CHAPTER II

AN ALGEBRAIC FORMULATION OF PATH PROBLEMS

In this chapter, the strong and weak closures of a stable element of a path algebra are defined and an arc-value matrix A of a digraph over a path algebra is given. When A is stable, $A^*\mathbf{b}$ where A^* is the strong closure of A and $\mathbf{b}$ is either a column vector of A or the unit matrix E of a path algebra $(M_n(P), \oplus, \otimes)$, is shown to be a solution of a matrix equation $\mathbf{y} = A\mathbf{y} \oplus \mathbf{b}$. The solution is determined by the Jordan elimination method. Then the entry a^*_{ij} or $\hat{a}_{ij}$ of $A^*\mathbf{b}$ is a solution of a path problem.

2.1 Arc-value Matrices

Let x be an element of a path algebra $(P, \oplus, \otimes)$. The **powers** of an element x are defined by

$$x^0 = e, \; x^k = x^{k-1}x \qquad \text{for all } k \in \mathbb{N}, \qquad (2.1.1)$$

where e is the unit of P.

An element x is said to be **stable** if there is $q \in \mathbb{N}_0$ such that $\oplus\sum_{k=0}^{q} x^k = \oplus\sum_{k=0}^{q+1} x^k$, and the **stability index** of an element x is the least such integer q if the element x is stable.

Let x be a stable element of a path algebra $(P, \oplus, \otimes)$ with the stability index q. Then

$$\oplus\sum_{k=0}^{q} x^k = \oplus\sum_{k=0}^{r} x^k \qquad \text{for all integers } r \geq q,$$

and

$$\oplus\sum_{k=1}^{q+1} x^k = \oplus\sum_{k=1}^{r} x^k \qquad \text{for all integers } r \geq q+1.$$

The **strong closure** of a stable element x of index q is defined as $\oplus\sum_{k=0}^{q} x^k$, denoted by x^*, and the **weak closure** of the element x is defined as $\oplus\sum_{k=1}^{q+1} x^k$, denoted by $x^\wedge$.

Let $M_n(P)$ be the set of all $n \times n$ matrices whose entries belong to P. Given any matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ in $M_n(P)$, the addition of A and B is defined by

$$A \oplus B = C$$

where $C = [c_{ij}]$ is the $n \times n$ matrix with entries $c_{ij} = a_{ij} \oplus b_{ij}$, and the multiplication of A and B is defined by

$$A \otimes B = AB = C$$

where $C = [c_{ij}]$ is the $n \times n$ matrix with entries $c_{ij} = \oplus\sum_{k=1}^{n}(a_{ik} \otimes b_{kj})$. Then $(M_n(P), \oplus, \otimes)$ is a path algebra whose zero and unit are $n \times n$ matrices

$$\Phi = \begin{bmatrix} \theta & \theta & \cdots & \theta \\ \theta & \theta & \cdots & \theta \\ \vdots & \vdots & & \vdots \\ \theta & \theta & \cdots & \theta \end{bmatrix} \text{ and } E = \begin{bmatrix} e & \theta & \cdots & \theta \\ \theta & e & \cdots & \theta \\ \vdots & \vdots & & \vdots \\ \theta & \theta & \cdots & e \end{bmatrix} \text{ respectively,}$$

where θ and e are the zero and unit of P respectively.

The arc-value matrix of a digraph $G = (X, U, v)$ over a path algebra $(P, \oplus, \otimes)$ is an $n \times n$ matrix $A = [a_{ij}]$ where

$$a_{ij} = \begin{cases} v(i, j) & \text{if } (i, j) \in U, \\ \theta & \text{otherwise,} \end{cases}$$

where θ is the zero of P. Thus, $A \in M_n(P)$.

Let A be any matrix belonging to $M_n(P)$. By (2.1.1), the powers of A are

$$A^0 = E, \; A^k = A^{k-1}A \qquad (k = 0, 1, 2, \ldots)$$

where E is the unit matrix of $M_n(P)$.

Let A be the arc-value matrix of a digraph G = (X, U, v) over a path algebra (P, ⊕, ⊗). For each $k \in \mathbb{N}_0$, the entry a_{ij}^k of A^k is given by

$$a_{ij}^k = \oplus\sum_{\mu \in S_{ij}^k} v(\mu) \qquad (2.1.2)$$

where S_{ij}^k = {μ / μ is a dipath of order k from node i to node j in G} [3].

For each $h \in \mathbb{N}_0$, if $A^{[h]}$ denotes $\oplus\sum_{k=0}^{h} A^k$ and $A^{[h]} = [a_{ij}^{[h]}]$ then from (1.2.1(iv)) and (2.1.2), $A^{[h]}$ has the entry

$$a_{ij}^{[h]} = \oplus\sum_{\mu \in T_{ij}^h} v(\mu), \qquad (2.1.3)$$

where T_{ij}^h = {μ / μ is a dipath of order r, 0 ≤ r ≤ h, from node i to node j in G}.

Since $M_n(P)$ is a path algebra, then all the definitions and results of stability can be applied to matrices. Therefore, if A is stable, then the entry a_{ij}^* of the strong closure A^* of A is given by

$$a_{ij}^* = \oplus\sum_{\mu \in T_{ij}^h} v(\mu) \qquad \text{for some } h \in \mathbb{N}_0, \qquad (2.1.4)$$

and the entry $\hat{a}_{ij}$ of the weak closure $\hat{A}$ of A is given by

$$\hat{a}_{ij} = \oplus\sum_{\mu \in W_{ij}^k} v(\mu) \qquad \text{for some } k \in \mathbb{N}, \qquad (2.1.5)$$

where W_{ij}^k = {μ / μ is a dipath of order r, 1 ≤ r ≤ k, from node i to node j in G}.

Some sufficient conditions for an arc-value matrix A of a digraph G = (X, U, v) over a path algebra (P, ⊕, ⊗) to be stable are the following statements [3].

(1) If for every elementary cycle γ in G, v(γ) ⊕ e = e where e is the unit of P, then A is stable.

(2) If A is nilpotent, i.e. $A^m = \Phi$ for some $m \in \mathbb{N}_0$, then A is stable.

The following examples show some arc-value matrices of digraphs over path algebras which satisfy the above conditions and they are used in solving certain path problems in the next section.

Example 2.1.1 Figure 2.1.1(a) shows a digraph $G = (X, U, v)$ over the path algebra $(P_2, \oplus, \otimes)$ and the arc-value matrix A of G is in Figure 2.1.1(b).

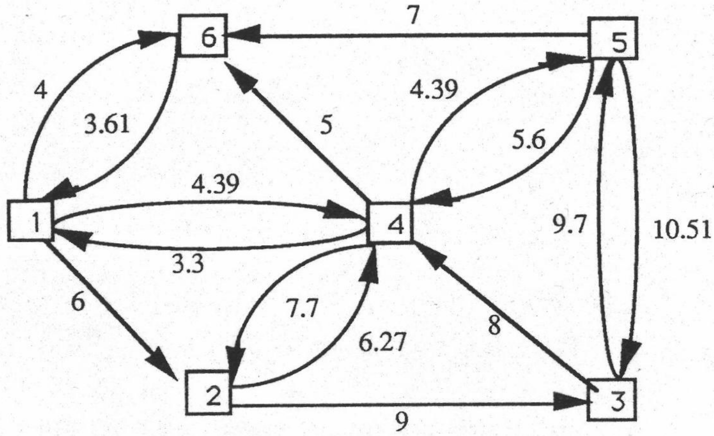

Figure 2.1.1(a)

$$\begin{bmatrix} \infty & 6 & \infty & 4.39 & \infty & 4 \\ \infty & \infty & 9 & 6.27 & \infty & \infty \\ \infty & \infty & \infty & 8 & 9.7 & \infty \\ 3.3 & 7.7 & \infty & \infty & 4.39 & 5 \\ \infty & \infty & 10.51 & 5.6 & \infty & 7 \\ 3.61 & \infty & \infty & \infty & \infty & \infty \end{bmatrix}$$

Figure 2.1.1(b)

Let $\gamma = (i_0, i_1)(i_1, i_2) \ldots (i_r, i_0)$ be an elementary cycle in G. Then $v(\gamma) = v(i_0, i_1) \otimes v(i_1, i_2) \otimes \ldots \otimes v(i_r, i_0)$. Since $v(i_0, i_1) \oplus 0 = \min\{v(i_0, i_1), 0\} = 0$, then

$$\begin{aligned} v(\gamma) \oplus 0 &= (v(i_0, i_1) \otimes v(i_1, i_2) \otimes \ldots \otimes v(i_r, i_0)) \oplus 0 \\ &= (v(i_0, i_1) \otimes v(i_1, i_2) \otimes \ldots \otimes v(i_r, i_0)) \oplus (0 \otimes (v(i_1, i_2) \otimes v(i_2, i_3) \otimes \ldots \otimes v(i_r, i_0))) \\ &= (v(i_0, i_1) \oplus 0) \otimes (v(i_1, i_2) \otimes v(i_2, i_3) \otimes \ldots \otimes v(i_r, i_0)) \\ &= 0 \otimes (v(i_1, i_2) \otimes v(i_2, i_3) \otimes \ldots \otimes v(i_r, i_0)) \\ &= 0. \end{aligned}$$

Therefore, A is stable.

Example 2.1.2 Figure 2.1.2(a) shows a digraph G = (X, U, v) over the path algebra (P_7, ⊕, ⊗), where $\Sigma = \{a, b, . . . , o\}$, and the arc-value matrix A of G is in Figure 2.1.2(b).

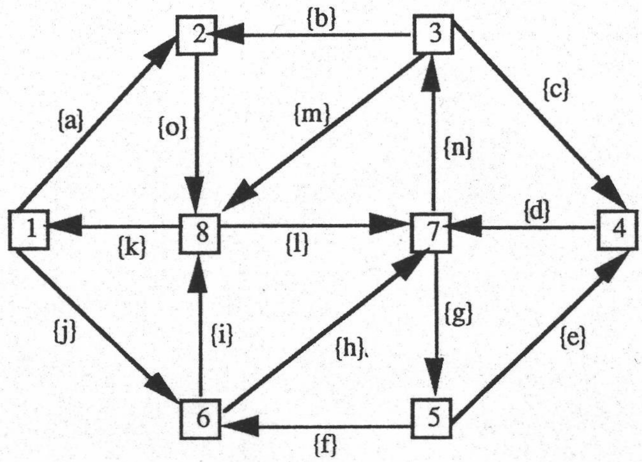

Figure 2.1.2(a)

$$\begin{bmatrix} \emptyset & \{a\} & \emptyset & \emptyset & \emptyset & \{j\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{o\} \\ \emptyset & \{b\} & \emptyset & \{c\} & \emptyset & \emptyset & \emptyset & \{m\} \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{d\} & \emptyset \\ \emptyset & \emptyset & \emptyset & \{e\} & \emptyset & \{f\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{h\} & \{i\} \\ \emptyset & \emptyset & \{n\} & \emptyset & \{g\} & \emptyset & \emptyset & \emptyset \\ \{k\} & \emptyset & \emptyset & \emptyset & \emptyset & \emptyset & \{l\} & \emptyset \end{bmatrix}$$

Figure 2.1.2(b)

Let t be the maximum order of simple dipaths in G. Let i, j ∈ X and let $S_{ij}^{t+1} = \{\mu \,/\, \mu$ is a dipath of order t + 1 from node i to node j in G$\}$. Then for each $\mu \in S_{ij}^{t+1}$, $v(\mu) = \emptyset$. Since $A^{t+1} = [a_{ij}^{t+1}]$ where $a_{ij}^{t+1} = \oplus\sum_{\mu \in S_{ij}^{t+1}} v(\mu)$, then $A^{t+1} = \Phi$. Therefore, A is stable.

2.2 Solutions of Path Problems

Suppose that the arc-value matrix A of a digraph G = (X, U, v) over a path algebra (P, ⊕, ⊗) is stable. Then $A^* = \oplus\sum_{k=0}^{q} A^k$ for some $q \in \mathbf{N}_0$, and the equation

$$\mathbf{y} = A\mathbf{y} \oplus \mathbf{b} \tag{2.2.1}$$

where **b** is the column vector of a specified matrix in a path algebra $(M_n(P), \oplus, \otimes)$, has a solution $y = A^*\mathbf{b}$ because $A(A^*\mathbf{b}) \oplus \mathbf{b} = (AA^* \oplus E)\mathbf{b} = A^*\mathbf{b}$.

Suppose that the arc-value matrix A is nilpotent. Then there is $q \in \mathbf{N}_0$ such that $A^q = \Phi$, and $A^* = \oplus\sum_{k=0}^{q-1} A^k$. Let $\mathbf{y}_0$ be an arbitrary solution of (2.2.1), i.e.

$$\mathbf{y}_0 = A\mathbf{y}_0 \oplus \mathbf{b}.$$

By substituting, we obtain

$$\mathbf{y}_0 = A(A\mathbf{y}_0 \oplus \mathbf{b}) \oplus \mathbf{b} = A^2\mathbf{y}_0 \oplus (E \oplus A)\mathbf{b},$$

and by repeated substitutions

$$\begin{aligned} \mathbf{y}_0 &= A^q\mathbf{y}_0 \oplus (\oplus\sum_{k=0}^{q-1} A^k)\mathbf{b} \\ &= (\oplus\sum_{k=0}^{q-1} A^k)\mathbf{b} \\ &= A^*\mathbf{b}. \end{aligned}$$

Therefore, the equation (2.2.1) has the unique solution $\mathbf{y} = A^*\mathbf{b}$.

Note that by the definitions of strong and weak closures of a stable element, the entry of the solution $\mathbf{y} = A^*\mathbf{b}$ is

$$y_{ij} = \begin{cases} a^*_{ij} & \text{if } \mathbf{b} \text{ is the } j\underline{th} \text{ column vector of the unit matrix E,} \\ \\ \hat{a}_{ij} & \text{if } \mathbf{b} \text{ is the } j\underline{th} \text{ column vector of the arc-value matrix A,} \end{cases}$$

where a^*_{ij} and $\hat{a}_{ij}$ are the entries of A^* and $\hat{A}$ respectively. From (2.1.4) and (2.1.5), we conclude that the entry y_{ij} of $\mathbf{y} = A^*\mathbf{b}$ is

$$y_{ij} = \begin{cases} \oplus\sum_{\mu \in T_{ij}^{h}} v(\mu) & \text{if } \mathbf{b} \text{ is the } j\underline{th} \text{ column vector of the unit matrix E,} \\ \\ \oplus\sum_{\mu \in W_{ij}^{k}} v(\mu) & \text{if } \mathbf{b} \text{ is the } j\underline{th} \text{ column vector of the arc-value matrix A.} \end{cases} \tag{2.2.2}$$

Therefore, a path problem is the determination of the entry of the solution $\mathbf{y} = A^{*}\mathbf{b}$ of the equation (2.2.1), where either $\mathbf{b}$ is the j<u>th</u> column vector of the unit matrix E or the j<u>th</u> column vector of the arc-value matrix A.

A solution $\mathbf{y} = A^{*}\mathbf{b}$ of the equation (2.2.1) can be determined by the following method [3].

The Jordan elimination method

Suppose that the arc-value matrix A of a digraph G = (X, U, v) over a path algebra (P, ⊕, ⊗), where X = {1, 2, . . . , n}, is stable. We denote the equation $\mathbf{y} = A\mathbf{y} \oplus \mathbf{b}$ by

$$\mathbf{y} = A^{(0)}\mathbf{y} \oplus \mathbf{b}^{(0)}.$$

From this equation we derive n new equations

$$\mathbf{y} = A^{(k)}\mathbf{y} \oplus \mathbf{b}^{(k)} \qquad (k = 1, 2, \ldots, n)$$

and we obtain the sequence of matrices $A^{(k)} = [a_{ij}^{(k)}]$ and column vectors $\mathbf{b}^{(k)} = [b^{(k)}]$

by using the following formulas

$$\left. \begin{array}{l} a_{ij}^{(k)} = \begin{cases} a_{ij}^{(k-1)} \oplus a_{ik}^{(k-1)}(a_{kk}^{(k-1)})^{*}\, a_{kj}^{(k-1)} & \text{if } i \neq k,\ j > k, \\ (a_{kk}^{(k-1)})^{*}\, a_{kj}^{(k-1)} & \text{if } i = k,\ j > k, \\ \theta & \text{otherwise,} \end{cases} \\ \\ b_{ij}^{(k)} = \begin{cases} (a_{kk}^{(k-1)})^{*}\, b_{kj}^{(k-1)} & \text{if } i = k, \\ b_{ij}^{(k-1)} \oplus a_{ik}^{(k-1)}\, (a_{kk}^{(k-1)})^{*}\, b_{kj}^{(k-1)} & \text{if } i \neq k, \end{cases} \end{array} \right\} \tag{2.2.3}$$

where θ is the zero of P, $a_{ij}^{(k-1)}$ and $b_{ij}^{(k-1)}$ are the entries of $A^{(k-1)}$ and $\mathbf{b}^{(k-1)}$ respectively. Then these equations $\mathbf{y} = A^{(k)}\mathbf{y} \oplus \mathbf{b}^{(k)}$ (k = 1, 2, . . . , n) have a solution $\mathbf{y} = A^{*}\mathbf{b}$ and in the final equation $\mathbf{y} = A^{(n)}\mathbf{y} + \mathbf{b}^{(n)}$, $A^{(n)} = \Phi$ and $\mathbf{y} = \mathbf{b}^{(n)}$.

Example 2.2.1 Enumeration of simple dipaths between two given nodes.

Let us consider a digraph G = (X, U) of Figure 2.2.1, whose arcs have distinct names from $\Sigma = \{a, b, . . . , o\}$. In order to determine the set of all non-null simple dipaths from node 3 to node 5 in G, a digraph G = (X, U, v) over a path algebra (P_7, $\oplus$, $\otimes$) is given as Figure 2.1.2(a).

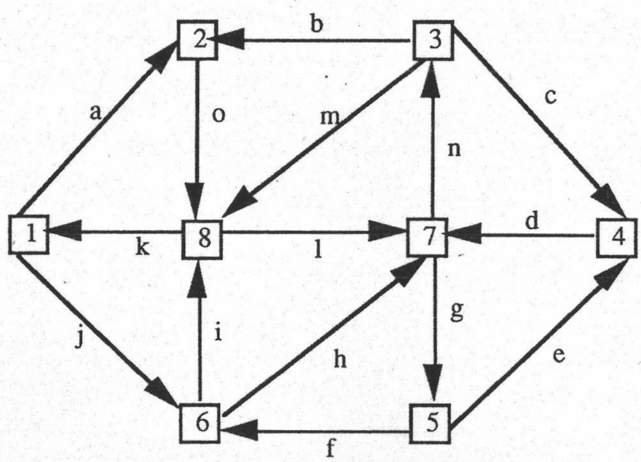

Figure 2.2.1

Recall that the set of all non-null simple dipaths from node i to node j is the formal sum $\oplus\sum_{\mu \in W_{ij}^{q}} v(\mu)$ where q is the maximum order of simple dipaths from node i to node j in G and $W_{ij}^{q} = \{\mu \,/\, \mu$ is a dipath of order r, $1 \le r \le q$, from node i to node j in G$\}$. Let the set of all non-null simple dipaths from node i to node 5 be denoted by y_{i5}. Therefore, $y_{i5} = \oplus\sum_{\mu \in W_{i5}^{q}} v(\mu)$ and y_{35} is the required answer.

From Example 2.1.2, the arc-value matrix A of G (Figure 2.1.2(b)) is stable. By (2.2.2), we have that y_{i5} is the entry of the solution $\mathbf{y} = A^{*}\mathbf{b}$ of the matrix equation $\mathbf{y} = A\mathbf{y} \oplus \mathbf{b}$ where

$$y = [y_{15}\ y_{25}\ y_{35}\ y_{45}\ y_{55}\ y_{65}\ y_{75}\ y_{85}]'$$

and $$\mathbf{b} = [\emptyset\ \emptyset\ \emptyset\ \emptyset\ \emptyset\ \emptyset\ \{g\}\ \emptyset]'$$

is the 5th column vector of A. The solution $\mathbf{y} = A^*\mathbf{b}$ can be determined by the Jordan elimination method as follows.

$$A^{(0)} = A \text{ and } \mathbf{b}^{(0)} = \mathbf{b}.$$

The successive vectors $\mathbf{b}^{(k)}$ (k = 1, 2, . . . , 8) obtained by the equation (2.2.3) are given below.

$$\mathbf{b}^{(1)} = \mathbf{b}^{(2)} = \mathbf{b}^{(3)} = \mathbf{b}^{(4)} = \mathbf{b}^{(5)} = \mathbf{b}^{(6)} = \begin{bmatrix} \phi \\ \phi \\ \phi \\ \phi \\ \phi \\ \phi \\ \{g\} \\ \phi \end{bmatrix}, \quad \mathbf{b}^{(7)} = \begin{bmatrix} \{jhg, jhncdg\} \\ \phi \\ \{cdg\} \\ \{dg\} \\ \{edg, fhg, fhncdg\} \\ \{hg, hncdg\} \\ \{g, ncdg\} \\ \{lg, kjhg, lncdg, kjhncdg\} \end{bmatrix},$$

$b^{(8)}$ =

{jhg,jhncdg,aolg,jilg,jhnmlg,jhnbolg,jikaolg,jhnmkaolg,aokjilg,aokjhnmlg,aokjhg,aolnmkjhg,aolncdg,jilncdg,jikaolncdg,aokjilncdg,aokjhncdg}

{olg,okjilg,okjhnmlg,okjhg,olnmkjhg,olncdg,okjilncdg,okjhncdg}

{cdg,mlg,bolg,cdnmlg,cdnbolg,mkaolg,cdnmkaolg,mkjilg,bokjilg,cdnmkjilg,cdnbokjilg,bokjhnmlg,mkjhnbolg,mkjhg,bokjhg,cdnmkjhg,cdnbokjhg,bolnmkjhg,mlnbokjhg,mlncdg,bolncdg,mkaolncdg,mkjilncdg,bokjilncdg,mkjhncdg,bokjhncdg}

{dg,dnmlg,dnbolg,dnmkaolg,dnmkjilg,dnbokjilg,dnmkjhg,dnbokjhg}

{edg,fhg,fhncdg,filg,ednmlg,fhnmlg,ednbolg,fhnbolg,fikaolg,ednmkaolg,fhnmkaolg,ednmkjilg,fhnmkjilg,ednbokjilg,fhnbokjilg,fikjhnmlg,fikjhnbolg,fikjhg,ednmkjhg,ednbokjhg,filnmkjhg,filnbokjhg,filncdg,fikaolncdg,fikjhncdg}

{hg,hncdg,ilg,hnmlg,hnbolg,ikaolg,hnmkaolg,hnmkjilg,hnbokjilg,ikjhnmlg,ikjhnbolg,ikjhg,ilnmkjhg,ilnbokjhg,ilncdg,ikaolncdg,ikjhncdg}

{g,ncdg,nmlg,nbolg,nmkaolg,nmkjilg,nbokjilg,nmkjhg,nbokjhg}

{lg,kaolg,kjilg,kjhnmlg,kjhnbolg,kjhg,lnmkjhg,lnbokjhg,lncdg,kaolncdg,kjilncdg,kjhncdg}

Therefore, the set of all non-null simple dipaths fron node 3 to node 5 is y_{35}.

y_{35} = {cdg,mlg,bolg,cdnmlg,cdnbolg,mkaolg,cdnmkaolg,mkjilg,bokjilg,cdnmkjilg,cdnbokjilg,bokjhnmlg,mkjhnbolg,mkjhg,bokjhg,cdnmkjhg,cdnbokjhg,bolnmkjhg,mlnbokjhg,mlncdg,bolncdg,mkaolncdg,mkjilncdg,bokjilncdg,mkjhncdg,bokjhncdg}.