CHAPTER III

DERIVATION OF THE FORM OF THE FUNCTION f

In this chapter we shall attempt to find the form of the function f by writing it as a power series, then substituting in the equation

$$f\left\{vf(u) + uf(v)\right\} = f(u)f(v) - (1 - f(v)^2)u/v, \quad (19)$$

and finally equating coefficients of the various products of the powers of u and v.

Let $f(v) = a_0 + a_1 v + a_2 v^2 + a_3 v^3 + a_4 v^4 + a_5 v^5 + a_6 v^6 + \ldots\ldots\ldots,$

and $f(u) = a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5 + a_6 u^6 + \ldots\ldots\ldots$

From the left hand side of eqn (19) we obtain

$$\text{L.H.S.} = f\left\{v(a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5 + a_6 u^6 + \ldots) + u(a_0 + a_1 v + a_2 v^2 + a_3 v^3 + a_4 v^4 + a_5 v^5 + a_6 v^6 + \ldots)\right\}$$

$$= f(a_0 u + a_0 v + 2a_1 uv + a_2 uv^2 + a_2 u^2 v + a_3 uv^3 + a_3 u^3 v + a_4 uv^4 + a_4 u^4 v + a_5 uv^5 + a_5 u^5 v + a_6 uv^6 + a_6 u^6 v + \ldots\ldots)$$

$$\begin{aligned} = a_0 &+ a_1(a_0 u + a_0 v + 2a_1 uv + \ldots\ldots\ldots + a_6 uv^6 + a_6 u^6 v + \ldots) \\ &+ a_2(a_0 u + a_0 v + 2a_1 uv + \ldots\ldots\ldots + a_6 uv^6 + a_6 u^6 v + \ldots)^2 \\ &+ a_3(a_0 u + a_0 v + 2a_1 uv + \ldots\ldots\ldots + a_6 uv^6 + a_6 u^6 v + \ldots)^3 \\ &+ a_4(a_0 u + a_0 v + 2a_1 uv + \ldots\ldots\ldots + a_6 uv^6 + a_6 u^6 v + \ldots)^4 \\ &+ a_5(a_0 u + a_0 v + 2a_1 uv + \ldots\ldots\ldots + a_6 uv^6 + a_6 u^6 v + \ldots)^5 \\ &+ a_6(a_0 u + a_0 v + 2a_1 uv + \ldots\ldots\ldots + a_6 uv^6 + a_6 u^6 v + \ldots)^6 \\ &+ \ldots \end{aligned}$$

$$
\begin{aligned}
\text{L.H.S.} = {} & a_o + a_o a_1 u + a_o a_1 v + (2a_1^2 + 2a_o^2 a_2)uv + a_o^2 a_2 u^2 + a_o^2 a_2 v^2 \\
& + (a_1 a_2 + 4a_o a_1 a_2 + 3a_o^2 a_3)uv^2 + (a_1 a_2 + 4a_o a_1 a_2 + 3a_o^2 a_3)u^2 v \\
& + (4a_1^2 a_2 + 4a_0 a_2^2 + 12a_o^2 a_1 a_3 + 6a_o^4 a_4)u^2 v^2 \\
& + (a_1 a_3 + 2a_o a_2^2 + 6a_o^2 a_1 a_3 + 4a_o^4 a_4)uv^3 + (a_1 a_3 + 2a_o a_2^2 + 6a_o^2 a_1 a_3 \\
& + 4a_o^4 a_4)u^3 v + (2a_o a_2 a_3 + 4a_1 a_2^2 + 9a_o^2 a_2 a_3 + 12a_o a_1^2 a_3 + 24a_o^3 a_1 a_4 \\
& + 10a_o^5 a_5)u^2 v^3 + (2a_o a_2 a_3 + 4a_1 a_2^2 + 9a_o^2 a_2 a_3 + 12a_o a_1^2 a_3 + 24a_o^3 a_1 a_4 \\
& + 10a_o^5 a_5)u^3 v^2 + (a_1 a_4 + 2a_o a_2 a_3 + 3a_o^2 a_2 a_3 + 8a_o^3 a_1 a_4 + 5a_o^5 a_5)uv^4 \\
& + (a_1 a_4 + 2a_o a_2 a_3 + 3a_o^2 a_2 a_3 + 8a_o^3 a_1 a_4 + 5a_o^5 a_5)u^4 v \\
& + (a_1 a_5 + 2a_o a_2 a_4 + 3a_o^2 a_3^2 + 4a_o^3 a_2 a_4 + 10a_o^4 a_1 a_5 + 6a_o^6 a_6)uv^5 \\
& + (a_1 a_5 + 2a_o a_2 a_4 + 3a_o^2 a_3^2 + 4a_o^3 a_2 a_4 + 10a_o^4 a_1 a_5 + 6a_o^6 a_6)u^5 v \\
& + (2a_2^3 + 6a_o^2 a_3^2 + 8a_1^3 a_3 + 24a_o a_1 a_2 a_3 + 48a_o^2 a_1^2 a_4 + 24a_o^3 a_2 a_4 + 60a_o^4 a_1 a_5 \\
& + 20a_o^6 a_6)u^3 v^3 + (a_2^3 + 2a_o a_2 a_4 + 4a_1 a_2 a_3 + 6a_o^2 a_3^2 + 12a_o a_1 a_2 a_3 \\
& + 24a_o^2 a_1^2 a_4 + 16a_o^3 a_2 a_4 + 40a_o^4 a_1 a_5 + 15a_o^6 a_6)u^2 v^4 + (a_2^3 + 2a_o a_2 a_4 \\
& + 4a_1 a_2 a_3 + 6a_o^2 a_3^2 + 12a_o a_1 a_2 a_3 + 24a_o^2 a_1^2 a_4 + 16a_o^3 a_2 a_4 + 40a_o^4 a_2 a_4 \\
& + 15a_o^6 a_6)u^4 v^2 + \ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

From the right hand side of eqn. (19) we obtain

$$
\begin{aligned}
\text{R.H.S.} = {} & (a_o + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5 + a_6 u^6 + \ldots)(a_o + a_1 v + a_2 v^2 \\
& + a_3 v^3 + a_4 v^4 + a_5 v^5 + a_6 v^6 + \ldots) - (1 - (a_o + a_1 v + a_2 v^2 + a_3 v^3 \\
& + a_4 v^4 + a_5 v^5 + a_6 v^6 + \ldots)^2)u/v
\end{aligned}
$$

$$\begin{aligned}
\text{R.H.S.} = {} & (a_o^2 - 1)u/v + a_o^2 + 3a_oa_1u + a_oa_1v + (2a_oa_2 + 2a_1^2)uv + a_oa_2u^2 \\
& + a_oa_2v^2 + (3a_1a_2 + 2a_oa_3)uv^2 + a_1a_2u^2v + a_oa_3u^3 + a_oa_3v^3 \\
& + (3a_1a_3 + 2a_oa_4 + a_2^2)uv^3 + a_1a_3u^3v + a_2^2u^2v^2 + a_2a_3u^2v^3 \\
& + a_2a_3u^3v^2 + (3a_1a_4 + 2a_oa_5 + 2a_2a_3)uv^4 + a_1a_4u^4v + a_oa_4u^4 \\
& + a_oa_4v^4 + (3a_1a_5 + 2a_oa_6 + 2a_2a_4 + a_3^2)uv^5 + a_1a_5u^5v + a_2a_4u^2v^4 \\
& + a_2a_4u^4v^2 + a_3^2u^3v^3 + \ldots\ldots\ldots\ldots .
\end{aligned}$$

We now put L.H.S. = R.H.S.

To find a_o, a_1, a_2, a_3, a_4, a_5, and a_6 we equate the coefficients of the various products of the powers of u and v.

From the coefficients of u/v we obtain

$$a_o^2 - 1 = 0$$

or

$$a_o^2 = 1$$

and from the constant terms we obtain

$$a_o^2 = a_o .$$

Therefore

$$a_o = 1. \tag{1}$$

From the coefficients of u we obtain

$$a_oa_1 = 3a_oa_1 \text{ ,}$$

and since $a_o = 1$ this becomes

$$a_1 = 0. \tag{2}$$

From the coefficients of uv we obtain a_2 = arbitrary .

From the coefficients of uv^2 we obtain

$$a_3 = 0. \tag{3}$$

From the coefficients of uv^3 we obtain

$$a_4 = -\tfrac{1}{2}a_2^2 . \tag{4}$$

From the coefficients of uv^4 we obtain

$$a_5 = 0 . \tag{5}$$

From the coefficients of uv^5 we obtain

$$a_2 a_4 + a_6 = 0, \tag{6}$$

and since by (4) $a_4 = -\tfrac{1}{2}a_2^2$, eqn. (6) gives

$$a_6 = \tfrac{1}{2}a_2^3 . \tag{7}$$

Hence from (1), (2), (3), (4), (5), and (7)

$$a_0 = 1, \quad a_1 = a_3 = a_5 = 0, \quad a_4 = -\tfrac{1}{2}a_2^2 , \text{ and } a_6 = \tfrac{1}{2}a_2^3 .$$

The power series expansion on v is

$$f(v) = 1 + a_2 v^2 - \tfrac{1}{2}a_2^2 v^4 + \tfrac{1}{2}a_2^3 v^6 - \ldots\ldots\ldots\ldots \tag{8}$$

From the symmetric assumption, $f(v) = f(-v)$, the series f must be even. This is in agreement with the series in (8), and provides a check on the above calculations.

We have only found the first 4 terms of the series in (8) but not the general term. The above work shows that all the coefficients may be written in terms of a_2, which is arbitrary. When $a_2 = 0$, the transformation is the Galilean transformation.