CHAPTER II

A METHOD TO SOLVE THE EQUATION OF A VIBRATING NON - UNIFORM BAR

We shall first start the method by considering a symmetrical non-uniform bar vibrating under its own weight; the load per unit length is equal to the inertial load due to its mass and acceleration[1]. The rate of change of the moment along the bar is equal to the shear. We find, by using these relations, that the Euler equation[2] for the bar should be

$$\frac{\partial^2}{\partial x^2}\left[S(x)\,K^2(x)\,\frac{\partial^2\psi}{\partial x^2}\right] = -\rho\,\frac{S(x)}{g}\,\frac{\partial^2\psi}{\partial t^2} \qquad (1)$$

To study the normal modes of the periodic vibration phenomena we can assume that

$$\psi = Y(x)\,e^{i\omega t}$$

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[1] WILLIAM T. THOMSON, Vibration Theory and Application (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1965), p.275.

[2] ARTHUR BRONWELL, Advanced Mathematics in Physics and Engineering (New York: McGraw-Hill Company, Inc., 1953), p.151-153.

and then the eq.(1) can be reduced to the form

$$\frac{d^2}{dx^2} \left| S(x) K^2(x) \frac{d^2y}{dx^2} \right| = \frac{\omega^2 \rho S(x)}{Q} \cdot Y \qquad (2)$$

which is the equation of motion for a non-uniform bar. In this case we have the following boundary conditions:

$$y''_{(0)} = y'''_{(0)} = 0$$

$$y(x_{10}) \neq 0, \quad y'(x_{10}) = 0$$

and $$y'''_{(x_{10})} \neq 0$$

In the case of a rectangular cross-section with constant base b and variable altitude a(x) we have $SK^2 = \frac{a^3 b}{12}$.

The next step is to substitute $SK^2 = \frac{a^3 b}{12}$ and $S = ab$ into equation (2). We have

$$\frac{d^2}{dx^2} \left(\frac{a^3(x)}{12} b \frac{d^2}{dx^2} y \right) = \omega^2 \rho ab\, y$$

or $$\frac{d^2}{dx^2} \left(a^3 (x) \frac{d^2 y}{dx^2} \right) = cf^2 a(x)\, y$$

where $c = \frac{48\pi^2 \rho}{Q}$, and for iron $\rho = 7.8\ gm/_{cm^3}$

$Q = 20 \times 10^{11}$ dyne/cm^2. Therefore $c = 0.185 \times 10^{-9}$ sec^2/cm^2

Putting $z = a^3 y''$, we have the simultaneous equations as follows:

$$y'' = \frac{z}{a^3} \qquad (4)$$

and

$$z'' = c\, f^2\, a\, y \qquad (5)$$

Assume, for the symmetrical bar, that the length of the bar is divided into 20 equal parts. If $y_0, y_1, y_2, \text{---} y_{20}$ denote the lateral displacements at the points $x_0, x_1, \text{---} x_{20}$, where x_0 and x_{20} are the end points. The finite difference approximation[3] at x_1 for the equations (4) and (5) are :

$$y_2 - 2y_1 + y_0 = h^2 \frac{z_1}{a_1^3}$$

and

$$z_2 - 2z_1 + z_0 = h^2\, cf^2\, a_1 y_1$$

or

$$y_2 = \frac{z_1}{a_1^3} + 2y_1 - y_0$$

$$z_2 = cf^2\, a_1 y_1 + 2z_1 - z_0$$

where $h = 1$

Using a step by step calculation we have

$$y_{i+1} = \frac{z_i}{(a_i^3)} + 2y_i - y_{i-1} \qquad (6)$$

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[3] F.B.HILDEBRAND, Introduction to Numerical Analysis (New York: McGraw-Hill book Company, Inc.,1956),p.91.

$$z_{i+1} = cf^2\, a_i y_i + 2z_i - z_{i-1} \qquad (7)$$

$$i = 1, 11 \;;\; y_{10} = y_{10}$$

$$y'_{10} = 0.5\,(y_{11} - y_9)$$

$$y''_{10} = y_{11} - 2y_{10} + y_9$$

$$y'''_{10} = 0.5\,(y_{12} - 2y_{11} + 2y_9 - y_8)$$

At the free end put $y_o = -1$. This implies

$y'_o = 0.5\,(y_1 - y_{-1}) = 0$. y_1 is to be varied.

The boundary conditions, require $z_8 = 0$, $z'_o = 0$. These imply $z_1 = z_{-1}$.

Therefore $\quad y_1 = -y_{-1} \qquad (8)$

and $\quad z_1 = \dfrac{cf^2 a_o y_o}{2} \qquad (9)$

Given $a(x_i)$ and assuming some trial values for the frequency, the system can be solved by means of the digital computer (IBM 1620) as follows:

$$a(x_i) = a(x_{i+1}) = 1.0$$

$F = 1000$, $DF = 1000$, $y_1 = -1.0$, $DY_1 = 0.2$ where F denotes frequency; DF denotes the increment of F; Y_1, denotes the displacement at x_1, DY_1 denotes the increment of Y_1.

The results obtained are shown in the tables 3 - 6; x_{10} is the center point. From these tables we can obtain the vibrating frequency and the mode shapes in the next chapter, which is necessary to refer frequently to mathematical concepts such as linear interpolation and inverse interpolation. For convenience these have been collected in the appendix.

Tables 3-6, the results obtained by using electronic computer (IBM 1620).

Table 3

F=1000	y_1	y_{10}	y'_{10}	y''_{10}	y'''_{10}
	-1.0	-1.7700	-3.1640	-0.09703	-0.02136
	-0.8	0.5239	0.03765	-0.03480	-0.004961
	-0.6	2.8180	0.39170	0.02742	0.01744
	-0.4	5.1120	0.74580	0.08964	0.03685
	-0.2	7.4070	1.09900	0.15180	0.05625
	0.0	9.7010	1.4540	0.21410	0.7566

Table 4

F=2000	y_1	y_{10}	y'_{10}	y''_{10}	y'''_{10}
	-1.0	-4.1700	-1.3450	-0.4430	-0.1206
	-0.8	-0.9762	-0.5109	-0.1799	-0.03199
	-0.6	2.2170	0.3231	0.08311	0.05662
	-0.4	5.4110	1.1570	0.3462	0.1452
	-0.2	8.6060	1.9910	0.6093	0.2338
	0.0	11.8000	2.8250	0.8724	0.3224

Table 5

F=4000	y_1	y_{10}	y'_{10}	y''_{10}	y'''_{10}
	-1.0	-15.1100	-6.6830	-2.6880	-1.0890
	-0.8	-8.0760	-3.6590	-1.4040	-0.5520
	-0.6	-1.0340	-0.6359	-0.1195	-0.01429
	-0.4	6.0080	2.3870	1.1650	0.5234
	-0.2	13.0500	5.4110	2.4490	1.0610
	0.0	20.0900	8.4350	3.7340	1.5980

Table 6

Mode shapes for F = 4000 Hz.

y_0	y_1	y_2	y_3	y_4	y_5
-1.00	-1.00	-1.0148000	-1.0888000	-1.2964380	-1.7443806
-1.00	-0.80	-0.6148000	-0.4828800	-0.4609180	-0.6198854
-1.00	-0.60	-0.2148000	0.1230400	0.3746019	0.5046097
-1.00	-0.40	0.1852000	0.7289600	1.2101219	1.6291049
-1.00	-0.20	0.5852999	1.3348800	2.0456420	2.7536003
-1.00	0.00	0.9852000	1.9048000	2.8811620	3.8780955

y_6	y_7	y_8	y_9	y_{10}
-2.577689	-3.9929777	-6.2632808	-9.7797440	-15.1189270
-1.0443966	-1.8374147	-3.1328170	-5.1188682	-8.0765640
0.4888754	0.3181476	-0.0023544	-0.4579942	1.0342051
2.0221478	2.4737107	3.1281094	4.2028816	6.0081570
3.5554202	4.6292736	6.2585729	8.8637570	13.0505190
5.0886925	6.7848370	9.3890380	13.5246350	20.0928830

By varying the inputs we generate another set of data as follows: F = 1000, DF = 100, Y_1 = 1.0, DY_1 = 0.1 The results obtained are shown in the table 7 - 14.

Table 7

F=1000	y_1	y_{10}	y'_{10}	y''_{10}	y'''_{10}
	-1.00	-1.770000	-0.316400	-0.097030	-0.021360
	-0.90	-0.623200	-0.139300	-0.065910	-0.011660
	-0.80	0.523900	0.037650	-0.034800	-0.001961
	-0.70	1.671000	0.214700	-0.003690	+0.007741
	-0.60	2.818000	0.391700	0.027420	0.017440
	-0.50	3.965000	0.568700	0.058530	0.027440

Table 8

F=1100	y_1	y_{10}	y'_{10}	y''_{10}	y'''_{10}
	-1.00	-1.934000	-0.384500	-0.118500	-0.026580
	-0.90	-0.755800	-0.191100	-0.080760	-0.014730
	-0.80	0.422400	0.002268	-0.042960	-0.002875
	-0.70	1.600000	0.195600	-0.005171	0.008980
	-0.60	2.779000	0.389000	0.032620	0.020830
	-0.50	3.957000	0.582500	0.070420	0.032690

Table 9

F=1200	y_1	y_{10}	y'_{10}	y''_{10}	y'''_{10}
	-1.00	-2.114000	-0.459800	-0.142600	-0.032590
	-0.90	-0.901700	-0.248400	-0.097420	-0.018330
	-0.80	0.310600	-0.037000	-0.052250	-0.004078
	-0.07	1.523000	0.174400	-0.070760	0.010180
	-0.60	2.735000	0.385800	0.038090	0.024440
	-0.50	3.947000	0.597200	0.083270	0.038700

Table 10

F=1300	y_1	y_{10}	y'_{10}	y''_{10}	y'''_{10}
	-1.00	-2.310000	-0.542400	-0.169200	-0.039480
	-0.90	-1.061000	-0.311300	-0.116000	-0.022550
	-0.80	0.188400	-0.080310	-0.062750	-0.005626
	-0.70	1.438000	0.150700	-0.009482	0.011300
	-0.60	2.687000	0.381800	0.0437800	0.028230
	-0.50	3.937000	0.612800	0.097050	0.045160

Table 11

F=1400	y_1	y_{10}	y'_{10}	y''_{10}	y'''_{10}
	-1.00	-0.523000	-0.632600	-0.198700	-0.047330
	-0.90	-1.334000	-0.380200	-0.136600	-0.027450
	-0.80	0.055640	-0.127800	-0.074560	-0.007581
	-0.70	1.345000	0.124500	-0.012470	0.012290
	-0.60	2.635000	0.376900	0.0496100	0.032160
	-0.50	3.925000	0.629300	0.111700	0.052040

Table 12

F=1500	y_1	y_{10}	y'_{10}	y''_{10}	y'''_{10}
	-1.00	-2.754000	-0.730500	-0.231100	-0.056230
	-0.90	-1.421000	-0.455000	-0.159400	-0.033120
	-0.80	-0.087990	-0.179600	-0.087810	-0.010000
	-0.70	1.245000	0.095760	-0.016140	0.013000
	-0.60	2.578000	0.371100	0.055510	0.036210
	-0.50	3.911000	0.646600	0.127100	0.059330

Table 13

F=1600	y_1	y_{10}	y'_{10}	y''_{10}	y'''_{10}
	-1.00	-3.001000	-0.836400	-0.266600	-0.066300
	-0.90	-1.622000	-0.536200	-0.184600	-0.039640
	-0.80	-0.242600	-0.236000	-0.102600	-0.012980
	-0.70	1.113600	0.064140	-0.020590	0.013670
	-0.60	2.516000	0.364300	0.064400	0.040330
	-0.50	3.896000	0.564500	0.143400	0.067000

Table 14

F=1700	y_1	y_{10}	y'_{10}	y''_{10}	y'''_{10}
	-1.00	-3.266000	-0.950500	-0.305300	-0.077650
	-0.90	-1.837000	-0.623800	-0.212200	-0.047110
	-0.80	-0.408500	-0.297100	-0.119000	-0.016580
	-0.70	1.020000	0.029540	-0.025930	0.013950
	-0.66	2.449000	0.356200	0.067200	0.044480
	-0.50	3.878000	0.682900	0.160300	0.075020