CHAPTER IV

THE THEORY OF $\vdash$ SYMBOL

4.1 The Symbol $\vdash$.

In the following, dots are sometimes used to separate the main parts of a complex expression.

For example $\vdash A \equiv t(A) \equiv 1$ is ambiguous. It has two meanings:

First $[\vdash A] \equiv [t(A) \equiv 1]$, which is the intended meaning.

Second $[\vdash A \equiv t(A)] \equiv [1]$, which is wrong.

Therefore we use dots to represent brackets, and we can write the above $\vdash A \;.\equiv.\; t(A) \equiv 1$

(1) Def. $\vdash A \;.\equiv.\; t(A) \equiv 1$ "A is true"

(2) Def. $A \vdash B \;.\equiv.\; \vdash A \Longrightarrow B$ ("B is deducable from A" or "If A is true then B is true" or "A yields B")

(3) Th. $A \vdash B \;.\equiv.\; \neg B \vdash \neg A$

(4) Def. $A, B \vdash C \;.\equiv.\; A \wedge B \vdash C$

(5) Th. $A, B \vdash C \;.\equiv.\; A \vdash B \Longrightarrow C$

The symbol $\vdash$ to the left of a statement indicates that the statement is true. Between statements it indicates that if the statements to the left are true then the statement to the right is true.

The following tautologies are useful in the process of deduction.

(6) Th. $\vdash A \Longrightarrow A$

(7) Th. $A \vdash A \vee B$

(8) Th. $A \Longrightarrow B \vdash \neg B \Longrightarrow \neg A$ (Contrapositive law)

(9) Th. $A \wedge B,\ B \Longrightarrow C \vdash A \wedge C$

(10) Th. $A \Longrightarrow B,\ B \Longrightarrow C \vdash A \Longrightarrow C$ (Hypothetical Syllogism)

(11) Th. $A,\ A \Longrightarrow B \vdash B$ (Rule of Detachment)

(12) Th. $A \Longrightarrow B,\ \neg B \vdash \neg A$ (Contradiction law)

(13) Th. $A \Longrightarrow B,\ \neg A \Longrightarrow B \vdash B$ (Proof by alternative)

(14) Th. $A \Longrightarrow B,\ A \Longrightarrow \neg B \vdash \neg A$ (Reduction to absurdity)

All these theorems may be written in the form $\vdash$, T where T is a tautology.

<u>Example of proof: th. (3)</u>

$A \vdash B$	$\equiv$	$\vdash A \Longrightarrow B$	(Def. 2)
	$\equiv$	$\vdash \neg B \Longrightarrow \neg A$	(Th. of compound statement)
	$\equiv$	$\neg B \vdash \neg A$	(Def. 2)

<u>Example of proof: th. (5)</u>

$A,\ B \vdash C$	$\equiv$	$A \wedge B \vdash C$	(Def. 4)
	$\equiv$	$\vdash (A \wedge B) \Longrightarrow C$	(Def. 2)
	$\equiv$	$\vdash \neg(A \wedge B) \vee C$	
	$\equiv$	$\vdash (\neg A \vee \neg B) \vee C$	
	$\equiv$	$\vdash \neg A \vee (\neg B \vee C)$	
	$\equiv$	$\vdash \neg A \vee (B \Longrightarrow C)$	
	$\equiv$	$\vdash A \Longrightarrow (B \Longrightarrow C)$	
	$\equiv$	$A \vdash (B \Longrightarrow C)$	

<u>Example of proof: th. (11)</u>

$$A,\ A \Longrightarrow B \vdash B \quad \equiv \quad A \wedge (A \Longrightarrow B) \vdash B \quad \text{(Def. 4)}$$

$$\equiv \ \vdash [A \wedge (A \Longrightarrow B)] \Longrightarrow B \quad \text{(Def. 2)}$$

But the truth value of the compound statement in the last line is $t\left[\{A \wedge (A \Longrightarrow B)\} \Longrightarrow B\right]$

$$\equiv t[\neg\{A \wedge (\neg A \vee B)\} \vee B]$$

$$\equiv t[\neg A \vee \neg(\neg A \vee B) \vee B]$$

$$\equiv t[\neg A \vee (A \wedge \neg B) \vee B]$$

$$\equiv t(\neg A) + t(A \wedge \neg B) + t(B)$$

$$\equiv a' + (a.b') + b$$

$$\equiv (a' + a).(a' + b') + b$$

$$\equiv (a' + a + b).(a' + b' + b)$$

$$\equiv 1.1$$

$$\equiv 1$$

But $\vdash (A \wedge (A \Longrightarrow B)) \Longrightarrow B \quad . \equiv . \quad t[(A \wedge (A \Longrightarrow B)) \Longrightarrow B] \equiv 1$

Hence the theorem is proved.

4.2 <u>Theory of Deduction.</u>

Before dealing with the predicate calculus we will introduce some theorems and one axiom containing the symbol $\vdash$

<u>Th. 1</u> $\quad \vdash A$ and $A \vdash B$, then $\vdash B$.

<u>Proof</u> $\quad \vdash A$ means $a \equiv 1$ and hence $a' \equiv 0$;

and $\quad A \vdash B$ means $a' + b \equiv 1$.

But $\quad a' \equiv 0$ so $0 + b \equiv 1$.

Hence $\quad b \equiv 1$

or $\quad \vdash B$

and the theorem is proved.

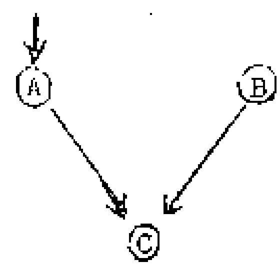
<u>Ax.</u> If $A \vdash B$ and $B \vdash C$, then $A \vdash C$.

The relations between A,B and C may be indicated informally by means of a sketch as follows (A) ⟶ (B) ⟶ (C) the arrows corresponding to the symbol $\vdash$.

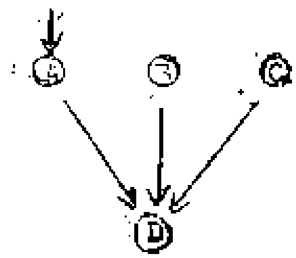
Similar sketches illustrate the other theorems.

The reason why we let this expression be an axiom is because we cannot prove it directly by using the theorems of identity and the rule of substitution.

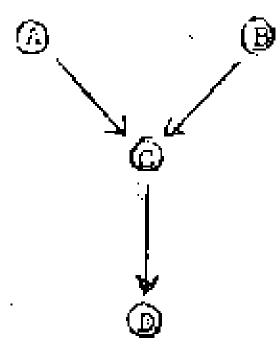
<u>Th. 2</u> If $\vdash A$ and $A,B \vdash C$, then $B \vdash C$.



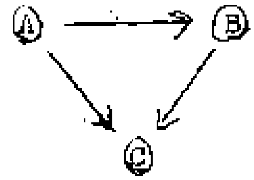
<u>Th. 3</u> If $\vdash A$ and $A,B,C \vdash D$, then $B,C \vdash D$.



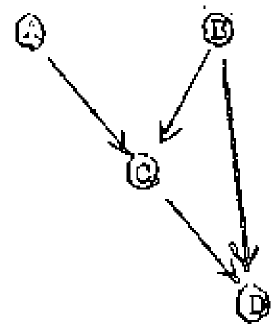
<u>Th. 4</u> If $A,B \vdash C$ and $C \vdash D$, then $A,B \vdash D$.



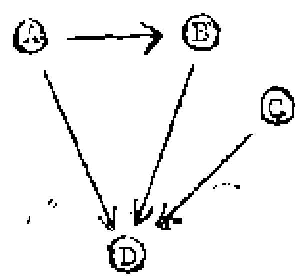
<u>Th. 5</u> If $A \vdash B$ and $A, B \vdash C$, then $A \vdash C$.



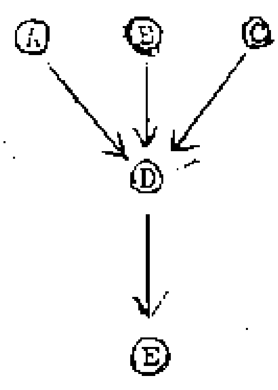
<u>Th. 6</u> If $A, B \vdash C$ and $B, C \vdash D$ then $A, B \vdash D$.



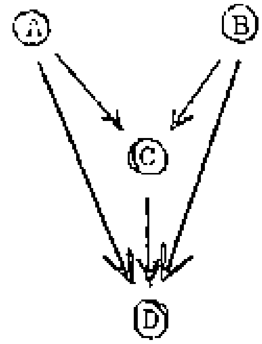
<u>Th. 7</u> If $A \vdash B$ and $A, B, C \vdash D$, then $A, C \vdash D$.



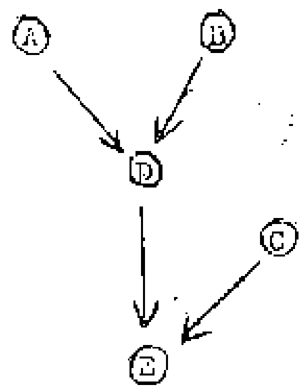
<u>Th. 8</u> If $A, B, C \vdash D$ and $D \vdash E$, then $A, B, C \vdash E$.



<u>Th. 9</u> If $A, B \vdash C$ and $A, B, C \vdash D$, then $A, B \vdash D$.



<u>Th. 10</u> If $A, B \vdash D$ and $D, C \vdash E$, then $A, B, C \vdash E$.



<u>Example of proof: th. 2</u>

$A, B \vdash C \quad \equiv \quad A \vdash B \Longrightarrow C$ (Th. of $\vdash$ symbol)

But from $\vdash A$ and $A \vdash B \Longrightarrow C$ we obtain $\vdash B \Longrightarrow C$ by theorem 1.

Since $\vdash B \Longrightarrow C \quad \equiv \quad B \vdash C$, the theorem is proved.

<u>Example of proof: th. 5</u>

$A, B \vdash C \quad \equiv \quad A \wedge B \vdash C$ (Def. 4)

$\equiv \quad B \wedge A \vdash C$ (Th. of compound statement)

$\equiv \quad B \vdash A \Longrightarrow C$ (Th. of $\vdash$ symbol)

But from $A \vdash B$ and $B \vdash A \Longrightarrow C$, we obtain $A \vdash A \Longrightarrow C$ by the axiom.

Since $A \vdash A \Longrightarrow C \quad \equiv \quad A \wedge A \vdash C$

$\equiv \quad A \vdash C$

The theorem is proved.