CHAPTER II

THE SUMS OF BASIC SEQUENCES AND THEIR PERIODS

First we consider any infinite sequence of binary digits

$$H = h_1h_2h_3h_4h_5h_6h_7h_8h_9h_{10} \dots$$

We may divide it into equal sets of consecutive digits. For example, H may be divided into sets containing three consecutive digits each thus :

$$H = h_1h_2h_3 \; ; \; h_4h_5h_6 ; \; h_7h_8h_9 ; \; \dots$$

We shall write

$$H^{(1)(3)} = h_1h_2h_3,$$

$$H^{(2)(3)} = h_4h_5h_6,$$

$$H^{(3)(3)} = h_7h_8h_9, \quad \text{and so on.}$$

Then $$H = H^{(1)(3)}\, H^{(2)(3)}\, H^{(3)(3)} \dots$$

In general, when the sequence H is divided into sets containing n consecutive digits each, we shall represent the m-th set by the symbol $H^{(m)(n)}$. Thus we may write

$$H = H^{(1)(n)}\, H^{(2)(n)}\, H^{(3)(n)} \dots$$

<u>Definition</u> A sequence H has period n if there exists a least positive integer n such that

$$H^{(1)(n)} = H^{(2)(n)} = H^{(3)(n)} = \dots = H^{(m)(n)} = \dots ,$$

for all positive integers m.

Definition H^* is defined to be the sequence obtained from H by replacing all 0's by 1's and all 1's by 0's. H^* will be called the complement of H.

Define

$$O = 0000000000 \ldots$$

$$I = 1111111111 \ldots$$

$$A_1 = 0101010101 \ldots$$

$$A_2 = 001100110011 \ldots$$

$$A_3 = 000111000111 \ldots$$

$$A_4 = 0000111100001111 \ldots$$

$$A_5 = 00000111110000011111 \ldots$$

$$A_6 = 000000111111000000111111 \ldots$$

$$\ldots$$

Definition The sequences A_k shall be called basic sequences.

Define

$$0 + 0 = 0,$$

$$0 + 1 = 1,$$

$$1 + 0 = 1,$$

$$1 + 1 = 0.$$

This definition of the operation denoted by the symbol + makes $\{0, 1 ; +\}$ an Abelian group.

Let H and K be two sequences

$$H = h_1 h_2 h_3 \ldots h_i \ldots$$

$$K = k_1 k_2 k_3 \ldots k_i \ldots$$

Then we define

$$H + K = h_1 + k_1, h_2 + k_2, h_3 + k_3, \ldots, h_i + k_i, \ldots,$$

where $h_i + k_i$ is computed as above.

In theorems 1 to 6 we give the basic properties of the + operation.

<u>Theorem 1</u> If H and K are any two sequences of binary digits, then

$$H + K = K + H.$$

<u>Proof</u> <u>Case 1</u> For all i, if $h_i = 0$, and $k_i = 0$, then

$$h_i + k_i = 0 + 0 = 0 = 0 + 0 = k_i + h_i.$$

<u>Case 2</u> For all i, if $h_i = 0$, and $k_i = 1$, then

$$h_i + k_i = 0 + 1 = 1 = 1 + 0 = k_i + h_i.$$

<u>Case 3</u> For all i, if $h_i = 1$, and $k_i = 0$, then

$$h_i + k_i = 1 + 0 = 1 = 0 + 1 = k_i + h_i.$$

<u>Case 4</u> For all i, if $h_i = 1$, and $k_i = 1$, then

$$h_i + k_i = 1 + 1 = 0 = 1 + 1 = k_i + h_i.$$

Hence $H + K = K + H$.

<u>Q.E.D</u>.

Note that this commutative law is a result of the fact that $\{0, 1, +\}$ is an Abelian group.

<u>Theorem 2</u> If H, K and R are any three sequences of binary digits, then $H + (K + R) = (H + K) + R$.

<u>Proof</u> <u>Case 1</u> For all i, if $h_i = 0$, $k_i = 0$, $r_i = 0$, then

$$h_i + (k_i + r_i) = 0 + (0 + 0) = 0 = (0+0)+0 = (h_i+k_i) + r_i.$$

<u>Case 2</u> For all i, if $h_i = 0$, $k_i = 0$, $r_i = 1$, then

$$h_i+(k_i+ r_i)=0+(0+1)=1=(0+0)+1=(h_i+k_i) + r_i \text{ .}$$

<u>Case 3</u> For all i, if $h_i = 0$, $k_i = 1$, $r_i = 0$, then

$$h_i+(k_i+ r_i)=0+(1+0)=1=(0+1)+0=(h_i+k_i) + r_i \text{ .}$$

<u>Case 4</u> For all i, if $h_i = 0$, $k_i = 1$, $r_i = 1$, then

$$h_i+ (k_i+ r_i)=0+(1+1)=0=(0+1)+1=(h_i+k_i) + r_i.$$

<u>Case 5</u> For all i, if $h_i = 1$, $k_i = 0$, $r_i = 0$, then

$$h_i+(k_i+ r_i) = 1+(0+0)=1=(1+0)+0 = (h_i+k_i) + r_i \text{ .}$$

<u>Case 6</u> For all i, if $h_i = 1$, $k_i = 0$, $k_i = 0$, $r_i = 1$, then

$$h_i+ (k_i+ r_i) = 1+(0+1)=0=(1+0)+1 = (h_i+ k_i) + r_i \text{ .}$$

<u>Case 7</u> For all i, if $h_i = 1$, $k_i = 1$, $r_i = 0$, then

$$h_i+ (k_i+ r_i) + 1+(1+0) = 0 =(1+1) + 0 = (h_i+k_i) + r_i.$$

<u>Case 8</u> For all i, if $h_i = 1$, $k_i = 1$, $r_i = 1$, then

$$h_i+ (k_i+ r_i) = 1+(1+1)=1=(1+1)+1 = (h_i+ k_i) + r_i.$$

Hence $H + (K + R) = (H + K) + R$.

<u>Q.E.D.</u>

Note that the associative law is a consequence of the fact that $\{0, 1, +\}$ is a group.

<u>Theorem 3</u> For any sequence H, $H + 0 = H$.

<u>Proof</u> For all i, if $h_i = 0$, then $h_i + 0_i = 0 + 0 = 0 = h_i$, and if $h_i = 1$, then $h_i + 0_i = 1 + 0 = 1 = h_i$.

Hence $H + O = H$.

Q.E.D.

Note that the sequence O is the additive identity.

Theorem 4 If H is any sequence, then $H + H = O$.

Proof For all i, if $h_i = 0$, then $h_i + h_i = 0 + 0 = 0 = 0_i$,

and if $h_i = 1$, then $h_i + h_i = 1 + 1 = 0 = 0_i$.

Hence $H + H = O$.

Q.E.D.

Note that the sequence H is the additive inverse of itself.

Theorem 5 If H^* is the complement of the sequence H, then

$$H + H^* = I.$$

Proof For all i, if $h_i = 0$, $h_i^* = 1$, then $h_i + h_i^* = 0+1=1= i_i$,

and if $h_i = 1$, $h_i^* = 0$, then $h_i + h_i = 1 + 0 = 1 = i_i$,

Hence $H + H^* = I$.

Q.E.D.

Theorem 6 If H^* and K^* are the complements of the sequences H and K respectively, then $H + K = H^* + K^*$.

Proof Since $H + K = H + K + O + O$, by theorem 3,

$= (H + K) + (H^* + H^*) + (K^* + K^*)$, by theorem 4,

$= (H + H^*) + (K + K^*) + (H^* + K^*)$, by theorem 2,

$= I + I + (H^* + K^*)$, by theorem 5,

$= O + H^* + K^*$, by theorem 4,

Hence $H + K = H^* + K^*$, by theorem 3.

Q.E.D.

We shall now examine the periods of the sums of basic sequences, and give general methods of finding these periods in theorems 7 and 8.

Consider the addition of any two different basic sequences. Table I is a list of all the sums $H_{i,j} = A_i + A_j$ and their periods for $i \neq j;\ i\ ,\ j \leq 6$.

Table I the sequences $H_{i,j}$ and their periods for $i \neq j;\ i,\ j \leq 6$.

Sequence	Period
$H_{1,2}$ = 0110,01100110...	4
$H_{1,3}$ = 010,010010...	3
$H_{1,4}$ = 01011010,0101101001011010...	8
$H_{1,5}$ = 01010,0101001010...	5
$H_{1,6}$ = 010101101010,010101101010010101101010...	12
$H_{2,3}$ = 001011110100,001011110100001011110100...	12
$H_{2,4}$ = 00111100,0011110000111100..	8
$H_{2,5}$ = 00110100111100101100,00110100111100101100...	20
$H_{3,4}$ = 000100110111111011001000,0001001101111111...	24
$H_{3,5}$ = 000100111011000 ,000100111011000...	15
$H_{3,6}$ = 000111111000,000111111000...	12
$H_{4,5}$ = 0000100011001110111111110111001100010000,00001...	40
$H_{4,6}$ = 000011001111111100110000,0000110011111111...	24
$H_{5,6}$ = 000001000011000111001111011111111111011110011 1 000110000100000,000001000011000...	60
$H_{2,6}$ = 001100,001100001100...	6

Let N be the set of all positive integers.

For any two different basic sequences A_i and A_j with periods $2i$ and $2j$ respectively, where i,j are in N, let L_{ij} be the least common multiple of $2i$ and $2j$,

and let $H_{i,j} = A_i + A_j$.

If we divide the sequences A_i and A_j into ~~sets~~ containing $L_{ij}/2$ consecutive digits, then we have,

if $q = L_{ij}/2$,

$$A_i = A_i^{(1)(q)} \; A_i^{(2)(q)} \; A_i^{(3)(q)} \; \ldots ,$$

$$A_j = A_j^{(1)(q)} \; A_j^{(2)(q)} \; A_j^{(3)(q)} \; \ldots ,$$

$$H_{i,j} = H_{i,j}^{(1)(q)} \; H_{i,j}^{(2)(q)} \; H_{i,j}^{(3)(q)} \; \ldots ,$$

where $H_{i,j}^{(m)(q)} = A_i^{(m)(q)} + A_j^{(m)(q)}$, for all m in N.

From table 1 we can see that the sequence $H_{i,j}$ has period either $P_{ij} = L_{ij}/2$ or $P_{ij} = L_{ij}$ determined as follows :

Let $q = L_{ij}/2$

(1) If both $(A_i^{(m)(q)})^* = A_i^{(m+1)(q)}$ and

$(A_j^{(m)(q)})^* = A_j^{(m+1)(q)}$, for all m in N,

then $P_{ij} = L_{ij}/2$.

(2) If either $(A_i^{(m)(q)})^* \neq A_i^{(m+1)(q)}$ or

$(A_j^{(m)(q)})^* \neq A_j^{(m+1)(q)}$, for all m in N, then $P_{ij} = L_{ij}$.

These results are consistent with the next theorem.

<u>Theorem 7</u> The period of $H_{i,j} = A_i + A_j$, where i, j are in N, and $i \neq j$, is given as follows :

Let $q = L_{ij}/2$

(1) If both $(A_i^{(m)(q)})* = A_i^{(m+1)(q)}$ and

$(A_j^{(m)(q)})* = A_j^{(m+1)(q)}$, for all m in N,

then $P_{ij} \leq L_{ij}/2$.

(2) If either $(A_i^{(m)(q)})* \neq A_i^{(m+1)(q)}$ or

$(A_j^{(m)(q)})* \neq A_j^{(m+1)(q)}$, for all m

in N, then $P_{ij} \leq L_{ij}$.

<u>Proof of (1)</u> By the hypothesis we have

$$A_i^{(m+1)(q)} + A_j^{(m+1)(q)} = (A_i^{(m)(q)})* + (A_j^{(m)(q)})* ,$$

for all m in N.

And by theorem 6, the right hand side is equal to

$$A_i^{(m)(q)} + A_j^{(m)(q)} , \text{ for all m in N .}$$

Therefore $H_{i,j}^{(m+1)(q)} = H_{i,j}^{(m)(q)}$, for all m in N.

Hence $P_{ij} \leq L_{ij}/2$ <u>Q.E.D.</u>

<u>Proof of (2)</u> By the hypothesis we have

$$A_i^{(m+1)(q)} + A_j^{(m+1)(q)} \neq (A_i^{(m)(q)})* + (A_j^{(m)(q)})*$$

for all m in N.

And by theorem 6, the right hand side is equal to

$$A_i^{(m)(q)} + A_j^{(m)(q)} , \text{ for all m in N.}$$

We have $H_{ij}^{(m+1)(q)} \neq H_{ij}^{(m)(q)}$, for all m in N.

But $A_i^{(m+1)(2q)} = A_i^{(m)(2q)}$, for all m in N.

and $A_j^{(m+1)(2q)} = A_j^{(m)(2q)}$, for all m in N .

Therefore $A_i^{(m+1)(2q)} + A_j^{(m+1)(2q)} = A_i^{(m)(2q)} + A_j^{(m)(2q)}$, for all m in N, and $H^{(m+1)(2q)} = H^{(m)(2q)}$, for all m in N.

Hence $P_{ij} \leq L_{ij}$.

Q.E.D.

In all the examples listed above the equalities hold, and it seems likely that the equalities hold in general.

Consider the addition of any three or more different basic sequences . Table 2 is a list of all the sums $A_i + A_j + \dots + A_k$ and their periods for $i \neq j \neq \dots \neq k$, $i, j, \dots, k \leq 6$.

Table 2 The sequences $H_{i,j,\dots,k}$ and their periods for $i \neq \dots \neq k$, $i, \dots, k \leq 6$.

Sequence			Period
$H_{1,2,3}$	=	011110100001,0111101000010...	12
$H_{1,2,4}$	=	01101001,01101001...	8
$H_{1,2,5}$	=	01100001101001111001,01100001101...	20
$H_{1,2,6}$	=	011001,011001011001...	6
$H_{1,3,4}$	=	010001100010101110011101,0100011...	24
$H_{1,3,5}$	=	010011101110010101100010001101,0100...	30
$H_{1,3,6}$	=	010010101101,010010101101...	12
$H_{1,4,5}$	=	0101110110011011101010100010011001000101,010111...	40
$H_{1,4,6}$	=	010110011010101001100101,0101100...	24
$H_{1,5,6}$	=	010100010110010010011010001010101011101001101101100101110101,010100010...	60
$H_{2,3,4}$	=	001000000100110111111011,001000000....	24
$H_{2,3,5}$	=	001010001000001100000100010100110101110111110011111011101011,00101000...	60
$H_{2,3,6}$	=	001011,001011001011...	6
$H_{2,4,5}$	=	0011101111111101110011000100000000100011,001110...	40
$H_{2,4,6}$	=	001111111100110000000011,0011111111...	24
$H_{2,5,6}$	=	001101110000001011111100010011,00110...	30

Sequences			Period
$H_{3,4,5}$	=	0001010010111111001110000110111101100001110011111101001010001110101101000000110001111001000 1001111000110000001011010111,...	120
$H_{3,4,6}$	=	0001000010001110111110111,00010000...	24
$H_{3,5,6}$	=	000110000100000000000100001100011100111101111111111011110011,0001100001...	60
$H_{4,5,6}$	=	0000101100111110110000000111000011100000001101111100110100001111010011000010011111111000 1111000111111100100000110010111,...	120
$H_{1,2,3,4}$	=	011101010001100010101110,01110101...	24
$H_{1,2,3,5}$	=	0111110111010110010100010000011000001000101001101011101111110,...	60
$H_{1,2,3,6}$	=	011110,011110011110...	6
$H_{1,2,4,5}$	=	0110111010101000100110010001010101110110,...	40
$H_{1,2,4,6}$	=	011010101001100101010110,...	24
$H_{1,2,5,6}$	=	011000100101011110101001000110,...	30
$H_{1,3,4,5}$	=	0100000111101010011011011011101000110100100110101000011111011011111000010101100100100 10001011100101101100101011110000010,...	120
$H_{1,3,4,6}$	=	010001011101101110100010,...	24
$H_{1,3,5,6}$	=	010011010001010101011101001101101100101110101010100010110010,...	60
$H_{1,4,5,6}$	=	0101111001101011100101010010010110110101011000101001100001011010000110010100011010101011 011010010010101001110101100111101 0,...	120

Sequence	Period
$H_{2,3,4,5}$ = 0010011110001100000010110101110001010010111111001110000110111101100001110011111101001 01000111010110100000011000111100100,...	120
$H_{2,3,4,6}$ = 001000111011110111000100,...	24
$H_{2,3,5,6}$ = 001010110111001100111011010100,...	30
$H_{2,4,5,6}$ = 0011100000000110111110011010000111101001100000100111111100011110001111111001000000011 00101111000010110011111011000000011100,...	120
$H_{3,4,5,6}$ = 0001011101001111000001110110110010010001111100001101000101111110100010110000111111000 1001001101101110000011110010111101000,...	120
$H_{1,2,3,4,5}$ = 0111001011011001010111110000010010000011110101001101101001110100011010010011010100000 1111101101111110000101011001001011000 1,...	120
$H_{1,2,3,4,6}$ = 011101101110100010010001,...	24
$H_{1,2,3,5,6}$ = 011111100010011001101110000001,...	30
$H_{1,2,4,5,6}$ = 0110110101011000101001100000101101000011001010001101010110110100100101010011101011 00 1111010010111110011010111001010101001001,...	120
$H_{1,3,4,5,6}$ = 0100001000011010010100100011100111000100101001011000010000101011110111100101101011 0 111000110001110110101101001111011101,...	120
$H_{2,3,4,5,6}$ = 00100100011111000011010001011111101000101100000111110001001001101101110000001111 0010 11101000000101110100111110000011 1011011,...	120
$H_{1,2,3,4,5,6}$ = 01110001001010010110000100001010111101111001011010110111000110001110110101101001111 0 1111010100001000011010010100100011 10,...	120

Theorem 8   Let $A_i, A_j, \dots, A_k, A_r, \dots$ be different basic sequences with periods $2i, 2j, \dots, 2k, 2r, \dots$ respectively, where $i, j, \dots, k, r, \dots$ are in N

Let $H_{i,j,\dots,k} = A_i + A_j + \dots + A_k$;

and let $P_{ij\dots k}$ be the period of the sequence $H_{i,j,\dots,k}$;

Let $L_{ij\dots kr}$ be the least common multiple of $P_{ij\dots k}$ and $2r$.

The period of the sequence $H_{i,j,\dots,k,r}$ is given as follows :

Let $q = L_{ij\dots kr}/2$

(1) If both $(H^{(m)(q)}_{i,j,\dots,k})^* = H^{(m+1)(q)}_{i,j,\dots,k}$ and

$(A_r^{(m)(q)})^* = A_r^{(m+1)(q)}$, for all m in N, then $P_{ij\dots kr} \leq L_{ij\dots kr}/2$.

(2) If either $(H^{(m)(q)}_{i,j,\dots,k})^* \neq H^{(m+1)(q)}_{i,j,\dots,k}$ or $(A_r^{(m)(q)})^* \neq A_r^{(m+1)(q)}$, for all m in N, then $P_{ij\dots kr} \nleq L_{ij\dots kr}$.

Proof   Similar to theorem 7.

As before we make the conjecture that the equalities hold in all cases,