CHAPTER II

ALGEBRA OF HYPERCOMPLEX NUMBER SYSTEMS

Definition I

A complex number is an ordered pair (x, y) of real numbers, x being called the real and y being the imaginary component of (x, y). Complex numbers are added and multiplied by the rules:

1. $(x, y) + (x', y') = (x + x', y + y')$
2. $(x, y) \cdot (x', y') = (xx' - yy', xy' + x'y)$ (2, p. 101)

Complex numbers can be manipulated by the laws of the algebra of real numbers by expressing the complex number (x, y) in the form $(x + iy)$ and using the rule $i^2 = -1$. The rules of equality, addition and multiplication may be written in the form,

$$(2.1)\ (x + iy) = (x' + iy') \text{ iff } x = x' \text{ and } y = y'$$

$$(2.2)\ (x + iy) \pm (x' + iy') = (x \pm x') + i\,(y \pm y')$$

$$(2.3)\ (x + iy) \cdot (x' + iy') = (xx' - yy') + i\,(xy' + x'y)$$

(2, p. 101)

With the above rules the set of complex numbers with the operations of addition and multiplication form a field. This can be proved form the definition of equality, addition, multiplication and the fact that the real number system is a field.

The complex number system is an extension of the real number system which is a complete ordered field. In modern algebra the complex number system is regarded as a two - dimensional vector space over a field of real or rational numbers with basis elements 1 and i. (6, p. 44).

Definition II

The quaternion number system is a four - dimensional vector space with basis elements 1, i, j and k over a field of real or complex numbers, in which the rule for multiplication is as given below. (8, p. 44).

A quaternion can be written $Q = 1.a + i.b + j.c + k.d$ where a, b, c, d are real or complex numbers. Using the rules $i^2 = j^2 = k^2 = -1$; $ij = k$, $jk = i$, $ki = j$, $ji = -k$, $kj = -i$, $ik = -j$ (1, p. 58) we can define multiplication for quaternions. The rules of equality, addition and multiplication may now be written in the form,

(2.4) $a_1 + ib_1 + jc_1 + kd_1 = a_2 + ib_2 + jc_2 + kd_2$

iff $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$ and $d_1 = d_2$

(2.5) $(a_1 + ib_1 + jc_1 + kd_1) \pm (a_2 + ib_2 + jc_2 + kd_2)$

$= (a_1 \pm a_2) + i(b_1 \pm b_2) + j(c_1 \pm c_2) + k(d_1 \pm d_2)$

(2.6) $(a_1 + ib_1 + jc_1 + kd_1) . (a_2 + ib_2 + jc_2 + kd_2)$

$= (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + i(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)$

$+ j(c_1a_2 + a_1c_2 + d_1b_2 - b_1d_2) + k(d_1a_2 + a_1d_2 + b_1c_2 - c_1b_2)$

According to the above rules, multiplication in the quaternion number system is associative but not commutative.

For example,

$ik = -j$

$ki = j$

therefore $ik \neq ki$,

and $(ik)k = (-j)k = -i$

$i(k.k) = i(-1) = -i$

there fore $(ik)k = i(k.k)$.

The system of quaternions also has an identity for multiplication, and every non zero quaternion has multiplicative inverses. The left and right inverse of each quaternion is unique by the following proof.

Proof.

$$\text{Let} \quad Q = a + ib + jc + kd$$

$$\text{and} \quad Q' = a' + ib' + jc' + kd',$$

where Q' is the right inverse of Q.

$$\text{then} \quad QQ' = 1.$$

$$\begin{aligned} QQ' = \; & (aa' - bb' - cc' - dd') \\ & + i\,(ba' + ab' - dc' + cd') \\ & + j\,(ca' + db' + ac' - bd') \\ & + k\,(da' - cb' + bc' + ad') \end{aligned}$$

Therefore

$$\begin{aligned} aa' - bb' - cc' - dd' &= 1 \\ ba' + ab' - dc' + cd' &= 0 \\ ca' + db' + ac' - bd' &= 0 \\ da' - cb' + bc' + ad' &= 0 \end{aligned}$$

We can solve these four equations for a', b', c', d'.

$$\text{The determinant} \begin{vmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{vmatrix} = (a^2 + b^2 + c^2 + d^2)^2 \quad \ldots.(1)$$

is not zero, except when $a = b = c = d = 0$. This shows us that the solutions a, b, c, d exist and are unique. This implies that Q has a unique right inverse Q^{-1}, such that $QQ^{-1} = 1$.

Similarly the right inverse of Q^{-1}, $(Q^{-1})^{-1}$ may be found and it is found to be Q. Therefore $Q^{-1}Q = 1$ and this implies that the right inverse of Q, Q^{-1} is also the left inverse of Q.

Because the quaternion number system is a four-dimensional vector space which is also a ring containing at least one element distinct from zero and in which the equations $ax = b$, $ya = b$, when $a \neq 0$ always have solutions x and y, the quaternion number system is called a skew field or a division algebra. (8, p.36)

Note : If any two of the coefficients of i, j or k are made zero for every quaternion in a set, then the set of quaternions so formed is isomorphic with a similar set of complex numbers.

Definition III

The Cayley number system is a hypercomplex system with eight basis elements $1, e_1, e_2, e_3 \ldots\ldots\ldots\ldots e_7$ having multiplication defined as below.

Let $a_0, a_1 \ldots\ldots\ldots\ldots a_7$ be a set of real or rational numbers. A Cayley number can be written $C = 1.\, a_0 + e_1a_1 + e_2a_2 + \ldots\ldots\ldots\ldots + e_7a_7$. The multiplication rules for the basis elements are

$$e_i^2 = -1 \qquad (i = 1, 2, \ldots\ldots\ldots\ldots 7)$$

$$e_i\, e_j = -e_je_i \qquad (j = 1, 2, \ldots\ldots\ldots\ldots 7)$$

$$e_1\, e_2 = e_3, \quad e_1e_4 = e_5, \quad e_1e_6 = e_7 = e_2e_5 = e_3e_4$$

$$e_2\, e_4 = -e_6 = e_5e_3$$

where the suffixes in each equation can be cyclically permuted.

From the above rules, we can see that multiplication in the Cayley number system is non-commutative and non-associative.

For example,

$$e_7e_5 \;=\; -\,e_2 \quad \text{and} \quad e_5e_7 \;=\; e_2 \,,$$

therefore $e_7e_5 \neq e_5e_7$

Also

$$e_4\,(e_7e_5) \;=\; e_4\,(-\,e_2) \;=\; -\,e_4e_2 \;=\; -\,e_6 \,,$$

and $(e_4e_7)\,e_5 \;=\; e_3e_5 \;=\; e_6$

therefore $e_4(e_7e_5) \neq (e_4e_7)\,e_5$

Note :

If $a_4 = a_5 = a_6 = a_7 = 0$ the set of Cayley numbers so formed is isomorphic with the set of quaternions.

The Cayley numbers have an identity for multiplication and each Cayley number has a left and right inverse.

Let C be any Cayley number

C' be its right inverse

Then $C \,.\, C' = 1$

$$\text{Let } C = a_o + e_1a_1 + e_2a_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ e_7a_7 \quad \text{and}$$

$$C' = a_o + e_1a_1 + e_2a_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ e_7a_7 \,, \text{ then}$$

$$CC' = A_o + e_1A_1 + e_2A_2 + e_3A_3 + \ldots\ldots\ldots\ldots\ e_7A_7 = 1 \quad \text{where}$$

$$A_o = (a_oa'_o - a_1a'_1 - a_2a'_2 - a_3a'_3 - a_4a'_4 - a_5a'_5 - a_6a'_6 - a_7a'_7)$$

$$A_1 = (a_1a'_o + a_oa'_1 - a_3a'_2 + a_2a'_3 - a_5a'_4 + a_4a'_5 - a_7a'_6 + a_6a'_7)$$

$$A_2 = (a_2a'_o + a_3a'_1 + a_oa'_2 - a_1a'_3 + a_6a'_4 - a_7a'_5 - a_4a'_6 + a_5a'_7)$$

$$A_3 = (a_3a'_o - a_2a'_1 + a_1a'_2 + a_oa'_3 - a_7a'_4 - a_6a'_5 + a_5a'_6 + a_4a'_7)$$

$$A_4 = (a_4a'_o + a_5a'_1 - a_6a'_2 + a_7a'_3 + a_oa'_4 - a_1a'_5 + a_2a'_6 - a_3a'_7)$$

$$A_5 = (a_5a'_o - a_4a'_1 + a_7a'_2 + a_6a'_3 + a_1a'_4 + a_oa'_5 - a_3a'_6 - a_2a'_7)$$

$$A_6 = (a_6a'_o + a_7a'_1 + a_4a'_2 - a_5a'_3 + a_2a'_4 + a_3a'_5 + a_oa'_6 - a_1a'_7)$$

$$A_7 = (a_7a'_o - a_6a'_1 - a_5a'_2 + a_4a'_3 + a_3a'_4 + a_2a'_5 + a_1a'_6 + a_oa'_7)$$

Therefore

$$A_o = 1$$

$$A_1 = A_2 = A_3 = A_4 = A_5 = A_6 = A_7 = 0$$

From these eight equations we can find the solutions a_1', a_2', a_3' .... a_7'

Consider the last seven equations, the trivial solutions are

$$a_o' = ka_o,\ a_1' = -ka_1,\ a_2' = -ka_2,\ a_3' = -ka_3,$$
$$a_4' = -ka_4,\ a_5' = -ka_5,\ a_6' = -ka_6,\ a_7' = -ka_7$$

where k is arbitrary constant. From the first equation the value of k is found to be $\dfrac{1}{a_o^2 + a_1^2 + a_2^2 + \ldots\ldots\ldots\ldots + a_7^2}$ C has a right inverse C^{-1}, such that $C\,C^{-1} = 1$.

Similarly the right inverse of C^{-1}, $(C^{-1})^{-1}$, may be found and it is found to be C. Therefore $C^{-1}C = 1$, and this implies that the right inverse of C, (C^{-1}), is also the left inverse of C.

Therefore the algebra of Cayley numbers is a non-associative division algebra.

<u>The relation between a hypercomplex number and its conjugate</u>

The product of a complex number and its conjugate is a real number which is always equal to the square of its modulus. Its inverse is the product of the reciprocal of the square of its modulus and its conjugate.

Similar facts hold for quaternions and Cayley numbers as the following work shows.

(2.7) Let $z = x + iy$ be a complex number,

then $\bar{z} = x - iy$ is its complex conjugate,

and $\bar{z}z = z\bar{z} = |z|^2 = x^2 + y^2$, where $|z|$ is the modulus of z.

(2.8) Let $Q = a + ib + jc + kd$ be a quaternion,

then $\bar{Q} = a - ib - jc - kd$ is its quaternion conjugate,

and $\bar{Q}Q = Q\bar{Q} = |Q|^2 = a^2 + b^2 + c^2 + d^2$, where $|Q|$ is the modulus of Q.

(2.9) Let $C = a_o + e_1a_1 + e_2a_2 + \dots\dots\dots + e_7a_7$ be a Cayley number

then $\bar{C} = a_o - e_1a_1 - e_2a_2 - \dots\dots\dots - e_7a_7$ is its Cayley conjugate,

and $\bar{C}C = C\bar{C} = |C|^2 = a_o^2 + a_1^2 + a_2^2 + \dots\dots\dots + a_7^2$

where $|C|$ is the modulus of C.

(2.10) Let z^{-1} be a right inverse of z

then $zz^{-1} = 1$

and $z^{-1} = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$;

(2.11) Let Q^{-1} be a right inverse of Q,

we have $QQ^{-1} = 1$, where

$$Q^{-1} = \frac{1}{Q} = \frac{\bar{Q}}{Q\bar{Q}} = \frac{\bar{Q}}{|Q|^2} ;$$

(2.12) and if C^{-1} is a right inverse of C, we have $CC^{-1} = 1$, where

$$C^{-1} = \frac{1}{C} = \frac{\bar{C}}{C\bar{C}} = \frac{\bar{C}}{|C|^2} .$$

Similarly if z^{-1}, Q^{-1}, C^{-1} are left inverse, we get the same expressions. All these results are easily proved from the definitions.

Theorem I

If Q_1 and Q_2 are any quaternions, then $\overline{Q_1Q_2} = \bar{Q}_2\bar{Q}_1$

Proof let
$$Q_1 = a_1 + ib_1 + jc_1 + kd_1$$
$$Q_2 = a_2 + ib_2 + jc_2 + kd_2$$
be the two quaternion numbers.

then
$$\bar{Q}_1 = a_1 - ib_1 - jc_1 - kd_1$$
$$\bar{Q}_2 = a_2 - ib_2 - jc_2 - kd_2$$
are their quaternion conjugates.

We have
$$Q_1Q_2 = (a_1 + ib_1 + jc_1 + kd_1)(a_2 + ib_2 + jc_2 + kd_2)$$
$$= A_o + iA_1 + jA_2 + kA_3$$

where
$$A_o = (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2)$$
$$A_1 = (b_1a_2 + a_1b_2 - d_1c_2 + c_1d_2)$$
$$A_2 = (c_1a_2 + d_1b_2 + a_1c_2 - b_1d_2), \quad A_3 = (d_1a_2 - c_1b_2 + b_1c_2 + a_1d_2$$

then
$$\overline{Q_1Q_2} = A_o - iA_1 - jA_2 - kA_3$$

but
$$\bar{Q}_2\bar{Q}_1 = A_o - iA_1 - jA_2 - kA_3$$

therefore
$$\overline{Q_1Q_2} = \bar{Q}_2\bar{Q}_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Theorem II

For any two Cayley numbers C_1 and C_2, $\overline{C_1C_2} = \bar{C}_2\bar{C}_1$

Proof Similar to the proof of theorem I