_{j=1}^{k} Y_{ij}.$$

Thus $T_1, T_2, \ldots, T_n$ are independent,

$$E(T_i) = 0, \quad |T_i| \le k\delta_0, \tag{4.6}$$

$$W_n = (W_{1n}, W_{2n}, \ldots, W_{kn}) \quad \text{and} \quad \sum_{j=1}^{k} W_{jn} = \sum_{i=1}^{n} T_i. \tag{4.7}$$

By the assumptions that Y_i has zero means and satisfies (4.1) and (4.2), we have

$$\sum_{i=1}^{n} Var(Y_{ij}) = 1 \quad \text{and} \quad Cov(Y_{ij}, Y_{ik}) = 0 \quad \text{for} \quad j \neq k.$$

Therefore

$$Var\left(\frac{1}{\sqrt{k}}\sum_{i=1}^{n} T_i\right) = \frac{1}{k}\sum_{i=1}^{n} Var(T_i) = \frac{1}{k}\sum_{j=1}^{k}\sum_{i=1}^{n} Var(Y_{ij}) = 1. \tag{4.8}$$

By Proposition 4.14, Theorem 4.5 and (4.6)-(4.8), we have

$$\begin{aligned}
&\sup_{r\in\mathbb{R}} |P(W_n \in A_k(r)) - \Phi_k(A_k(r))| \\
&= \sup_{r\in\mathbb{R}} \left| P\left(\sum_{j=1}^{k} W_{jn} \le r\right) - \Phi_1\left(\frac{r}{\sqrt{k}}\right)\right| \\
&= \sup_{r\in\mathbb{R}} \left| P\left(\sum_{i=1}^{n} T_i \le r\right) - \Phi_1\left(\frac{r}{\sqrt{k}}\right)\right| \\
&= \sup_{r\in\mathbb{R}} \left| P\left(\frac{1}{\sqrt{k}}\sum_{i=1}^{n} T_i \le \frac{r}{\sqrt{k}}\right) - \Phi_1\left(\frac{r}{\sqrt{k}}\right)\right| \\
&\le 3.3\sqrt{k}\delta_0.
\end{aligned} \tag{4.9}$$

For the second part, by Theorem 4.6 and (4.9), we have

$$\begin{aligned}
|P(W_n \in A_k(r)) - \Phi_k(A_k(r))| &= \left| P\left(\frac{1}{\sqrt{k}}\sum_{i=1}^{n} T_i \le \frac{r}{\sqrt{k}}\right) - \Phi_1\left(\frac{r}{\sqrt{k}}\right)\right| \\
&\le \frac{C\sqrt{k}\delta_0}{(1 + |\frac{r}{\sqrt{k}}|^3)} \\
&= \frac{Ck^2\delta_0}{[(\sqrt{k})^3 + |r|^3]}
\end{aligned}$$

for all real numbers r. Hence, the proof is completed. □

Proof of Theorem 4.11

Proof. For each $i = 1, 2, \ldots, n$, define T_i as in Theorem 4.10.

Thus, by the inequality

$$\left|\sum_{j=1}^{k} Y_{ij}\right|^p \leq k^p \sum_{j=1}^{k} |Y_{ij}|^p, \tag{4.10}$$

we obtain that

$$E|T_i|^p = E|\sum_{j=1}^{k} Y_{ij}|^p \leq k^p \sum_{j=1}^{k} E|Y_{ij}|^p < \infty.$$

So, by (4.3), (4.6), (4.8)–(4.10) and Theorem 4.8, we have

$$\begin{aligned}\sup_{r\in\mathbb{R}} |P(W_n \in A_k(r)) - \Phi_k(A_k(r))| &= \sup_{r\in\mathbb{R}} \left| P\left(\frac{1}{\sqrt{k}} \sum_{i=1}^{n} T_i \leq \frac{r}{\sqrt{k}}\right) - \Phi_1\left(\frac{r}{\sqrt{k}}\right)\right| \\ &\leq 75(4)^{p-1} \sum_{i=1}^{n} E\left|\frac{T_i}{\sqrt{k}}\right|^p \\ &\leq 75(4)^{p-1} k^{\frac{p}{2}} \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^p.\end{aligned}$$

For a non-uniform bound, by (4.4), (4.6), (4.8)–(4.10) and Theorem 4.9, we have

$$\begin{aligned}|P(W_n \in A_k(r)) - \Phi_k(A_k(r))| &= \left| P\left(\frac{1}{\sqrt{k}} \sum_{i=1}^{n} T_i \leq \frac{r}{\sqrt{k}}\right) - \Phi_1\left(\frac{r}{\sqrt{k}}\right)\right| \\ &\leq \frac{5^p C}{(1+|\frac{r}{\sqrt{k}}|)^p} \sum_{i=1}^{n} E\left|\frac{T_i}{\sqrt{k}}\right|^p \\ &\leq \frac{C(5k)^p}{(\sqrt{k}+|r|)^p} \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^p\end{aligned}$$

for all real numbers r. Hence, the proof is completed. □

Proof of Theorem 4.12

Proof. By Theorem 4.1, Theorem 4.2 and the same argument as in Theorem 4.11, we have the theorem. □

Remark 4.15. *The assumptions* (4.1) *and* (4.2) *in all of the above theorems can be extended to*

$$\frac{1}{k} Var\left(\sum_{j=1}^{k}\sum_{i=1}^{n} Y_{ij}\right) = 1.$$

CHAPTER V

UNIFORM BERRY-ESSEEN BOUNDS ON SOME BOREL SETS IN $\mathbb{R}^k$

For each $n, k \in \mathbb{N}$ and $i = 1, 2, \ldots, n$, let $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{ik})$ be independent random vectors in $\mathbb{R}^k$ with zero vector means,

$$\sum_{i=1}^{n} EY_{ij}^2 = 1 \text{ for } j = 1, 2, \ldots, k \text{ and}$$

$$EY_{ij}Y_{il} = 0 \text{ for } j \neq l.$$

Define

$$W_n = \sum_{i=1}^{n} Y_i.$$

Let F_n be the distribution of W_n and Φ_k the standard Gaussian distribution in $\mathbb{R}^k$. Assume that the third moments are finite. Götze [14] used the Stein's method to find bounds on multivariate normal approximation. His uniform bound on all measurable convex sets C in $\mathbb{R}^k$ is

$$|F_n(C) - \Phi_k(C)| \leq C_k \gamma_3 \tag{5.1}$$

where $\gamma_3 = \sum_{i=1}^{n} E||Y_i||^3$, $||\cdot||$ is the Euclidean norm in $\mathbb{R}^k$ and

$$C_k = 124.4 a_k \sqrt{k} + 10.7,$$

wheren $a_k = 2.04, 2.4, 2.69, 2.94$ for $k = 2, 3, 4, 5$, respectively and $a_k \leq 1.27\sqrt{k}$ for $k \geq 6$. His estimation is of order $O(n^{-\frac{1}{2}})$. In 2009, Reinert and Röllin [22] used the same method as in [8] with a new Stein's equation to estimate the bounds of the approximation. The estimation in [22] is of order $O(n^{-\frac{1}{4}})$, but their result can be applied to the case that Y_i, $i = 1, 2, \ldots, n$, may be dependent random vectors.

In this chapter, we will use the Stein's method to find bounds on multivariate normal approximation on the sets

$$B_k(r) = \{x \in \mathbb{R}^k \mid x_1^2 + x_2^2 + \cdots + x_k^2 \le r^2\} \text{ for } r > 0,$$
$$A_k(r) = \{x \in \mathbb{R}^k \mid x_1 + x_2 + \cdots + x_k \le r\} \text{ for } r \in \mathbb{R} \text{ and}$$
$$R_k(r) = \{x \in \mathbb{R}^k \mid |x_j| \le r_j, j = 1, 2, \ldots, k\} \text{ where } r = (r_1, r_2, \ldots, r_k)$$
$$\text{and } r_j > 0 \text{ for all } j = 1, 2, \ldots, k.$$

In our theorems, we assume further that all components of Y_i are independent for all $i = 1, 2, \ldots, n$. The results are as follows:

Theorem 5.1. *Let $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{ik}), i = 1, 2, \ldots, n$, be independent random vectors in $\mathbb{R}^k$ with zero means and Y_{ij} are independent for all $j = 1, 2, \ldots, k$. Define $W_n = \sum_{i=1}^{n} Y_i$. Let F_n be the distribution function of W_n. Assume that $\sum_{i=1}^{n} EY_{ij}^2 = 1$ for $j = 1, 2, \ldots, k$ and $\sum_{j=1}^{k} E|Y_{ij}|^3 < \infty$ for $i = 1, 2, \ldots, n$. Then*

$$|F_n(B_k(r)) - \Phi_k(B_k(r))| \le C\beta_3$$

where $C = \dfrac{4.55}{k} + \dfrac{3}{k\sqrt{k}}$ and $\beta_3 = \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^3$.

Theorem 5.2. *Under the assumptions of Theorem 5.1, we have*

$$|F_n(A_k(r)) - \Phi_k(A_k(r))| \le C\beta_3$$

where $C = \dfrac{4.55}{k} + \dfrac{3}{k\sqrt{k}}$ and $\beta_3 = \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^3$.

The order of the estimations in Theorem 5.1 and Theorem 5.2 are $O(n^{-\frac{1}{2}})$ which is better than the result in [22] and the constants are smaller than the constant in (5.1). In addition, the constant in Theorem 5.2 is smaller than the constant in Theorem 4.12 for $k \ge 7$.

Corollary 5.3. *Let X_i, $i = 1, 2, \ldots, n$, be independent random variables with zero means and $\sum_{i=1}^{n} EX_i^2 = 1$. Define $W_n = \sum_{i=1}^{n} X_i$. Let F_n be the distribution function of W_n. If $E|X_i|^3 < \infty$ for $i = 1, 2, \ldots, n$, then*

$$|F_n(x) - \Phi_1(x)| \leq 7.55 \sum_{i=1}^{n} E|X_i|^3.$$

Theorem 5.4. *Under the assumption of Theorem 5.1, we have*

$$|F_n(R_k(r)) - \Phi_k(R_k(r))| \leq C\beta_3$$

where $C = \dfrac{4.55}{k} + \dfrac{3}{k\sqrt{k}}$ *and* $\beta_3 = \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^3.$

The technique used in all of the above theorems is the Stein's method. An information of this method, which is needed to prove these results, has already been given in Chapter III. In the next section, we will give the proofs of our results.

5.1 Proof of Main Results

In this section, we will give the uniform bounds of the distribution approximation of W_n by Φ_k. We use the idea in [12] to prove our results. The Stein's method using concentration inequality approach is applied. The key of this approach is the concentration inequality.

Proposition 5.5. *(Concentration inequality)*
Let X_i, $i = 1, 2, \ldots, n$, be independent random variables with zero means and

$$\sum_{j=1}^{n} EX_j^2 = 1.$$

Let $\gamma = \sum_{j=1}^{n} E|X_j|^3$ *and* $W^{(i)} = \sum_{j=1}^{n} X_j - X_i$. *Then*

$$P(a \leq W^{(i)} \leq b) \leq \sqrt{2}(b-a) + (1+\sqrt{2})\gamma$$

for all reals $a < b$ and for every $i = 1, 2, \ldots, n$.

Proof. See also [12] pp. 32–33. □

To prove our theorems, we introduce the following notations.

For $k, n \in \mathbb{N}$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$, let

$$W_{nj} = \sum_{i=1}^{n} Y_{ij}, \;\; W_{nj}^{(i)} = W_{nj} - Y_{ij}, \quad \text{and} \quad W_n = (W_{n1}, W_{n2}, \ldots, W_{nk}).$$

We are now ready to prove our main results.

Proof of Theorem 5.1

Proof. Firstly, we will prove the theorem in the case of $k = 2$. Let f_r be the solution of (3.3) with respect to the indicator test function on $B_2(r)$ and $f_{r_{w_1}}, f_{r_{w_2}}$ partial derivatives of f_r with respect to w_1 and w_2, respectively. Thus, by (3.3),

$$P(W_n \in B_2(r)) - \Phi_2(B_2(r)) = \frac{1}{\sqrt{2}}(S_1 - T_1) + \frac{1}{\sqrt{2}}(S_2 - T_2) \tag{5.2}$$

where

$$\begin{aligned} S_1 &= Ef_{r_{w_1}}(W_{n1}, W_{n2}), & T_1 &= EW_{n1} f_r(W_{n1}, W_{n2}), \\ S_2 &= Ef_{r_{w_2}}(W_{n1}, W_{n2}), \quad \text{and} & T_2 &= EW_{n2} f_r(W_{n1}, W_{n2}). \end{aligned}$$

The theorem is proved when we give a bound on the right handside of (5.2). To estimate $|S_1 - T_1|$, let

$$K_{ij}(t) = EY_{ij}[I(0 \le t \le Y_{ij}) - I(Y_{ij} \le t < 0)]$$

for $t \in \mathbb{R}$, $i = 1, 2, \ldots, n$, $j = 1, 2$ where I is the indicator function on Ω. We can follow the idea from [12] to show that

$$K_{ij}(t) \ge 0 \quad \text{for all } t \in \mathbb{R}, \tag{5.3}$$

$$\sum_{i=1}^{n} E\int_{-\infty}^{\infty} K_{ij}(t)dt = \sum_{i=1}^{n} EY_{ij}^2 = 1, \tag{5.4}$$

$$\sum_{i=1}^{n} E\int_{-\infty}^{\infty} (|Y_{ij}|+|t|)K_{ij}(t)dt = \frac{3}{2}\sum_{i=1}^{n} E|Y_{ij}|^3, \tag{5.5}$$

$$S_1 = \sum_{i=1}^{n} E\int_{-\infty}^{\infty} f_{r_{w_1}}(W_{n1}^{(i)} + Y_{i1}, W_{n2})K_{i1}(t)dt, \;\text{ and} \tag{5.6}$$

$$T_1 = \sum_{i=1}^{n} E\int_{-\infty}^{\infty} f_{r_{w_1}}(W_{n1}^{(i)} + t, W_{n2})K_{i1}(t)dt. \tag{5.7}$$

Thus, by (3.6) and (5.6)–(5.7),

$$\begin{aligned} S_1 - T_1 &= \sum_{i=1}^{n} E \int_{-\infty}^{\infty} [f_{r_{w_1}}(W_{n1}^{(i)} + Y_{i1}, W_{n2}) - f_{r_{w_1}}(W_{n1}^{(i)} + t, W_{n2})] K_{i1}(t) dt \\ &= \frac{1}{\sqrt{2}} R_1 + \frac{1}{2} R_2 \end{aligned} \tag{5.8}$$

where

$$\begin{aligned} R_1 &= \sum_{i=1}^{n} E \int_{-\infty}^{\infty} [h_{B_2(r)}(W_{n1}^{(i)} + Y_{i1}, W_{n2}) - h_{B_2(r)}(W_{n1}^{(i)} + t, W_{n2})] K_{i1}(t) dt \\ R_2 &= \sum_{i=1}^{n} E \int_{-\infty}^{\infty} [(W_{n1}^{(i)} + Y_{i1} + W_{n2}) f_r(W_{n1}^{(i)} + Y_{i1}, W_{n2}) \\ &\qquad - (W_{n1}^{(i)} + t + W_{n2}) f_r(W_{n1}^{(i)} + t, W_{n2})] K_{i1}(t) dt. \end{aligned}$$

For $i = 1, 2, \ldots, n$, let

$$\begin{aligned} A_{i1} &= \Big\{ w \in \Omega \mid -t + \alpha(w) < W_{n1}^{(i)}(w) \leq -Y_{i1}(w) + \alpha(w) \Big\} \text{ and} \\ B_{i1} &= \Big\{ w \in \Omega \mid -Y_{i1}(w) - \alpha(w) \leq W_{n1}^{(i)}(w) < -t - \alpha(w) \Big\} \end{aligned}$$

where $\alpha(w) = \sqrt{r^2 - W_{n2}^2(w) I(w \in \Lambda)}$ and $\Lambda = \{w \in \Omega \mid W_{n2}^2(w) \leq r^2\}$.
To find an upper bound of R_1, we will show that

$$\Big\{ w \in \Omega \mid h_{B_2(r)}(W_{n1}^{(i)} + Y_{i1}, W_{n2})(w) - h_{B_2(r)}(W_{n1}^{(i)} + t, W_{n2})(w) = 1 \Big\} \subseteq A_{i1} \cup B_{i1}. \tag{5.9}$$

To prove (5.9), let $w \in \Omega$ be such that

$$h_{B_2(r)}(W_{n1}^{(i)} + Y_{i1}, W_{n2})(w) - h_{B_2(r)}(W_{n1}^{(i)} + t, W_{n2})(w) = 1 \text{ and } w \notin A_{i1}.$$

Thus $h_{B_2(r)}(W_{n1}^{(i)} + Y_{i1}, W_{n2})(w) = 1$, $h_{B_2(r)}(W_{n1}^{(i)} + t, W_{n2})(w) = 0$ and $w \in \Lambda$. Suppose that $w \notin B_{i1}$. Then

$$W_{n1}^{(i)}(w) < -Y_{i1}(w) - \alpha(w) \quad \text{or} \quad W_{n1}^{(i)}(w) \geq -t - \alpha(w).$$

If $W_{n1}^{(i)}(w) < -Y_{i1}(w) - \alpha(w)$, then $W_{n1}^{(i)}(w) + Y_{i1}(w) < -\alpha(w)$. Thus

$$(W_{n1}^{(i)}(w) + Y_{i1}(w))^2 + W_{n2}^2(w) > r^2.$$

This contradicts to $h_{B_2(r)}(W_{n1}^{(i)} + Y_{i1}, W_{n2})(w) = 1$. Therefore

$$W_{n1}^{(i)}(w) \geq -Y_{i1}(w) - \alpha(w).$$

Assume that $W_{n1}^{(i)}(w) \geq -t - \alpha(w)$. Since $w \notin A_{i1}$, we have

$$-t - \alpha(w) \leq W_{n1}^{(i)}(w) \leq -t + \alpha(w) \quad \text{or} \quad W_{n1}^{(i)}(w) > -Y_{i1}(w) + \alpha(w).$$

If $-t - \alpha(w) \leq W_{n1}^{(i)}(w) \leq -t + \alpha(w)$, then

$$(W_{n1}^{(i)}(w) + t)^2 + W_{n2}^2(w) \leq r^2.$$

This contradicts to $h_{B_2(r)}(W_{n1}^{(i)} + t, W_{n2})(w) = 0$. If $W_{n1}^{(i)}(w) > -Y_{i1}(w) + \alpha(w)$, then

$$(W_{n1}^{(i)}(w) + Y_{i1}(w))^2 + W_{n2}^2(w) > r^2.$$

This contradicts to $h_{B_2(r)}(W_{n1}^{(i)} + Y_{i1}, W_{n2})(w) = 1$. Hence $w \in B_{i1}$. This proves (5.9).

From (5.9) and the fact that $h_{B_2(r)}$ is the indicator function, we obtain

$$h_{B_2(r)}(W_{n1}^{(i)} + Y_{i1}, W_{n2}) - h_{B_2(r)}(W_{n1}^{(i)} + t, W_{n2}) \leq I(A_{i1} \cup B_{i1}).$$

Thus, by (5.3),

$$\begin{aligned} R_1 &\leq \sum_{i=1}^{n} E \int_{-\infty}^{\infty} (I(A_{i1} \cup B_{i1})K_{i1}(t)dt \\ &\leq \sum_{i=1}^{n} E \int_{-\infty}^{\infty} (I(A_{i1}) + I(B_{i1})K_{i1}(t)dt \\ &\leq \sum_{i=1}^{n} E \int_{-\infty}^{0} I(A_{i1})K_{i1}(t)dt + \sum_{i=1}^{n} E \int_{0}^{\infty} I(B_{i1})K_{i1}(t)dt \end{aligned} \tag{5.10}$$

where

$$I(A_{i1})K_{i1}(t) = 0 \text{ for } t \in [0, \infty) \quad \text{and} \quad I(B_{i1}K_{i1}(t) = 0 \text{ for } t \in (-\infty, 0].$$

By Proposition 5.5, we obtain

$$\begin{aligned} &\sum_{i=1}^{n} E \int_{-\infty}^{0} I(A_{i1})K_{i1}(t)dt \\ &= \sum_{i=1}^{n} E \int_{-\infty}^{0} E^{Y_{i1}, W_{n2}} I(A_{i1})K_{i1}(t)dt \\ &= \sum_{i=1}^{n} E \int_{-\infty}^{0} P(A_{i1} \mid Y_{i1}, W_{n2})K_{i1}(t)dt \\ &\leq \sum_{i=1}^{n} E \int_{-\infty}^{0} \left[\sqrt{2}(|Y_{i1} - t|) + (1 + \sqrt{2}) \sum_{m=1}^{n} E|Y_{m1}|^3 \right] K_{i1}(t)dt. \end{aligned} \tag{5.11}$$

Similarly, we have

$$\begin{aligned}
&\sum_{i=1}^{n} E\int_{0}^{\infty} I(B_{i1})K_{i1}(t)dt \\
&\leq \sum_{i=1}^{n} E\int_{0}^{\infty} \left[\sqrt{2}(|Y_{i1}-t|)+(1+\sqrt{2})\sum_{m=1}^{n} E|Y_{m1}|^3\right] K_{i1}(t)dt
\end{aligned} \tag{5.12}$$

By (5.4)–(5.5) and (5.10)–(5.12), we obtain that

$$\begin{aligned}
R_1 &\leq \sum_{i=1}^{n} E\int_{-\infty}^{\infty} \sqrt{2}(|Y_{i1}|+|t|)K_{i1}(t)dt \\
&\quad + (1+\sqrt{2})\sum_{m=1}^{n} E|Y_{m1}|^3 \sum_{i=1}^{n} E\int_{-\infty}^{\infty} K_{i1}(t)dt \\
&\leq \frac{3\sqrt{2}}{2}\sum_{i=1}^{n} E|Y_{i1}|^3 + (1+\sqrt{2})\sum_{i=1}^{n} E|Y_{i1}|^3 \\
&\leq 4.55\sum_{i=1}^{n} E|Y_{i1}|^3.
\end{aligned} \tag{5.13}$$

In order to prove

$$|R_1| \leq 4.55\sum_{i=1}^{n} E|Y_{i1}|^3, \tag{5.14}$$

it remains to show that

$$R_1 \geq -4.55\sum_{i=1}^{n} E|Y_{i1}|^3. \tag{5.15}$$

This inequality holds when we follow an argument as (5.13) and use the relation that

$$\left\{w\in\Omega \mid h_{B_2(r)}(W_{n1}^{(i)}+Y_{i1}, W_{n2})(w) - h_{B_2(r)}(W_{n1}^{(i)}+t, W_{n2})(w) = -1\right\} \subseteq C_{i1}\cup D_{i1} \tag{5.16}$$

where

$$\begin{aligned}
C_{i1} &= \left\{w\in\Omega \mid -Y_{i1}(w)+\alpha(w) < W_{n1}^{(i)}(w) \leq -t+\alpha(w)\right\} \text{ and} \\
D_{i1} &= \left\{w\in\Omega \mid -t-\alpha(w) \leq W_{n1}^{(i)}(w) < -Y_{i1}(w)-\alpha(w)\right\}.
\end{aligned}$$

This relation is proved by using the same argument as (5.9). The relation (5.16) implies that

$$h_{B_2(r)}(W_{n1}^{(i)}+Y_{i1}, W_{n2}) - h_{B_2(r)}(W_{n1}^{(i)} + t, W_{n2}) \geq -I(C_{i1} \cup D_{i1}) \text{ and then}$$
$$R_1 \geq -\sum_{i=1}^{n} E\int_{-\infty}^{\infty}(I(C_{i1}) + I(D_{i1})K_{i1}(t)dt.$$

By the same argument as (5.13), we have (5.15) and hence (5.14).

Next, we estimate the bound R_2. Since g_1 in Proposition 3.6 is piecewise continuous, by Proposition 3.6, (5.3),(5.5) and the fundamental theorem of calculus, we have

$$\begin{aligned}|R_2| = \Bigg|\sum_{i=1}^{n} E\int_{-\infty}^{\infty}\Big[&(W_{n1}^{(i)} + Y_{i1} + W_{n2})f_r(W_{n1}^{(i)} + Y_{i1}, W_{n2})\\ &- (W_{n1}^{(i)} + t + W_{n2})f_r(W_{n1}^{(i)} + t, W_{n2})\Big]K_{i1}(t)dt\Bigg|\\ = \Bigg|\sum_{i=1}^{n} E\int_{-\infty}^{\infty}\int_{t}^{Y_{i1}} &g_1(W_{n1}^{(i)} + u, W_{n2})duK_{i1}(t)dt\Bigg|\\ \leq 2\sum_{i=1}^{n} E\int_{-\infty}^{\infty}&(|Y_{i1}| + |t|)K_{i1}(t)dt\\ \leq 3\sum_{i=1}^{n} E|Y_{i1}|^3. &\qquad (5.17)\end{aligned}$$

Combining (5.8), (5.14) and (5.17) yields

$$|S_1 - T_1| \leq \frac{1}{\sqrt{2}}|R_1| + \frac{1}{2}|R_2| \leq 4.72\sum_{i=1}^{n} E|Y_{i1}|^3. \qquad (5.18)$$

Similarly, we obtain that

$$|S_2 - T_2| \leq 4.72\sum_{i=1}^{n} E|Y_{i2}|^3. \qquad (5.19)$$

Hence, by (5.2), (5.18)–(5.19), theorem 5.1 is proved in case of $k = 2$. For multidimensional case, we use the same argument as in the case that $k = 2$. The results on multidimensional case are as follow:

$$\begin{aligned}P(W_n \in B_k(r)) - \Phi_k(B_k(r)) &= \frac{1}{\sqrt{k}}\sum_{m=1}^{k}[S_m - T_m]\\ &= \frac{1}{k}\left[\sum_{m=1}^{k} R_{m1} + \frac{1}{\sqrt{k}}\sum_{m=1}^{k} R_{m2}\right] \qquad (5.20)\end{aligned}$$

where

$$\begin{aligned}
S_m &= Ef_{r_{w_m}}(W_{n1}, W_{n2}, \ldots, W_{nk}),\\
T_m &= EW_{nm}f_r(_{n1}, W_{n2}, \ldots, W_{nk}),\\
R_{m1} &= \sum_{i=1}^{n} E\int_{-\infty}^{\infty}\Big[h_{B_k(r)}(W_{n1}, W_{n2}, \ldots, W_{nm}^{(i)} + Y_{im}, \ldots, W_{nk})\\
&\qquad - h_{B_2(r)}(W_{n1}, W_{n2}, \ldots, W_{nm}^{(i)} + t, \ldots, W_{nk})\Big]K_{im}(t)dt\\
R_{m2} &= \sum_{i=1}^{n} E\int_{-\infty}^{\infty}\Big[(\sum_{\substack{l=1\\ l\neq m}}^{k} W_{nl} + (W_{nm}^{(i)} + Y_{im}))f_r(W_{n1}, W_{n2}, \ldots, W_{nm}^{(i)} + Y_{im}, \ldots, W_{nk})\\
&\qquad - (\sum_{\substack{l=1\\ l\neq m}}^{k} W_{nl} + (W_{nm}^{(i)} + t))f_r(W_{n1}, W_{n2}, \ldots, W_{nm}^{(i)} + t, \ldots, W_{nk})\Big]K_{im}(t)dt.
\end{aligned}$$

For $m = 1, 2, \ldots, k$, we follow the argument as in (5.14) and (5.17) and then

$$|R_{m1}| \leq 4.55\sum_{i=1}^{n} E|Y_{im}|^3 \quad \text{and} \quad |R_{m2}| \leq 3\sum_{i=1}^{n} E|Y_{im}|^3. \tag{5.21}$$

Combining (5.20)–(5.21), we obtain that

$$|P(W_n \in B_k(r)) - \Phi_k(B_k(r))| \leq \left(\frac{4.55}{\sqrt{k}} + \frac{3}{k\sqrt{k}}\right)\beta_3$$

Hence, the theorem for multidimensional case is proved. □

Remark 5.6. *In case of* $k = 2$, $P(W_n \notin B_2(r))$ *converges weakly to* $e^{-\frac{r^2}{2}}$ *for all* $r > 0$.

The convergence holds due to the relation

$$F_n(B_2(r)) - \Phi_2(B_2(r)) = P(W_n \notin (B_2(r)) - (1 - \Phi_2(B_2(r)))$$

and equation (3.14) in Chapter III, i.e.

$$\Phi_2(B_2(r)) = 1 - e^{-\frac{r^2}{2}}.$$

Proof of Theorem 5.2.

Proof. We follow the argument of Theorem 5.1 by using the relations that

$\left\{w \in \Omega \mid h_{B_{i1}(r)}(W_{n1}^{(i)} + Y_{i1}, W_{n2})(w) - h_{B_{i1}(r)}(W_{n1}^{(i)} + t, W_{n2})(w) = 1\right\} \subseteq E_{i1}$ and

$\left\{w \in \Omega \mid h_{B_{i1}(r)}(W_{n1}^{(i)} + Y_{i1}, W_{n2})(w) - h_{B_{i1}(r)}(W_{n1}^{(i)} + t, W_{n2})(w) = -1\right\} \subseteq F_{i1}$

where
$$E_{i1} = \left\{w \in \Omega \mid r - W_{n2}^{(i)}(w) - t < W_{n1}^{(i)}(w) \le r - W_{n2}^{(i)}(w) - Y_{i1}(w)\right\},$$
$$F_{i1} = \left\{w \in \Omega \mid r - W_{n2}^{(i)}(w) - Y_{i1}(w) < W_{n1}^{(i)}(w) \le r - W_{n2}^{(i)}(w) - t\right\}.$$

The estimations are
$$|R_1| \le 4.55 \sum_{i=1}^{n} E|Y_{i1}|^3 \text{ and } |R_2| \le 3 \sum_{i=1}^{n} E|Y_{i1}|^3.$$

Hence, the theorem is proved for $k = 2$. For multidimensional case, we use the same technique as in (5.21) and then have
$$|R_{m1}| \le 4.55 \sum_{i=1}^{n} E|Y_{im}|^3 \quad \text{and} \quad |R_{m2}| \le 3 \sum_{i=1}^{n} E|Y_{im}|^3$$

Hence, Theorem 5.2 is proved. □

Proof of Corollary 5.3.

Proof. Corollary 5.3 is immediately obtained from Theorem 5.2. □

Proof of Theorem 5.4.

Proof. We use the same idea as in Theorem 5.2 with the relations that

$\left\{w \in \Omega \mid h_{R_2(r)}(W_{n1}^{(i)} + Y_{i1}, W_{n2})(w) - h_{R_2(r)}(W_{n1}^{(i)} + t, W_{n2})(w) = 1\right\} \subseteq G_{i1} \cup H_{i1}$ and

$\left\{w \in \Omega \mid h_{R_2(r)}(W_{n1}^{(i)} + Y_{i1}, W_{n2})(w) - h_{R_2(r)}(W_{n1}^{(i)} + t, W_{n2})(w) = -1\right\} \subseteq I_{i1} \cup J_{i1}$

where

$$G_{i1} = \left\{w \in \Omega \mid r_1 - t < W_{n1}^{(i)}(w) \le r_1 - Y_{i1}(w)\right\},$$
$$H_{i1} = \left\{w \in \Omega \mid -r_1 - Y_{i1}(w) \le W_{n1}^{(i)}(w) < -r_1 - t\right\},$$
$$I_{i1} = \left\{w \in \Omega \mid r_1 - Y_{i1}(w) < W_{n1}^{(i)}(w) \le r_1 - t\right\}, \text{ and}$$
$$J_{i1} = \left\{w \in \Omega \mid -r_1 - t \le W_{n1}^{(i)}(w) < -r_1 - Y_{i1}(w)\right\}.$$

□

CHAPTER VI

NON-UNIFORM BERRY-ESSEEN BOUND ON THE CLOSED SPHERE IN $\mathbb{R}^k$

In this chapter, we adopt the same notations as in chapter V.

In 1967, Bahr [1] obtained a non-uniform bound on multivariate normal approximation for multidimensional Berry-Esseen theorem. He gave a bound of the estimation on the closed sphere

$$B_k(r) = \{x \in \mathbb{R}^k \mid x_1^2 + x_2^2 + \cdots + x_k^2 \leq r^2\}$$

for some positive real numbers r depending on n. The result is obtained under the assumption that $Y_i's$ are identically distributed and the s^{th} moments is finite,

$$E\left(\sum_{j=1}^{k} Y_{ij}^2\right)^{\frac{s}{2}} < \infty,$$

for an integer $s \geq 3$ and $i = 1, 2, \ldots, n$. The result is stated as follows:

Theorem 6.1. *Let M be a covariance matrix of $\sqrt{n}Y_i$ for $i = 1, 2, \ldots, n$. If the s^{th} moments of Y_i are finite for an integer $s \geq 3$, then there exists a positive constant C_k (depends on k) such that*

$$|F_n(B_k(r)) - \Phi_k(B_k(r))| \leq \frac{C_k \cdot d(n)}{r^s n^{\frac{s-2}{2}}} \quad \text{for} \quad r \geq \left(\frac{5}{4}m(s-2)\log n\right)^{\frac{1}{2}},$$

where m is the largest eigenvalue of the covariance matrix M, $d(n)$ is a function bounded by 1 and $\lim_{n\to\infty} d(n) = 0$.

For $r < \left(\frac{5}{4}m(s-2)\log n\right)^{\frac{1}{2}}$, Bahr gave a bound of the estimation when the limit distribution is the chi-square $\chi^2(k)$ with degree of freedom k. We state here the result.

Theorem 6.2. *If the forth moments of $Y_i's$ are finite for $i = 1, 2, \ldots, n$ and the covariance matrix M is the identity matrix, then there exists a positive constant C_k (depends on k) such that*

$$|F_n(B_k(r)) - \chi^2(k)(r^2)| \leq \frac{C_k(1 + r^{k+2})}{e^{\alpha r^2} n^{\frac{k}{k+1}}} + O\left(\frac{(\log n)^{\frac{k-1}{4}}}{n}\right) \quad \textit{for} \ \ r < \left(\frac{5}{2}\log n\right)^{\frac{1}{2}},$$

where $\alpha = \dfrac{1}{8}$ *if* $k = 2$ *and* $\alpha = \dfrac{k}{2(k+1)}$ *if* $k \geq 3$.

In this chapter, we will give a non-uniform bound of this convergence without assumming that $Y_i's$ are identically distributed. We assume that all components of $Y_i's$ are independent and

$$\sum_{j=1}^{k} E|Y_{ij}|^3 < \infty,$$

for $i = 1, 2, \ldots, n$. The following theorem is the main result.

Theorem 6.3. *Let $Y_i = (Y_{i1}, Y_{i2}, \ldots, Y_{ik}), i = 1, 2, \ldots, n$ be independent random vectors in $\mathbb{R}^k$ with zero means and Y_{ij} are independent for all $j = 1, 2, \ldots, k$. Define $W_n = \sum_{i=1}^{n} Y_i$. Let F_n be the distribution function of W_n. Assume that $\sum_{i=1}^{n} EY_{ij}^2 = 1$ for $j = 1, 2, \ldots, k$ and $\sum_{j=1}^{k} E|Y_{ij}|^3 < \infty$ for $i = 1, 2, \ldots, n$. Then there exists a positive constant C_k (depends on k) such that*

$$|F_n(B_k(r)) - \Phi_k(B_k(r))| \leq \frac{C_k \beta_3}{1 + r^3}$$

for all positive real numbers r, where $\beta_3 = \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^3$.

The order of convergence in the statement of Theorem 6.3 is $O(n^{-\frac{1}{2}})$ which is better than that in Theorem 6.2 and its result is obtained for all positive real numbers r which is broader than the result in Theorem 6.1.

The contents in this chapter are organized into two sections. The first section, Auxiliary Results, contains propositions which is used to prove our result. In the latter section, Proof of the Main Result, gives a proof of the result.

6.1 Auxiliary Results

In this section, two propositions required in the proof of main theorem is presented. Proposition 6.4 gives the inequalities of the truncated random vectors while Proposition 6.5 gives an effective tool, non-uniform concentration inequality, for proving our main result.

Apart from the notations given in chapter V, we further introduce the following notations. For $i = 1, 2, ..., n$, $j = 1, 2, ..., k$, $u \in \mathbb{R}$ and $r > 0$, let

$$\overline{Y}_{ij} = Y_{ij} I(|Y_{ij}| \le 1 + \frac{r}{4}), \quad \overline{W}_{nj} = \sum_{i=1}^{n} \overline{Y}_{ij},$$

$$\overline{W}_{nj}^{(i)} = \overline{W}_{nj} - \overline{Y}_{ij}, \quad \overline{W}_{n} = (\overline{W}_{n1}, \overline{W}_{n2}, \ldots, \overline{W}_{nk}) \text{ and}$$

$$\overline{W}_{nj,u}^{(i)} = (\overline{W}_{n1}, \overline{W}_{n2}, \ldots, \overline{W}_{nj}^{(i)} + u, \ldots, \overline{W}_{nk}).$$

Proposition 6.4. *Let* $\beta_3 = \sum_{j=1}^{k} \sum_{i=1}^{n} E|Y_{ij}|^3$. *Then*

(1) $\sum_{j=1}^{k} E\overline{W}_{nj}^4 \le \left(2 + \frac{r}{4}\right) \beta_3 + 3k.$

(2) Let g_j *be defined as in Proposition 3.6 for* $j = 1, 2, \ldots, k$. *Then, there exists an absolute constant* C_k *(depends on* k*) such that*

$$E|g_j(\overline{W}_{nj,u}^{(i)})| \le \frac{C_k}{1 + r^4}$$

for $r \ge 4$, $|u| \le \frac{r}{4}$ *and* $(1 + r)\beta_3 < 1$.

Proof. 1) By Proposition 2.1 in [27], we have

$$E\overline{W}_{nj}^4 \le \left(1 + \frac{r}{4}\right) \gamma_j + 1 + \frac{\eta_j \gamma_j}{1 + \frac{r}{4}} + \left(\frac{\eta_j}{1 + \frac{r}{4}}\right)^2 + \left(\frac{\eta_j}{1 + \frac{r}{4}}\right)^4 \tag{6.1}$$

where

$$\eta_j = \sum_{i=1}^{n} EY_{ij}^2 I\left(|Y_{ij}| \ge 1 + \frac{r}{4}\right), \text{ and } \gamma_j = \sum_{i=1}^{n} E|Y_{ij}|^3 I\left(|Y_{ij}| < 1 + \frac{r}{4}\right).$$

By the inequalities:

$$\eta_j \le \sum_{i=1}^{n} EY_{ij}^2 = 1 \text{ and } \gamma_j \le \sum_{i=1}^{n} E|Y_{ij}|^3,$$

(6.1) becomes

$$E\overline{W}_{nj}^4 \le \left(1+\frac{r}{4}\right)\sum_{i=1}^{n} E|Y_{ij}|^3 + \sum_{i=1}^{n} E|Y_{ij}|^3 + 3$$
$$\le \left(2+\frac{r}{4}\right)\sum_{i=1}^{n} E|Y_{ij}|^3 + 3. \tag{6.2}$$

Thus,

$$\sum_{j=1}^{k} E\overline{W}_{nj}^4 \le \left(2+\frac{r}{4}\right)\sum_{j=1}^{k}\sum_{i=1}^{n} E|\overline{Y}_{ij}|^3 + 3k$$
$$\le \left(2+\frac{r}{4}\right)\beta_3 + 3k.$$

Hence, (1) is proved.

2) By (1) and the assumption that $(1+r)\beta_3 < 1$, we have

$$\sum_{j=1}^{k} E\overline{W}_{nj}^4 \le C_k \tag{6.3}$$

for some positive constant C_k. From this inequality and Proposition 3.6(2), we obtain (2) and hence the proposition. □

Proposition 6.5 is a non-uniform concentration inequality which is the essential inequality for this approach. We prove this proposition by applying the concentration inequality in [10].

Proposition 6.5. *For $j = 1, 2, \ldots, k$ and $m = 1, 2, \ldots, n$, let*

$$T_{nj}^{(m)} = \sum_{\substack{i=1 \\ i\neq m}}^{n} \frac{\overline{Y}_{ij} - E\overline{Y}_{ij}}{\sqrt{Var(\overline{W}_{nj})}}.$$

Then there exists an absolute constant C such that

$$P(a \le T_{nj}^{(m)} \le b) \le \frac{C}{(1+a)^3}\{b - a + \beta_{j,3}\}$$

for all reals $0 \le a < b < \infty$ where

$$\beta_{j,3} = \frac{1}{\left(\sqrt{Var(\overline{W}_{nj})}\right)^3}\sum_{i=1}^{n} |\overline{Y}_{ij} - E\overline{Y}_{ij}|^3.$$

Proof. For $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, k$, let

$$\overline{X}_{ij} = \frac{\overline{Y}_{ij} - E\overline{Y}_{ij}}{\sqrt{Var(\overline{W}_{nj})}}.$$

By Proposition 3.4 in [10], we obtain that for $m = 1, 2, \ldots, n$,

$$P(a \leq T_{nj}^{(m)} \leq b) \leq C\left\{\frac{b-a}{(1+a)^3} + \delta_{j,a}\right\} \tag{6.4}$$

where

$$\delta_{j,a} = \sum_{i=1}^{n}\left\{\frac{E\overline{X}_{ij}^2 I(|\overline{X}_{ij}| > 1+a)}{(1+a)^2} + \frac{E|\overline{X}_{ij}|^3 I(|\overline{X}_{ij}| \leq 1+a)}{(1+a)^3}\right\}.$$

The proof is completed by (6.4) and the inequality

$$\begin{aligned}
&\sum_{i=1}^{n}\left\{\frac{E\overline{X}_{ij}^2 I(|\overline{X}_{ij}| > 1+a)}{(1+a)^2} + \frac{E|\overline{X}_{ij}|^3 I(|\overline{X}_{ij}| \leq 1+a)}{(1+a)^3}\right\} \\
&\leq \sum_{i=1}^{n}\left\{\frac{E|\overline{X}_{ij}|^3 I(|\overline{X}_{ij}| > 1+a)}{(1+a)^3} + \frac{E|\overline{X}_{ij}|^3 I(|\overline{X}_{ij}| \leq 1+a)}{(1+a)^3}\right\} \\
&= \frac{1}{(1+a)^3}\sum_{i=1}^{n} E|\overline{X}_{ij}|^3 \\
&= \frac{\beta_{j,3}}{(1+a)^3}.
\end{aligned}$$

□

6.2 Proof of Main Result

In this section, we will give a non-uniform bound of multivariate normal approximation on the set of closed sphere $B_k(r)$. The used technique is the concentration inequality approach. This proof is based on an idea of [12]. The positive constant C in the proof has different values in different places.

Proof of Theorem 6.1

Proof. If $r < 4$ then by Theorem 5.1, we have

$$|F_n(B_k(r)) - \Phi_k(B_k(r))| \leq \frac{C_k\beta_3}{1+r^3}$$

for some positive constant C_k. Next, assume that $r \geq 4$. We observe that

$$|P(W_n \in B_k(r)) - \Phi_k(B_k(r))| \leq |P(W_n \in B_k(r)) - P(\overline{W}_n \in B_k(r))| + |P(\overline{W}_n \in B_k(r)) - \Phi_k(B_k(r))|. \tag{6.5}$$

Firstly, we will find a bound of the first term on the right side of (6.5). Note that

$$\begin{aligned}
&P(W_n \in B_k(r)) - P(\overline{W}_n \in B_k(r)) \\
&= P(W_n \in B_k(r), W_n = \overline{W}_n) + P(W_n \in B_k(r), W_n \neq \overline{W}_n) \\
&\quad - P(\overline{W}_n \in B_k(r)) \\
&\leq P(W_n \neq \overline{W}_n)
\end{aligned}$$

and

$$\begin{aligned}
P(W_n \in B_k(r)) - P(\overline{W}_n \in B_k(r)) \\
&= P(W_n \in B_k(r)) - P(\overline{W}_n \in B_k(r), W_n = \overline{W}_n) \\
&\quad - P(\overline{W}_n \in B_k(r), W_n \neq \overline{W}_n) \\
&\geq -P(W_n \neq \overline{W}_n).
\end{aligned}$$

We can conclude from these two inequalities that

$$|P(W_n \in B_k(r)) - P(\overline{W}_n \in B_k(r))| \leq P(W_n \neq \overline{W}_n) \tag{6.6}$$

Note that

$$W_n = \overline{W}_n \quad \text{if} \quad \max_{\substack{1\leq i\leq n \\ 1\leq j\leq k}} |Y_{ij}| \leq 1 + \frac{r}{4}.$$

Then,

$$\begin{aligned}
P(W_n \neq \overline{W}_n) &\leq P\left(\max_{\substack{1\leq i\leq n \\ 1\leq j\leq k}} |Y_{ij}| > 1 + \frac{r}{4}\right) \\
&\leq \sum_{j=1}^{k}\sum_{i=1}^{n} P\left(|Y_{ij}| > 1 + \frac{r}{4}\right) \\
&\leq \frac{1}{\left(1+\frac{r}{4}\right)^3} \sum_{j=1}^{k}\sum_{i=1}^{n} E|Y_{ij}|^3
\end{aligned}$$

$$\leq \frac{C}{(1+r)^3} \sum_{j=1}^{k} \sum_{i=1}^{n} E|Y_{ij}|^3$$
$$= \frac{C\beta_3}{1+r^3} \tag{6.7}$$

where Chebyshev's inequality is used in the third inequality. Therefore, by (6.5)–(6.7),

$$|P(W_n \in B_k(r)) - \Phi_k(B_k(r))| \leq \frac{C\beta_3}{1+r^3} + |P(\overline{W}_n \in B_k(r)) - \Phi_k(B_k(r))|. \tag{6.8}$$

To prove our theorem, it remains to estimate the second term of (6.8).

If $(1+r)\beta_3 \geq 1$, by (3.30) and (6.3), we have

$$\begin{aligned}
&|P(\overline{W}_n \in B_k(r)) - \Phi_k(B_k(r))| \\
&\leq P(\overline{W}_n \notin B_k(r)) + (1 - \Phi_k(B_k(r)) \\
&= P\left(\sum_{j=1}^{k} \overline{W}_{nj}^2 > r^2\right) + (1 - \Phi_k(B_k(r)) \\
&\leq \frac{1}{r^4} E\left(\sum_{j=1}^{k} \overline{W}_{nj}^2\right)^2 + \frac{C_k}{1+r^6} \\
&\leq \frac{kC_k}{1+r^4} \sum_{j=1}^{k} E\overline{W}_{nj}^4 + \frac{C_k(1+r)\beta_3}{1+r^6} \\
&\leq \frac{C_k(1+r)\beta_3}{1+r^4} + \frac{C_k\beta_3}{1+r^5} \\
&\leq \frac{C_k\beta_3}{1+r^3}.
\end{aligned}$$

Next, assume that $(1+r)\beta_3 < 1$. In this case, we will prove the theorem in case that $k = 2$. For multidimensional case, we use the same argument. Let f_r be the solution of (3.3) with respect to the indicator test function on $B_2(r)$ and $f_{r_{w_1}}, f_{r_{w_2}}$ partial derivatives of f_r with respect to w_1 and w_2, respectively. Thus, by (3.3),

$$P(\overline{W}_n \in B_2(r)) - \Phi_2(B_2(r)) = \frac{1}{\sqrt{2}}(U_1 - V_1) + \frac{1}{\sqrt{2}}(U_2 - V_2) \tag{6.9}$$

where

$$U_1 = Ef_{r_{w_1}}(\overline{W}_{n1}, \overline{W}_{n2}), \qquad V_1 = E\overline{W}_{n1} f(\overline{W}_{n1}, \overline{W}_{n2}),$$
$$U_2 = Ef_{r_{w_2}}(\overline{W}_{n1}, \overline{W}_{n2}), \qquad V_2 = E\overline{W}_{n2} f(\overline{W}_{n1}, \overline{W}_{n2}).$$

To estimate the right handside of (6.9), let

$$M_{ij}(t) = E\overline{Y}_{ij}[I(0 \le t \le \overline{Y}_{ij}) - I(\overline{Y}_{ij} \le t < 0)].$$

for $t \in \mathbb{R}$, $i = 1, 2, \ldots, n$,$j = 1, 2$ where I is the indicator function on Ω. We can follow the idea from [12] to show that

$$M_{ij}(t) \ge 0 \quad \text{for all } t \in \mathbb{R}, \tag{6.10}$$

$$\sum_{i=1}^{n} E\int_{-\infty}^{\infty} M_{ij}(t)dt = \sum_{i=1}^{n} E\overline{Y}_{ij}^{2} = 1 - \sum_{i=1}^{n} EY_{ij}^{2} I(|Y_{ij}| > 1 + \frac{r}{4}) \le 1, \tag{6.11}$$

$$\sum_{i=1}^{n} E\int_{-\infty}^{\infty} |t| M_{ij}(t)dt = \frac{1}{2}\sum_{i=1}^{n} E|\overline{Y}_{ij}|^{3}, \tag{6.12}$$

$$\sum_{i=1}^{n} E\int_{-\infty}^{\infty} |\overline{Y}_{ij}| M_{ij}(t)dt = \sum_{i=1}^{n} E|\overline{Y}_{ij}|^{3} \quad \text{and} \tag{6.13}$$

$$V_1 = \sum_{i=1}^{n} E\int_{-\infty}^{\infty} f_{r_{w_1}}(\overline{W}_{1n}^{(i)} + t, \overline{W}_{2n}) M_{i1}(t)dt + \sum_{i=1}^{n} E\overline{Y}_{i1} f(\overline{W}_{1n}^{(i)}, \overline{W}_{2n}). \tag{6.14}$$

Thus, by (6.11) and (6.14), we have

$$\begin{aligned}
U_1 - V_1 &= Ef_{r_{w_1}}(\overline{W}_{n1}, \overline{W}_{n2}) - \sum_{i=1}^{n} E\int_{-\infty}^{\infty} f_{r_{w_1}}(\overline{W}_{n1}^{(i)} + t, \overline{W}_{n2}) M_{i1}(t)dt \\
&\quad - \sum_{i=1}^{n} E\overline{Y}_{i1} f(\overline{W}_{n1}^{(i)}, \overline{W}_{n2}) \\
&= E(f_{r_{w_1}}(\overline{W}_{n1}, \overline{W}_{n2}) \\
&\quad + \sum_{i=1}^{n} E\int_{-\infty}^{\infty} [f_{r_{w_1}}(\overline{W}_{n1}^{(i)} + \overline{Y}_{i1}, \overline{W}_{n2}) - f_{r_{w_1}}(\overline{W}_{n1}^{(i)} + t, \overline{W}_{n2})] M_{i1}(t)dt \\
&\quad - Ef_{r_{w_1}}(\overline{W}_{n1}^{(i)} + \overline{Y}_{i1}, \overline{W}_{n2})[1 - \sum_{i=1}^{n} EY_{i1}^{2} I\left(|Y_{i1}| > 1 + \frac{r}{4}\right)] \\
&\quad - \sum_{i=1}^{n} E\overline{Y}_{i1} f(\overline{W}_{n1}^{(i)}, \overline{W}_{n2}) \\
&= \sum_{i=1}^{n} E\int_{-\infty}^{\infty} [f_{r_{w_1}}(\overline{W}_{n1}^{(i)} + \overline{Y}_{i1}, \overline{W}_{n2}) - f_{r_{w_1}}(\overline{W}_{n1}^{(i)} + t, \overline{W}_{n2})] M_{i1}(t)dt \\
&\quad + Ef_{r_{w_1}}(\overline{W}_{n1}^{(i)} + \overline{Y}_{i1}, \overline{W}_{n2}) \sum_{i=1}^{n} EY_{i1}^{2} I\left(|Y_{i1}| > 1 + \frac{r}{4}\right) \\
&\quad - \sum_{i=1}^{n} E\overline{Y}_{i1} f(\overline{W}_{n1}^{(i)}, \overline{W}_{n2}) \\
&=: R_1 + R_2 + R_3,
\end{aligned}$$

where

$$R_1 = \sum_{i=1}^{n} E \int_{-\infty}^{\infty} [f_{r_{w_1}}(\overline{W}_{n1}^{(i)} + \overline{Y}_{i1}, \overline{W}_{n2}) - f_{r_{w_1}}(\overline{W}_{n1}^{(i)} + t, \overline{W}_{n2})] M_{i1}(t) dt,$$

$$R_2 = E f_{r_{w_1}}(\overline{W}_{n1}^{(i)} + \overline{Y}_{i1}, \overline{W}_{n2}) \sum_{i=1}^{n} E Y_{i1}^2 I\left(|Y_{i1}| > 1 + \frac{r}{4}\right),$$

$$R_3 = -\sum_{i=1}^{n} E\overline{Y}_{i1} f(\overline{W}_{n1}^{(i)}, \overline{W}_{n2}).$$

By Proposition 3.5(2), we get

$$\begin{aligned} |R_2| &\le \frac{C}{1+r^2} \sum_{i=1}^{n} E Y_{i1}^2 I\left(|Y_{i1}| > 1 + \frac{r}{4}\right) \\ &\le \frac{C}{1+r^3} \sum_{i=1}^{n} E|Y_{i1}|^3 I\left(|Y_{i1}| > 1 + \frac{r}{4}\right) \\ &\le \frac{C}{1+r^3} \sum_{i=1}^{n} E|Y_{i1}|^3. \end{aligned} \tag{6.15}$$

Similarly, by the independence of $\overline{Y}_{i1}, \overline{W}_{n1}^{(i)}$ and $\overline{W}_{n2}$, Proposition 3.5(1) and

$$0 = EY_{i1} = EY_{i1} I\left(Y_{i1} \le 1 + \frac{r}{4}\right) + EY_{i1} I\left(Y_{i1} > 1 + \frac{r}{4}\right), \tag{6.16}$$

we obtain

$$\begin{aligned} |R_3| &\le \frac{C}{1+r^2} \sum_{i=1}^{n} |E\overline{Y}_{i1}| \\ &\le \frac{C}{1+r^2} \sum_{i=1}^{n} E|Y_{i1}| I\left(Y_{i1} > 1 + \frac{r}{4}\right) \\ &\le \frac{C}{1+r^3} \sum_{i=1}^{n} E|Y_{i1}|^3. \end{aligned} \tag{6.17}$$

Next, we will find a bound of R_1. By (3.6), R_1 can be written as

$$R_1 = \frac{1}{\sqrt{2}} R_{11} + \frac{1}{2} R_{12} \tag{6.18}$$

where

$$R_{11} = \sum_{i=1}^{n} E \int_{-\infty}^{\infty} [h_{B_2(r)}(\overline{W}_{n1}^{(i)} + \overline{Y}_{i1}, \overline{W}_{n2}) - h_{B_2(r)}(\overline{W}_{n1}^{(i)} + t, \overline{W}_{n2})] M_{i1}(t) dt,$$

$$\begin{aligned} R_{12} = \sum_{i=1}^{n} E \int_{-\infty}^{\infty} &[(\overline{W}_{n1}^{(i)} + \overline{Y}_{i1} + \overline{W}_{n2}) f(\overline{W}_{n1}^{(i)} + \overline{Y}_{i1}, \overline{W}_{n2}) \\ &- (\overline{W}_{n1}^{(i)} + t + \overline{W}_{n2}) f(\overline{W}_{n1}^{(i)} + t, \overline{W}_{n2})] M_{i1}(t) dt. \end{aligned}$$

For $i = 1, 2, \ldots, n$, let $T_{n1}^{(i)}$ be defined as in Proposition 6.5,

$$A_{i1} = \left\{ w \in \Omega \mid \frac{-t + \alpha_i^-(w)}{\sqrt{Var(\overline{W}_{n1})}} < T_{n1}^{(i)}(w) \le \frac{-\overline{Y}_{i1}(w) + \alpha_i^-(w)}{\sqrt{Var(\overline{W}_{n1})}} \right\},$$

$$B_{i1} = \left\{ w \in \Omega \mid \frac{-\overline{Y}_{i1}(w) - \alpha_i^+(w)}{\sqrt{Var(\overline{W}_{n1})}} < T_{n1}^{(i)}(w) \le \frac{-t - \alpha_i^+(w)}{\sqrt{Var(\overline{W}_{n1})}} \right\},$$

$\alpha_i^+(w) = \sqrt{r^2 - \overline{W}_{n2}^2(w) I(w \in \Lambda)} + E\overline{W}_{n1}^{(i)}$ and

$\alpha_i^-(w) = \sqrt{r^2 - \overline{W}_{n2}^2(w) I(w \in \Lambda)} - E\overline{W}_{n1}^{(i)}$ where $\Lambda = \left\{ w \in \Omega \mid \overline{W}_{n2}^2(w) \le r^2 \right\}$.

Obviously, $A_{i1} \cap B_{i1} = \varnothing$. By the same argument as (5.9), we obtain the relation

$$\left\{ w \in \Omega \mid h_{B_2(r)}(\overline{W}_{n1}^{(i)} + \overline{Y}_{i1}, \overline{W}_{n2})(w) - h_{B_2(r)}(\overline{W}_{n1}^{(i)} + t, \overline{W}_{n2})(w) = 1 \right\} \subseteq A_{i1} \cup B_{i1}. \tag{6.19}$$

Thus, by (6.10),

$$\begin{aligned} R_{11} &\le \sum_{i=1}^{n} E \int_{-\infty}^{\infty} I(A_{i1} \cup B_{i1}) M_{i1}(t) dt \\ &= \sum_{i=1}^{n} E \int_{-\infty}^{\infty} I(A_{i1} \cup B_{i1}) I\left(\overline{W}_{n2}^2(w) \le \frac{r^2}{4} \right) M_{i1}(t) dt \\ &\quad + \sum_{i=1}^{n} E \int_{-\infty}^{\infty} I(A_{i1} \cup B_{i1}) I\left(\overline{W}_{n2}^2(w) > \frac{r^2}{4} \right) M_{i1}(t) dt. \end{aligned} \tag{6.20}$$

We will find a bound of the first term of (6.20) by using the non-uniform concentration inequality in Proposition 6.5. Note that

$$\begin{aligned} &\sum_{i=1}^{n} E \int_{-\infty}^{\infty} I(A_{i1} \cup B_{i1}) I\left(\overline{W}_{n2}^2(w) \le \frac{r^2}{4} \right) M_{i1}(t) dt \\ &\le \sum_{i=1}^{n} E \int_{-\infty}^{\infty} I(A_{i1}) I\left(\overline{W}_{n2}^2(w) \le \frac{r^2}{4} \right) M_{i1}(t) dt \\ &\quad + \sum_{i=1}^{n} E \int_{-\infty}^{\infty} I(B_{i1}) I\left(\overline{W}_{n2}^2(w) \le \frac{r^2}{4} \right) M_{i1}(t) dt \\ &\le \sum_{i=1}^{n} E \int_{-\infty}^{0} I(A_{i1}) I\left(\overline{W}_{n2}^2(w) \le \frac{r^2}{4} \right) M_{i1}(t) dt \\ &\quad + \sum_{i=1}^{n} E \int_{0}^{\infty} I(B_{i1}) I\left(\overline{W}_{n2}^2(w) \le \frac{r^2}{4} \right) M_{i1}(t) dt \end{aligned} \tag{6.21}$$

where we used the fact that

$$I(A_{i1})M_{i1}(t) = 0 \text{ for } t \in [0,\infty) \quad \text{and} \quad I(B_{i1})M_{i1}(t) = 0 \text{ for } t \in (-\infty, 0]$$

in the last equality. To estimate (6.21), we use the inequality

$$|E\overline{W}_{n1}^{(i)}| = \left| \sum_{\substack{l=1 \\ l\neq i}}^{n} EY_{l1} I\left(|Y_{l1}| \leq 1 + \frac{r}{4}\right) \right| \leq \sum_{i=1}^{n} EY_{l1}^2 I\left(|Y_{l1}| > 1 + \frac{r}{4}\right) \leq 1 \quad (6.22)$$

where the first inequality is obtained from (6.16). In addition, we note from this inequality that

$$\begin{aligned} Var(\overline{W}_{n1}) &\leq E(\overline{W}_{n1})^2 \\ &\leq \sum_{l=1}^{n} E\overline{Y}_{l1}^2 + \left|\sum_{l=1}^{n} E\overline{Y}_{l1}\right| \left|\sum_{\substack{m=1 \\ m\neq l}}^{n} E\overline{Y}_{m1}\right| \\ &\leq \sum_{\substack{l=1 \\ l\neq i}}^{n} E\overline{Y}_{l1}^2 + |E\overline{W}_{n1}| \left|E\overline{W}_{n1}^{(l)}\right| \\ &\leq \sum_{i=1}^{n} EY_{l1}^2 I\left(|Y_{i1}| \leq 1 + \frac{r}{4}\right) + \sum_{i=1}^{n} EY_{i1}^2 I\left(|Y_{l1}| > 1 + \frac{r}{4}\right) \\ &= 1. \end{aligned} \quad (6.23)$$

Assume that $t < 0$ and $w \in A_{i1} \cap \left\{ w \mid \overline{W}_{2n}^2(w) \leq \frac{r^2}{4} \right\}$. We note from (6.22) that for $r \geq 4$,

$$-t + \alpha_i^-(w) = -t + \sqrt{r^2 - \overline{W}_{n2}^2(w)} - E\overline{W}_{n1}^{(i)} > \frac{\sqrt{3}r}{2} - |E\overline{W}_{n1}^{(i)}| > 0 \quad (6.24)$$

and

$$\begin{aligned} \sqrt{Var(\overline{W}_{n1})} + \left(-t + \sqrt{r^2 - \overline{W}_{n2}^2 I\left(\overline{W}_{n2}^2 \leq \frac{r^2}{4}\right)} - E\overline{W}_{n1}^{(i)}\right) \\ \geq \frac{\sqrt{3}r}{2} - |E\overline{W}_{n1}^{(i)}| \\ \geq C(1+r) \end{aligned} \quad (6.25)$$

for some absolute constant C. By (6.23),(6.25) and Proposition 6.5, we obtain that

$$E^{\overline{Y}_{i1},\overline{W}_{n2}} I(A_{i1}) I\left(\overline{W}_{n2}^2 \le \frac{r^2}{4}\right)$$

$$= P\left(\frac{-t+\alpha_i^-(w)}{\sqrt{Var(\overline{W}_{n1})}} < T_{n1}^{(i)}(w) \le \frac{-\overline{Y}_{i1}(w)+\alpha_i^-(w)}{\sqrt{Var(\overline{W}_{n1})}} \mid \overline{Y}_{i1}, \overline{W}_{n2}\right) I\left(\overline{W}_{n2}^2 \le \frac{r^2}{4}\right)$$

$$\le \frac{Var(\overline{W}_{n1})(|\overline{Y}_{i1}|+|t|)+\left(\sqrt{Var(\overline{W}_{n1})}\right)^3\beta_{1,3}}{\left[\left(\sqrt{Var(\overline{W}_{n1})}\right)^3+\left(-t+\sqrt{r^2-\overline{W}_{n2}^2 I(\Lambda)}-E\overline{W}_{n1}^{(i)}\right)\right]^3} \times I\left(\overline{W}_{n2}^2 \le \frac{r^2}{4}\right)$$

$$\le \frac{Var(\overline{W}_{n1})(|\overline{Y}_{i1}|+|t|)+\left(\sqrt{Var(\overline{W}_{n1})}\right)^3\beta_{1,3}}{\left[\left(\sqrt{Var(\overline{W}_{n1})}\right)^3+\left(-t+\sqrt{r^2-\overline{W}_{n2}^2 I(\overline{W}_{n2}^2 \le \frac{r^2}{4})}-E\overline{W}_{n1}^{(i)}\right)\right]^3}$$

$$\le \frac{C}{(1+r)^3}[Var(\overline{W}_{n1})(|\overline{Y}_{i1}|+|t|)+(Var(\overline{W}_{n1}))^3\beta_{1,3}]$$

$$\le \frac{C}{(1+r)^3}\left[(|\overline{Y}_{i1}|+|t|)+\sum_{l=1}^n |\overline{Y}_{l1}-E\overline{Y}_{l1}|^3\right] \tag{6.26}$$

where $\beta_{1,3}$ is defined as in Proposition 6.5. Note that we can apply Proposition 6.5 because of (6.24). Thus, by (6.26),

$$\sum_{i=1}^n E\int_{-\infty}^0 I(A_{i1}) I\left(\overline{W}_{n2}^2 \le \frac{r^2}{4}\right) M_{i1}(t)dt$$

$$= \sum_{i=1}^n E\int_{-\infty}^0 E^{\overline{Y}_{i1},\overline{W}_{n2}} I(A_{i1}) I\left(\overline{W}_{n2}^2 \le \frac{r^2}{4}\right) M_{i1}(t)dt$$

$$\le \frac{C}{(1+r)^3}\sum_{i=1}^n E\int_{-\infty}^0 (|\overline{Y}_{i1}|+|t|)M_{i1}(t)dt$$

$$+ \frac{C}{(1+r)^3}\sum_{i=1}^n E\int_{-\infty}^0 \sum_{l=1}^n |\overline{Y}_{l1}-E\overline{Y}_{l1}|^3 M_{i1}(t)dt$$

$$\le \frac{C}{1+r^3}\sum_{i=1}^n |\overline{Y}_{i1}|^3 + \frac{C}{1+r^3}\sum_{i=1}^n |\overline{Y}_{i1}|^3$$

$$\le \frac{C}{1+r^3}\sum_{i=1}^n |\overline{Y}_{i1}|^3. \tag{6.27}$$

Assume that $t \ge 0$ and $w \in B_{i1} \cap \{w \mid \overline{W}_{2n}^2(w) \le \frac{r^2}{4}\}$. By (6.22), we obtain

$$-t-\alpha_i^+(w) = -t-\sqrt{r^2-\overline{W}_{n2}^2(w)}-E\overline{W}_{n1}^{(i)} < -\frac{\sqrt{3}r}{2}-\overline{W}_{n1}^{(i)} < 0.$$

Therefore, we can apply Proposition 6.5 to $-T_{n1}^{(i)}(w)$ and use the same argument as (6.27). We have

$$\sum_{i=1}^{n} E\int_{0}^{\infty} I(B_{i1})I\left(\overline{W}_{n2}^{2}(w) \le \frac{r^2}{4}\right) M_{i1}(t)dt$$
$$\le C\sum_{i=1}^{n} E\int_{0}^{\infty} I\left(\overline{W}_{n2}^{2} \le \frac{r^2}{4}\right)$$
$$\times \frac{Var(\overline{W}_{n1})(|\overline{Y}_{i1}|+|t|) + (Var(\overline{W}_{n1}))^3\beta_{1,3}}{\left[\left(\sqrt{Var(\overline{W}_{n1})}\right)^3 + \left(-t+\sqrt{r^2 - \overline{W}_{n2}^{2} I(\overline{W}_{n2}^{2} \le \frac{r^2}{4})} - E\overline{W}_{n1}^{(i)}\right)\right]^3} M_{i1}(t)dt$$
$$\le \frac{C}{(1+r)^3}\sum_{i=1}^{n} E\int_{0}^{\infty} (|\overline{Y}_{i1}|+|t|)M_{i1}(t)dt$$
$$+ \frac{C}{(1+r)^3}\sum_{i=1}^{n} E\int_{0}^{\infty} \sum_{l=1}^{n} |\overline{Y}_{l1} - E\overline{Y}_{l1}|^3 M_{i1}(t)dt$$
$$\le \frac{C}{1+r^3}\sum_{i=1}^{n} |\overline{Y}_{i1}|^3. \tag{6.28}$$

By (6.21), (6.27)–(6.28), we have

$$\sum_{i=1}^{n} E\int_{-\infty}^{\infty} I(A_{i1}\cup B_{i1})I\left(\overline{W}_{n2}^{2} \le \frac{r^2}{4}\right) M_{i1}(t)dt \le \frac{C}{1+r^3}\sum_{i=1}^{n} |\overline{Y}_{i1}|^3. \tag{6.29}$$

Next, we will find a bound of the second term of (6.20) by using the uniform concentration inequality. By the same argument as (6.21), we have

$$\sum_{i=1}^{n} E\int_{-\infty}^{\infty} I(A_{i1}\cup B_{i1})I\left(\overline{W}_{n2}^{2}(w) > \frac{r^2}{4}\right) M_{i1}(t)dt$$
$$= \sum_{i=1}^{n} E\int_{-\infty}^{0} I(A_{i1})I\left(\overline{W}_{n2}^{2}(w) > \frac{r^2}{4}\right) M_{i1}(t)dt$$
$$+ \sum_{i=1}^{n} E\int_{0}^{\infty} I(B_{i1})I\left(\overline{W}_{n2}^{2}(w) > \frac{r^2}{4}\right) M_{i1}(t)dt \tag{6.30}$$

By Proposition 5.5, Proposition 6.4(1), (6.11)–(6.13), Chebyshev's inequality and the same argument as in (5.11), we get

$$\sum_{i=1}^{n} E\int_{-\infty}^{0} I(A_{i1})I\left(\overline{W}_{n2}^{2}(w) > \frac{r^2}{4}\right) M_{i1}(t)dt$$
$$\leq \sum_{i=1}^{n} EI\left(\overline{W}_{n2}^{2} > \frac{r^2}{4}\right)\left[\frac{\sqrt{2}(|\overline{Y}_{i1}|+|t|)}{\sqrt{Var(\overline{W}_{n1})}}\int_{-\infty}^{0} M_{i1}(t)dt\right.$$
$$\left.+\frac{1+\sqrt{2}}{\left(\sqrt{Var(\overline{W}_{n1})}\right)^3}\int_{-\infty}^{0}\sum_{k=1}^{n} E|\overline{Y}_{k1}-E\overline{Y}_{k1}|^3 M_{i1}(t)dt\right]$$
$$\leq \frac{CE(|\overline{W}_{n2}|^4)}{1+r^4}\sum_{i=1}^{n} E|\overline{Y}_{i1}|^3 + \frac{CE(|\overline{W}_{n2}|^4)}{1+r^4}\sum_{i=1}^{n} E|\overline{Y}_{i1}-E\overline{Y}_{i1}|^3$$
$$\leq \frac{C[(2+\frac{r}{4})\beta_3+3k]}{1+r^4}\sum_{i=1}^{n} E|\overline{Y}_{i1}|^3$$
$$\leq \frac{C}{1+r^3}\sum_{i=1}^{n} E|\overline{Y}_{i1}|^3 \tag{6.31}$$

where we used the assumption that $(1+r)\beta_3 < 1$ in the last inequality. By the same argument as (6.31), we have

$$\sum_{i=1}^{n} E\int_{0}^{\infty} I(B_{i1})I\left(\overline{W}_{n2}^{2} > \frac{r^2}{4}\right) M_{i1}(t)dt \leq \frac{C}{1+r^3}\sum_{i=1}^{n} E|\overline{Y}_{i1}|^3. \tag{6.32}$$

Therefore, by (6.30)–(6.32), we obtain

$$\sum_{i=1}^{n} E\int_{0}^{\infty} I(A_{i1}\cup B_{i1})I\left(\overline{W}_{n2}^{2}(w) > \frac{r^2}{4}\right) M_{i1}(t)dt \leq \frac{C}{1+r^3}\sum_{i=1}^{n} E|\overline{Y}_{i1}|^3. \tag{6.33}$$

By (6.20), (6.29) and (6.33), we have

$$R_{11} \leq \frac{C}{1+r^3}\sum_{i=1}^{n} E|\overline{Y}_{i1}|^3. \tag{6.34}$$

To prove

$$|R_{11}| \leq \frac{C}{1+r^3}\sum_{i=1}^{n} E|\overline{Y}_{i1}|^3, \tag{6.35}$$

it remains to show that

$$R_{11} \geq -\frac{C}{1+r^3}\sum_{i=1}^{n} E|\overline{Y}_{i1}|^3. \tag{6.36}$$

This equation is proved by the same argument as (6.34) and using the following relation,

$$\Big\{w \in \Omega \mid h_{B_2(r)}(W_{n1}^{(i)} + Y_{i1}, W_{n2})(w) - h_{B_2(r)}(W_{n1}^{(i)} + t, W_{n2})(w) = -1\Big\} \subseteq E_{i1} \cup F_{i1} \tag{6.37}$$

where

$$C_{i1} = \left\{ w \in \Omega \mid \frac{-\overline{Y}_{i1}(w) + \alpha_i^-(w)}{\sqrt{Var(\overline{W}_{n1})}} < T_{n1}^{(i)}(w) \le \frac{-t + \alpha_i^-(w)}{\sqrt{Var(\overline{W}_{n1})}} \right\} \text{ and}$$

$$D_{i1} = \left\{ w \in \Omega \mid \frac{-t - \alpha_i^+(w)}{\sqrt{Var(\overline{W}_{n1})}} < T_{n1}^{(i)}(w) \le \frac{-\overline{Y}_{i1}(w) - \alpha_i^+(w)}{\sqrt{Var(\overline{W}_{n1})}} \right\}.$$

We have (6.36) and hence (6.35). To prove our theorem, it remains to estimate R_{12}. By Proposition 3.6 and the Fundamental Theorem of Calculus, we have

$$\begin{aligned}
|R_{12}| \le & \Big| \sum_{i=1}^{n} \int_{-\infty}^{\infty} E\Big\{ I(t \le \overline{Y}_{i1}) \Big[E^{\overline{Y}_{i1}}(\overline{W}_{n1}^{(i)} + Y_{i1} + \overline{W}_{n2}) f(\overline{W}_{n1}^{(i)} + Y_{i1}, \overline{W}_{n2}) \\
& \qquad - E(\overline{W}_{n1}^{(i)} + t + \overline{W}_{n2}) f(\overline{W}_{n1}^{(i)} + t, \overline{W}_{n2}) \Big] \Big\} M_{i1}(t) dt \Big| \\
& + \Big| \sum_{i=1}^{n} \int_{-\infty}^{\infty} E\Big\{ I(t > \overline{Y}_{i1}) \Big[E^{\overline{Y}_{i1}}(\overline{W}_{n1}^{(i)} + Y_{i1} + \overline{W}_{n2}) f(\overline{W}_{n1}^{(i)} + Y_{i1}, \overline{W}_{n2}) \\
& \qquad - E(\overline{W}_{n1}^{(i)} + t + \overline{W}_{n2}) f(\overline{W}_{n1}^{(i)} + t, \overline{W}_{n2}) \Big] \Big\} M_{i1}(t) dt \Big| \\
\le & \, 2 \sum_{i=1}^{n} E \int_{-\infty}^{\infty} \int_{t}^{\overline{Y}_{i1}} E^{\overline{Y}_{i1}} |g_1(\overline{W}_{n1,u}^{(i)}, \overline{W}_{n2})| M_{i1}(t) du dt \\
\le & \, \frac{C}{1 + r^3} \sum_{i=1}^{n} E \int_{-\infty}^{\infty} (|\overline{Y}_{i1}| + |t|) M_{i1}(t) dt \\
\le & \, \frac{C}{1 + r^3} \sum_{i=1}^{n} E|\overline{Y}_{i1}|^3.
\end{aligned} \tag{6.38}$$

By (6.15), (6.17)–(6.18), (6.35) and (6.38), we have

$$|U_1 - V_1| \le \frac{C}{1 + r^3} \sum_{i=1}^{n} E|\overline{Y}_{i1}|^3. \tag{6.39}$$

By the same way as (6.39), we have

$$|U_2 - V_2| \le \frac{C}{1 + r^3} \sum_{i=1}^{n} E|\overline{Y}_{i2}|^3. \tag{6.40}$$

By (6.9), (6.39) and (6.40), we complete the proof of theorem 6.3. □

REFERENCES

[1] Bahr, B.V.: On the central limit theorem in R_k, *Ark. Mat.* **7**, 61–69 (1967).

[2] Bahr, B.V.: Multi-dimensional integral limit theorems, *Ark. Mat.* **7**, 71–88 (1967).

[3] Barbour, A.D.: Stein's method for diffusion approximations, *Probab. Theory Relat. Field* **84**, 297–322 (1990).

[4] Bergström, H.: On the central limit theorem in $R_k, k > 1$, *Skand. Akt.* **28**, 106–127 (1945).

[5] Berry, A.C.: The accuracy of the Gaussian approximation to the sum of independent variables, *Trans.Amer.Math.Soc.* **49**, 122–136 (1941).

[6] Bhattacharya, R.N.: Rates of weak convergence for the multi-dimensional central limit theorem, *Theory Probab. Appl.* **15**, 68–86 (1970).

[7] Chaidee, N.: Non-uniform Bounds in Normal Approximation for Matrix Correlation Statistics and Independent Bounded Random Variables, Ph.D. thesis, Chulalongkorn university (2005).

[8] Chatterjee, S., Meckes, E.: Multivariate normal approximation using exchangeable pairs, *Alea* **4**, 257–283 (2008).

[9] Chen, L.H.Y.: Poisson approximation for dependent trials. *Ann. Probab.* **3**(3), 534–545 (1975).

[10] Chen, L.H.Y., Shao, Q.M.: A non-uniform Berry-Esseen bound via Stein's method. *Probab.Theory Relat. Field* **120**, 236–254 (2001).

[11] Chen, L.H.Y., Shao, Q.M.: Normal approximation under local dependence, *Ann. Probab.* **32**, 1985–2028 (2004).

[12] Chen, L.H.Y., Shao, Q.M.: Stein's method for normal approximation, *An Introduction to Stein's Method* **4**, 1–59 (2005), Singapore University Press, Singapore.

[13] Esseen, C.G.: Fourier analysis of distribution functions. A Mathematical Study of the Laplace Gaussian Law, *Acta Math.* **77**, 1–125 (1945).

[14] Götze, F.: On the rate of convergence in the multivariate CLT, *Ann. Probab.* **19**(2), 724–739 (1991).

[15] Holmes, S.: Steins method for birth and death chains. *In Steins method: expository lectures and applications*, IMS Lecture Notes **46**, 45–67 (2004), Beachwood, OH.

[16] Luk, H.M.: Steins Method for the Gamma Distribution and Related Statistical Applications. Ph.D. thesis, University of Southern California (1994).

[17] Nagaev, S.V.: Some limit theorems for large deviations. *Theory Probab. Appl.* **10**, 214–235 (1965).

[18] Neammanee, K., Thongtha, P.: Improvement of the non-uniform version of Berry-Esseen inequality via Paditz-Siganov Theorems, *JIPAM* **8**(4), 1–10 (2007).

[19] Paditz, L.: On the analytical structure of the constant in the nonuniform version of the Esseen inequality, *Statistics* **20**, 453–464 (1989).

[20] Pickett, A.: Rates of Convergence of χ^2 Approximations via Steins Method, Ph.D. thesis, University of Oxford (2004).

[21] Rao, R.R.: , On the central limit theorem in R_k, *Bull. Amer. Math. Soc.* **67**, 359–361 (1961).

[22] Reinert, G., Röllin, A.: Multivariate normal approximation with Stein's method of exchangeable pairs under a general linearity condition, *Ann. Probab.* **37**(6), 2150–2173 (2009).

[23] Shevtsova, I.G.: An improvement of convergence rate estimates in the Lyapunov theorem. *Dokl. Math.* **82**(3), 862–864 (2010).

[24] Siganov, I.S.: Refinnement of the upper bound of the constant in the central limit theorem, *J. Sov. Math.* **35**, 2545–2550 (1986).

[25] Stein, C.: A bound for the error in the normal approximation to the distribution of a sum of dependent random variables, *Proc. Sixth Berkeley Symp. Math. Stat. Prob.* **2**, 583–602 (1972), Univ. California Press. Berkeley, CA.

[26] Stein, C.: *Approximation Computation of Expectations*, IMS Lecture Notes **7**, (1986), Hayward, CA.

[27] Thongtha, P., Neammanee, K.: Refinement on the constants in the non-uniform version of the Berry-Esseen theorem, TJM **5**, 1–13 (2007).

VITA

Name	Mr. Dawud Thongtha
Date of Birth	24 April 1983
Place of Birth	Bangkok, Thailand
Education	B.Sc.(Mathematics), Chulalongkorn University, 2004
	M.Sc.(Mathematics), Chulalongkorn University, 2006