CHAPTER III

THE THEORY OF COMPOUND STATEMENTS

3.1 Compound Statements.

<u>Basic concepts and symbols:</u>

Arbitary Statements	A, B, C, ...
Negation ("not")	$\neg$
Conjunction ("and")	$\wedge$
Disjunction ("or")	$\vee$

<u>Formation rules.</u>

$\neg$A is a statement ("not A").

A$\wedge$B is a compound statement ("A and B").

A$\vee$B is a compound statement ("A or B").

A and B are called the constituent statements of the compound statements A$\wedge$B and A$\vee$B.

<u>Examples of interpretation.</u>

Suppose A means " $x < 1$ "

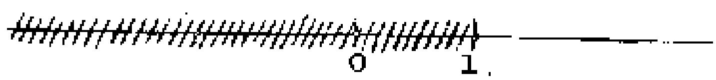

Fig. 1

(x is somewhere in the shaded region, which does not include 1)

and B means " $x \neq 0$ "

Fig. 2

(x is somewhere in the shaded region, which does not include 0).

Then ¬A means " $x \geq 1$ "

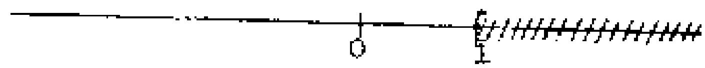

Fig. 3

(x is somewhere in the shaded region, which includes 1)

A∧B means " $x < 1$ and $x \neq 0$ "

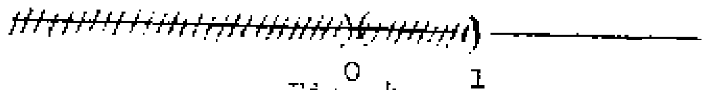

Fig. 4

(x is somewhere in the shaded region, which does not include either 1 or 0)

A∨B means "$x < 1$ or $x \neq 0$"

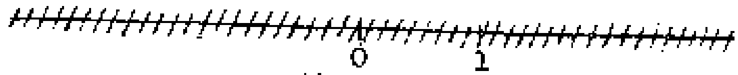

Fig. 5

(x is somewhere in the shaded region, which includes all values of x)

¬(A∧B) means "It is not the case that $x < 1$ and $x \neq 0$."

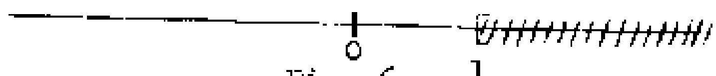

Fig. 6

(x is 0 or is somewhere in the shaded region, which includes 1)

¬(A∨B) means "It is not the case that $x < 1$ or $x \neq 0$." There is no value of x which satisfies these conditions.

<u>Theory</u>

(1)	Ax.	$\neg(\neg A)$	$\equiv$	A
(2)	Ax.	$\neg(A \wedge B)$	$\equiv$	$\neg A \vee \neg B$
(3)	Th.	$\neg(A \vee B)$	$\equiv$	$\neg A \wedge \neg B$
(4)	Ax.	$A \wedge A$	$\equiv$	A
(5)	Ax.	$A \wedge B$	$\equiv$	$B \wedge A$
(6)	Ax.	$(A \wedge B) \wedge C$	$\equiv$	$A \wedge (B \wedge C)$
(7)	Th.	$A \vee A$	$\equiv$	A
(8)	Th.	$A \vee B$	$\equiv$	$B \vee A$
(9)	Th.	$(A \vee B) \vee C$	$\equiv$	$A \vee (B \vee C)$
(10)	Ax.	$A \wedge (B \vee C)$	$\equiv$	$(A \wedge B) \vee (A \wedge C)$
(11)	Th.	$A \vee (B \wedge C)$	$\equiv$	$(A \vee B) \wedge (A \vee C)$
(12)	Def.	$A \Longrightarrow B$	$\equiv$	$\neg A \vee B$

($A \Longrightarrow B$ is read "if A then B" or "A implies B.")

(13)	Th.	$A \Longrightarrow B$	$\equiv$	$\neg B \Longrightarrow \neg A$
(14)	Def.	$A \Longleftrightarrow B$	$\equiv$	$(A \Longrightarrow B) \wedge (B \Longrightarrow A)$

($A \Longleftrightarrow B$ is read "A if and only if B.")

(15)	Th.	$A \Longleftrightarrow B$	$\equiv$	$\neg A \Longleftrightarrow \neg B$

First of all we shall show that the set of axioms is consistent by using a model as follows.

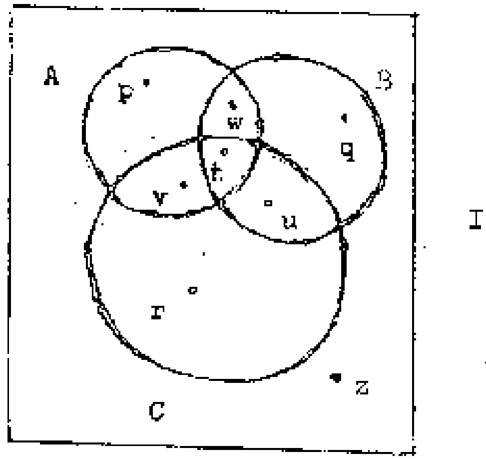

Fig. 7

Let I denote the set $\{p, q, r, t, u, v, w, z\}$. (See figure 7)

Let A correspond to the set $\{p, t, v, w\}$. This may be written $A \longleftrightarrow \{p, t, v, w\}$.

Also let $B \longleftrightarrow \{q, t, u, w\}$,

and $C \longleftrightarrow \{r, t, u, v\}$.

Let $\neg$ correspond to the operation of taking all the elements in I that are not in set corresponding to the expression following the $\neg$ symbol.

e. g. $\neg A \longleftrightarrow \{q, r, u, z\}$ etc.

Let $\wedge$ correspond to the operation of taking all the elements that are common to sets corresponding the statements on each side of the $\wedge$ symbol.

e. g. $A \wedge B \longleftrightarrow \{t, w\}$.

Let $\vee$ correspond to the operation of putting together all the elements from the two sets corresponding to the statements on each side of $\vee$ symbol.

e. g. $A \vee B \longleftrightarrow \{p, q, t, u, v, w\}$.

Consider Ax. (1), and the correspondences

$$A \longleftrightarrow \{p, t, v, w\},$$

$$\neg A \longleftrightarrow \{q, r, u, z\}.$$

and $$\neg(\neg A) \longleftrightarrow \{p, t, v, w\}.$$

Since the sets corresponding to $\neg(\neg A)$ and A are the same, Ax. (1): $\neg(\neg A) \equiv A$ corresponds to a true statement in the model.

Also Ax. (2) corresponds to a true statement in the model. For, consider the correspondences

$$A \wedge B \longleftrightarrow \{t, w\},$$

$$\neg(A \wedge B) \longleftrightarrow \{p, q, r, u, v, z\},$$

$$\neg A \longleftrightarrow \{q, r, u, z\},$$

$$\neg B \longleftrightarrow \{p, r, v, z\},$$

and $$\neg A \vee \neg B \longleftrightarrow \{p, q, r, u, v, z\}.$$

Since the sets corresponding to $\neg(A \wedge B)$ and $\neg A \vee \neg B$ are the same, Ax. (2): $\neg(A \wedge B) = \neg A \vee \neg B$ corresponds to a true statement in the model.

Similarly we can show that the axioms (4), (5), (6), (10) correspond to true statements in the model. Therefore the set of axioms is consistent.

<u>Example of proof: th. (3)</u>

Let	A	$\equiv$	$\neg C$	------------------ (1)
and	B	$\equiv$	$\neg D$	------------------ (2)
Then	$\neg (A \vee B)$	$\equiv$	$\neg (\neg C \vee \neg D)$	(Substitution)
		$\equiv$	$\neg [\neg (C \wedge D)]$	(Ax. 2)
		$\equiv$	$C \wedge D$ ---------(3)	(Ax. 1)
But	$\neg X$	$\equiv$	$\neg X$	(Identity Th. 3)
Therefore	$\neg (\neg C)$	$\equiv$	$\neg A$	(Substituting from (1))
and	$\neg (\neg D)$	$\equiv$	$\neg B$	(Substituting from (2))
Therefore	C	$\equiv$	$\neg A$	(Ax. 1)
and	D	$\equiv$	$\neg B$	(Ax. 1)
Therefore	$\neg (A \vee D)$	$\equiv$	$\neg A \wedge \neg B$	(Substituting in (3))

<u>Example of proof : th.(13)</u>

$A \Longrightarrow B$	$\equiv$	$\neg A \vee B$	(Def. 12)
	$\equiv$	$B \vee \neg A$	(Th. 8)
	$\equiv$	$\neg (\neg B) \vee \neg A$	(Ax. 1)
	$\equiv$	$\neg B \Longrightarrow \neg A$	(Def. 12)

This proof is given in less detail than the previous one.

3.2 <u>The Truth Value Calculus.</u>

<u>Basic concepts and symbols:</u>

Truth values : 0 (false), 1 (true)

Truth function: t

<u>Formation rule.</u> t (A) is the truth value of A.

Defs.

a	$\equiv$	$t(A)$
a'	$\equiv$	$t(\neg A)$
$a.b$	$\equiv$	$t(A \wedge B)$
$a + b$	$\equiv$	$t(A \vee B)$.

a, a', $a.b$, $a + b$ are called truth functions.

Theory (This is a two-valued Boolean Algebra)

(1)	Th.	$(a')'$	$\equiv$	a
(2)	Th.	$(a.b)'$	$\equiv$	$a' + b'$
(3)	Th.	$(a + b)'$	$\equiv$	$a'.b'$
(4)	Th.	$a.a$	$\equiv$	a
(5)	Th.	$a.b$	$\equiv$	$b.a$
(6)	Th.	$(a.b).c$	$\equiv$	$a.(b.c)$
(7)	Th.	$a + a$	$\equiv$	a
(8)	Th.	$a + b$	$\equiv$	$b + a$
(9)	Th.	$(a + b) + c$	$\equiv$	$a + (b + c)$
(10)	Th.	$a.(b + c)$	$\equiv$	$(a.b) + (a.c)$
(11)	Th.	$a + (b.c)$	$\equiv$	$(a + b).(a + c)$
(12)	Ax.	$0'$	$\equiv$	1
(13)	Th.	$1'$	$\equiv$	0
(14)	Ax.	$a.a'$	$\equiv$	0
(15)	Th.	$a + a'$	$\equiv$	1
(16)	Ax.	$a.0$	$\equiv$	0
(17)	Ax.	$a.1$	$\equiv$	a
(18)	Th.	$a + 0$	$\equiv$	a
(19)	Th.	$a + 1$	$\equiv$	1

Example of interpretation.

By Ax. (14) $A \wedge \neg A$ is always false. If A means "$x > 3$" then $A \wedge \neg A$ means "$x > 3$ and $x \not> 3$" which is false.

Example of proof : th. (2)

$$
\begin{aligned}
(a.b)' &= [t(A \wedge B)]' && \text{(Def. of a.b)} \\
&= t[\neg(A \wedge B)] && \text{(Def. of a')} \\
&= t(\neg A \vee \neg B) && \text{(Th. of compound statement)} \\
&= t(\neg A) + t(\neg B) && \text{(Def. of a + b)} \\
&= a' + b' && \text{(Def. of a')}
\end{aligned}
$$

Note Theorems (1) to (11) follow from the theory of compound statements.

Before we prove the theorems (13), (15), (18), (19) we have to show that the set of axioms is consistent by using a model as follows.

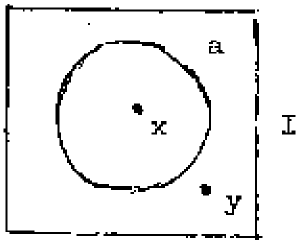

fig. 8

Let I denote the set $\{x, y\}$. (See figure 8)

Let $a \longleftrightarrow \{x\}$.

Let ' correspond to the same operation as that corresponding to $\neg$ in our model for proving the consistency of the axioms for compound statements and let . correspond the same operation as that corresponding to $\wedge$

Let $0 \longleftrightarrow \{ \quad \}$ (the empty set)

and $1 \longleftrightarrow \{x, y\}$.

Consider Ax. (12), and the correspondences

$$0 \longleftrightarrow \{ \quad \},$$
$$0' \longleftrightarrow \{x, y\},$$

and

$$1 \longleftrightarrow \{x, y\}.$$

Since the sets corresponding to $0'$ and 1 are the same, Ax. (12): $0' \equiv 1$ corresponds to a true statement in the model.

Also Ax. (14) corresponds to a true statement in the model. For, consider the correspondences

$$a \longleftrightarrow \{x\},$$
$$a' \longleftrightarrow \{y\},$$
$$a.a' \longleftrightarrow \{ \quad \},$$

and

$$0 \longleftrightarrow \{ \quad \}.$$

Since the sets corresponding to $a.a'$ and 0 are the same, Ax. (14): $a.a' \equiv 0$ corresponds to a true statement in the model.

Similarly we can show that the axioms (16), (17) correspond to true statements in the model. Therefore the set of axioms is consistent.

Example of proof: th. (18)

$b.1$	$\equiv$	b	(Ax. 17)
$(b.1)'$	$\equiv$	b'	$(a' = a')$
$b' + 1'$	$\equiv$	b'	(Th. 2)
$b' + 0$	$\equiv$	b'	(Th. 13)
$a + 0$	$\equiv$	a	(Putting $b' = a$)

3.3 Tautologies.

If a truth function containing arbitary truth values is identical with 1 (whatever values the arbitary truth values have) then the corresponding compound statement is called a tautology.

The following procedure may always be used to find out whether or not a given compound statement is a tautology.

Step 1. Write down the corresponding truth function.

Step 2. Take all dashes ($'$) inside brackets using theorems (1) or (2) or (3).

Step 3. Take all plus sign (+) inside brackets using theorem (11).

Step 4. Simplify the brackets using theorems (15) or (19);

Step 5. Simplify the whole expression using axiom (17)

If the final result is 1, the compound statement is a tautology otherwise is not.

For a proof that this procedure always works: p. 51 A.H Basson and D.J.O' Connor .

Example $((A \Longrightarrow B) \wedge \neg B) \Longrightarrow \neg A$ is a tautology.

Proof.

Step 1.

$$t\left[((A \Longrightarrow B) \wedge \neg B) \Longrightarrow \neg A\right]$$

$$= t\left[\neg((A \Longrightarrow B) \wedge \neg B) \vee \neg A\right] \quad (\text{Def. of } A \Longrightarrow B)$$

$$= \left[t((\neg A \vee B) \wedge \neg B)\right]' + \left[t(A)\right]' \quad (\text{Def. of } a' \text{ and } a + b)$$

$$= \left[t(\neg A \vee B).t(\neg B)\right]' + \left[t(A)\right]' \quad (\text{Def of } a.b)$$

$$= \left[\{t(\neg A) + t(B)\}.\{t(B)\}'\right]' + \left[t(A)\right]' \quad (\text{Def. of } a + b \text{ and } a')$$

$$= \left[\{\{t(A)\}' + t(B)\}.\{t(B)\}'\right]' + \left[t(A)\right]'$$

$$= \left[(a' + b).b'\right]' + a' \quad (\text{Def. of } a)$$

<u>Step 2.</u>	=	$\left[(a' + b)' + (b')'\right] + a'$	(Th. 2)
	=	$\left[\{(a')'.b'\} + (b')'\right] + a'$	(Th. 3)
	=	$\left[(a.b') + b\right] + a'$	(Th. 1)
<u>Step 3.</u>	=	$\left[(a + b).(b' + b)\right] + a'$	(Th. 11)
	=	$\left[(a + b) + a'\right].\left[(b' + b) + a'\right]$	(Th. 11)
<u>Step 4.</u>	=	$\left[(a + b) + a'\right].\left[a' + (b' + b)\right]$	(Th. 8)
	=	$\left[b + (a + a')\right].\left[a' + (b' + b)\right]$	(Th. 9)
	=	$(b + 1).(a + 1)$	(Th. 15)
	=	1.1	(Th. 19)
<u>Step 5.</u>	=	1	(Ax. 17)

Since the final answer is 1, it follows that $((A \Longrightarrow B) \wedge \neg B) \Longrightarrow \neg A$ is a tautology.