CHAPTER III

MATRICES AS Γ−SEMIGROUPS

In this chapter, we deal with sets of $m \times n$ matrices over $\mathbb{R}$. For fixed $T \subseteq M_{mn}(\mathbb{R})$, our goal is to find all subsets Γ of $M_{nm}(\mathbb{R})$ such that T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$.

Recall that if $[b_{pq}] \in M_{nm}(\mathbb{R})$ and $[a_{rs}], [c_{uv}] \in M_{mn}(\mathbb{R})$, then

$$[a_{rs}][b_{pq}][c_{uv}] = \left[\sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} a_{r\beta} b_{\beta\alpha} c_{\alpha v} \right].$$

Theorem 3.1. *Let $i \in N_m$, $j \in N_n$ and $T = \{ \ [a_{rs}] \in M_{mn}(\mathbb{R}) \mid a_{ij} = 0 \ \}$. Then T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$ if and only if*

$$\Gamma \subseteq \{ \ [b_{pq}] \in M_{nm}(\mathbb{R}) \mid b_{pq} = 0 \quad \text{if} \quad p \neq j \quad \text{and} \quad q \neq i \ \}.$$

Proof. Let $M = \{ \ [b_{pq}] \in M_{nm}(\mathbb{R}) \mid b_{pq} = 0 \text{ if } p \neq j \text{ and } q \neq i \ \}$.

First, assume that T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$. Let $[b_{pq}] \in \Gamma$. Fix $p \neq j$ and $q \neq i$. Let $[a_{r\beta}], [c_{\alpha v}] \in T$ be such that

$$a_{i\beta} = \delta_{p\beta} \qquad \text{and} \qquad c_{\alpha j} = \delta_{\alpha q} \ \text{ for all } \beta \in N_n \text{ and } \alpha \in N_m.$$

Then $[a_{r\beta}][b_{pq}][c_{\alpha v}] \in T$. Thus

$$0 = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} \delta_{p\beta} b_{\beta\alpha} \delta_{\alpha q} = b_{pq}.$$

Hence $[b_{pq}] \in M$.

Conversely, assume that $\Gamma \subseteq M$. Let $[a_{r\beta}], [c_{\alpha v}] \in T$ and $[b_{pq}] \in \Gamma$. We claim that the $(i, j)-$entry of $[a_{r\beta}][b_{pq}][c_{\alpha v}]$ is zero. Since $b_{\beta\alpha} = 0$ for all $\beta \neq j$ and $\alpha \neq i$ and $a_{ij} = c_{ij} = 0$, it follows that

$$\sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} = \sum_{\substack{\alpha=1\\ \alpha\neq i}}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} + \sum_{\beta=1}^{n} a_{i\beta}b_{\beta i}c_{ij} = \sum_{\substack{\alpha=1\\ \alpha\neq i}}^{m} a_{ij}b_{j\alpha}c_{\alpha j} = 0.$$

Therefore, T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$. □

Theorem 3.2. *Let $i, p \in N_m$ and $j, q \in N_n$ be such that $i \neq p$ and $j \neq q$. Let $T = \{\ [a_{rs}] \in M_{mn}(\mathbb{R}) \mid a_{ij} = 0 = a_{pq}\ \}$. Then T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$ if and only if*

$$\Gamma \subseteq \{\ [b_{\alpha\beta}] \in M_{nm}(\mathbb{R}) \mid\ b_{jp} \textit{ and } b_{qi} \textit{ are arbitrary and 0 otherwise}\ \}.$$

Proof. Let $M = \{\ [b_{\alpha\beta}] \in M_{nm}(\mathbb{R}) \mid b_{jp}$ and b_{qi} are arbitrary and 0 otherwise$\}$.

First, assume that T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$. Let $[b_{tk}] \in \Gamma$. Fix $(t, k) \neq (j, p)$ and $(t, k) \neq (q, i)$. Then for any $[a_{rs}], [c_{uv}] \in T$,

$$\sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} = 0 = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{p\beta}b_{\beta\alpha}c_{\alpha q}.$$

Case 1. $k \neq i$ and $t \neq j$.

Let $[a_{r\beta}], [c_{\alpha v}] \in T$ be such that $a_{i\beta} = \delta_{t\beta}$ and $c_{\alpha j} = \delta_{\alpha k}$ for all $\beta \neq j$ and $\alpha \neq i$. Then

$$0 = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} = \sum_{\substack{\alpha=1\\ \alpha\neq i}}^{m}\sum_{\substack{\beta=1\\ \beta\neq j}}^{n} \delta_{t\beta}b_{\beta\alpha}\delta_{\alpha k} = b_{tk}.$$

Case 2. $k = i$ or $t = j$.

Subcase 2.1 $k = i$.

Then $t \neq q$. Let $[a_{r\beta}], [c_{\alpha v}] \in T$ be such that $a_{p\beta} = \delta_{t\beta}$ and $c_{\alpha q} = \delta_{\alpha i}$ for all

$\beta \neq q$ and $\alpha \neq p$. Then

$$0 = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{p\beta}b_{\beta\alpha}c_{\alpha q} = \sum_{\substack{\alpha=1\\ \alpha\neq p}}^{m}\sum_{\substack{\beta=1\\ \beta\neq q}}^{n} \delta_{t\beta}b_{\beta\alpha}\delta_{\alpha i} = b_{ti}.$$

Subcase 2.2 $t = j$.

Then $k \neq p$. Similarly to Subcase 2.1, by choosing $[a_{r\beta}], [c_{\alpha v}] \in T$ such that $a_{p\beta} = \delta_{j\beta}$ and $c_{\alpha q} = \delta_{\alpha k}$ for all $\beta \neq q$ and $\alpha \neq p$, we obtain that $b_{jk} = 0$.

Hence, we can conclude from all cases that $[b_{tk}] \in M$.

Conversely, assume that $\Gamma \subseteq M$. Let $[a_{r\beta}], [c_{\alpha v}] \in T$ and $[b_{\beta\alpha}] \in \Gamma$. Since $b_{\beta\alpha} = 0$ for all $\beta \neq j$ and $\alpha \neq i$ and $a_{ij} = c_{ij} = 0$, we have

$$\begin{aligned}\sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} &= \sum_{\substack{\alpha=1\\ \alpha\neq i}}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} + \sum_{\beta=1}^{n} a_{i\beta}b_{\beta i}c_{ij} \\ &= \sum_{\substack{\alpha=1\\ \alpha\neq i}}^{m}\sum_{\substack{\beta=1\\ \beta\neq j}}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} + \sum_{\substack{\alpha=1\\ \alpha\neq i}}^{m} a_{ij}b_{j\alpha}c_{\alpha j} + \sum_{\beta=1}^{n} a_{i\beta}b_{\beta i}c_{ij} = 0.\end{aligned}$$

Since $b_{\beta\alpha} = 0$ for all $\beta \neq q$ and $\alpha \neq p$ and $a_{pq} = c_{pq} = 0$, we have

$$\begin{aligned}\sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{p\beta}b_{\beta\alpha}c_{\alpha q} &= \sum_{\substack{\alpha=1\\ \alpha\neq p}}^{m}\sum_{\beta=1}^{n} a_{p\beta}b_{\beta\alpha}c_{\alpha q} + \sum_{\beta=1}^{n} a_{p\beta}b_{\beta p}c_{pq} \\ &= \sum_{\substack{\alpha=1\\ \alpha\neq p}}^{m}\sum_{\substack{\beta=1\\ \beta\neq q}}^{n} a_{p\beta}b_{\beta\alpha}c_{\alpha q} + \sum_{\substack{\alpha=1\\ \alpha\neq p}}^{m} a_{pq}b_{q\alpha}c_{\alpha q} + \sum_{\beta=1}^{n} a_{p\beta}b_{\beta p}c_{pq} = 0.\end{aligned}$$

Thus $[a_{r\beta}][b_{\beta\alpha}][c_{\alpha v}] \in T$. Therefore, T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$. □

Theorem 3.3. *Let $i \in N_m$, $j, t \in N_n$ and $j \neq t$. Let $T = \{\ [a_{rs}] \in M_{mn}(\mathbb{R}) \mid a_{ij} = 0 = a_{it}\ \}$. Then T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$ if and only if*

$$\Gamma \subseteq \{\ [b_{pq}] \in M_{nm}(\mathbb{R}) \mid b_{pq} = 0 \ \ \textit{if} \ \ p \neq j, t \ \ \textit{and} \ \ q \neq i\ \}.$$

Proof. Let $M = \{ [b_{pq}] \in M_{nm}(\mathbb{R}) \mid b_{pq} = 0 \text{ if } p \neq j, t \text{ and } q \neq i \}$.

First, assume that T is a Γ−subsemigroup of $M_{mn}(\mathbb{R})$. Let $[b_{pq}] \in \Gamma$. Fix $p \neq j, t$ and $q \neq i$. Let $[a_{r\beta}], [c_{\alpha v}] \in T$ be such that $a_{i\beta} = \delta_{p\beta}$ and $c_{\alpha j} = \delta_{\alpha q}$ for all $\beta \neq j, t$ and $\alpha \neq i$. Then $[a_{r\beta}][b_{pq}][c_{\alpha v}] \in T$. Thus

$$0 = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} = \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{m} \sum_{\substack{\beta=1 \\ \beta\neq j,t}}^{n} \delta_{p\beta} b_{\beta\alpha} \delta_{\alpha q} = b_{pq}.$$

Thus, $[b_{pq}] \in M$.

Conversely, assume that $\Gamma \subseteq M$. Let $[a_{rs}], [c_{uv}] \in T$ and $[b_{pq}] \in \Gamma$. Since $a_{ij} = a_{it} = c_{ij} = c_{it} = 0$ and $\sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} = a_{ij} b_{j\alpha} + a_{it} b_{t\alpha} = 0$ where $\alpha = 1, 2, \ldots, \hat{i}, \ldots, m$, we have

$$\sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} = \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} + \sum_{\beta=1}^{n} a_{i\beta} b_{\beta i} c_{ij} = 0, \qquad \text{and}$$

$$\sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha t} = \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha t} + \sum_{\beta=1}^{n} a_{i\beta} b_{\beta i} c_{it} = 0.$$

Hence $[a_{rs}][b_{pq}][c_{uv}] \in T$. Therefore, T is a Γ−subsemigroup of $M_{mn}(\mathbb{R})$. □

The following proposition is also useful for obtaining results in this chapter. Notice that this proposition is analogous to Proposition 2.1.1 and Proposition 2.2.1 which are major tools in studying Γ−subsemigroups of $\mathbb{R}$ under usual addition and multiplication, respectively.

Proposition 3.4. *Let T and Γ be nonempty subsets of $M_{mn}(\mathbb{R})$ and $M_{nm}(\mathbb{R})$, respectively. Then T is a Γ−subsemigroup of $M_{mn}(\mathbb{R})$ if and only if T^{tr} is a Γ^{tr}−subsemigroup of $M_{nm}(\mathbb{R})$.*

Proof. First, assume that T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$. Then $T\Gamma T \subseteq T$. Thus $T^{tr}\Gamma^{tr}T^{tr} = (T\Gamma T)^{tr} \subseteq T^{tr}$. Thus T^{tr} is a $\Gamma^{tr}-$subsemigroup of $M_{nm}(\mathbb{R})$. The converse is obtained similarly. □

Theorem 3.3 and Proposition 3.4 give the following result.

Corollary 3.5. *Let $i, t \in N_m$ and $j \in N_n$ and $i \neq t$. Let $T = \{ [a_{rs}] \in M_{mn}(\mathbb{R}) \mid a_{ij} = 0 = a_{tj} \}$. Then T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$ if and only if*

$$\Gamma \subseteq \{ [b_{pq}] \in M_{nm}(\mathbb{R}) \mid b_{pq} = 0 \ \ if \ \ p \neq j, \ q \neq i, t \}.$$

Proof. Let $M = \{ [b_{pq}] \in M_{nm}(\mathbb{R}) \mid b_{pq} = 0 \text{ if } p \neq j, \ q \neq i, t \}$.

First, assume that T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$. By Proposition 3.4, T^{tr} is a $\Gamma^{tr}-$subsemigroup of $M_{nm}(\mathbb{R})$. Note that $T^{tr} = \{ [a_{rs}] \in M_{nm}(\mathbb{R}) \mid a_{ji} = 0 = a_{jt} \}$ and $M^{tr} = \{ [b_{pq}] \in M_{mn}(\mathbb{R}) \mid b_{pq} = 0 \ if \ p \neq i, t \text{ and } q \neq j\}$. It follows from Theorem 3.3 that $\Gamma^{tr} \subseteq M^{tr}$. Thus $\Gamma \subseteq M$.

Conversely, assume that $\Gamma \subseteq M$. Then $\Gamma^{tr} \subseteq M^{tr}$ so that T^{tr} is a $\Gamma^{tr}-$subsemigroup of $M_{nm}(\mathbb{R})$. Therefore T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$. □

Theorem 3.6. *Let $\lambda \in \mathbb{R}\backslash\{0\}$, $i \in N_m$ and $j \in N_n$. Let $T = \{ [a_{rs}] \in M_{mn}(\mathbb{R}) \mid a_{ij} = \lambda \}$. Then T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$*

$$if \ and \ only \ if \ \ \Gamma = \{ [b_{pq}] \in M_{nm}(\mathbb{R}) \mid b_{ji} = \frac{1}{\lambda} \ \ and \ 0 \ otherwise \}.$$

Proof. First, assume that T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$. Let $[b_{xy}] \in \Gamma$. For each $[a_{r\beta}], [c_{\alpha v}] \in T$, we have

$$\lambda = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} = \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} + \sum_{\beta=1}^{n} a_{i\beta}b_{\beta i}c_{ij}. \tag{3.1}$$

Choose $[c_{\alpha v}] \in T$ such that $c_{\alpha j} = 0$ where $\alpha \neq i$. Then

$$\lambda = \lambda \sum_{\beta=1}^{n} a_{i\beta} b_{\beta i}$$

so that

$$\sum_{\beta=1}^{n} a_{i\beta} b_{\beta i} = 1. \tag{3.2}$$

From (3.1),(3.2) and the fact that $c_{ij} = \lambda$,

$$0 = \sum_{\substack{\alpha=1 \\ \alpha \neq i}}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} \quad \text{for all } [a_{rs}], [c_{uv}] \in T. \tag{3.3}$$

Suppose there exists $t \in N_n \backslash \{j\}$ such that $b_{ti} \neq 0$. Then choose $[a_{r\beta}] \in T$ such that

$$a_{i\beta} = \begin{cases} \dfrac{2 - \lambda b_{ji}}{b_{ti}} & \text{, if } \beta = t, \\ 0 & \text{, if } \beta \in N_n \backslash \{j, t\}. \end{cases} \tag{3.4}$$

From (3.2) and (3.4), $1 = a_{ij} b_{ji} + a_{it} b_{ti} = 2$ which is a contradiction. Hence $b_{ti} = 0$ for all $t \in N_n \backslash \{j\}$. From this result together with (3.2) and the fact that $a_{ij} = \lambda$, we see that $b_{ji} = \dfrac{1}{\lambda}$.

Next, suppose that there exists $p \in N_m \backslash \{i\}$ such that $b_{jp} \neq 0$. Choose

$$a_{i\beta} = 0 \quad \text{for all } \beta \neq j \quad \text{and} \quad c_{\alpha j} = \begin{cases} \dfrac{1}{\lambda b_{jp}} & \text{, if } \alpha = p, \\ 0 & \text{, if } \alpha \in N_m \backslash \{i, p\}. \end{cases}$$

Then from (3.4)

$$0 = \sum_{\substack{\alpha=1 \\ \alpha \neq i}}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} = \sum_{\beta=1}^{n} a_{i\beta} b_{\beta p} c_{pj} = a_{ij} b_{jp} c_{pj} = \lambda b_{jp} c_{pj} = 1$$

which is a contradiction. Hence $b_{jp} = 0$ for all $p \in N_m \backslash \{i\}$. As a result,

$$0 = \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{m} \sum_{\substack{\beta=1 \\ \beta\neq j}}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} + \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{m} a_{ij} b_{j\alpha} c_{\alpha j} = \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{m} \sum_{\substack{\beta=1 \\ \beta\neq j}}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} \quad \text{for all } [a_{rs}], [c_{uv}] \in T.$$

To show that $b_{tp} = 0$ for all $t \in N_n \backslash \{j\}$ and $p \in N_m \backslash \{i\}$, we suppose not. Then there exist $t \in N_n \backslash \{j\}$ and $p \in N_m \backslash \{i\}$ such that $b_{tp} \neq 0$. Setting $[a_{r\beta}], [c_{\alpha v}] \in T$ such that

$$a_{i\beta} = \begin{cases} \dfrac{1}{b_{tp}} & , \text{ if } \beta = t, \\ 0 & , \text{ if } \beta \in N_n \backslash \{j, t\}, \end{cases} \quad \text{and} \quad c_{\alpha j} = \begin{cases} 1 \text{ , if } & \alpha = p, \\ 0 \text{ , if } & \alpha \in N_m \backslash \{i, p\}. \end{cases}$$

We obtain that

$$0 = \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{m} \sum_{\substack{\beta=1 \\ \beta\neq j}}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} = \sum_{\substack{\beta=1 \\ \beta\neq j}}^{n} a_{i\beta} b_{\beta p} c_{pj} = a_{it} b_{tp} = 1.$$

This leads to a contradiction. Hence $b_{tp} = 0$ for all $t \in N_n \backslash \{j\}$ and $p \in N_m \backslash \{i\}$ as desired. Therefore $b_{xy} = 0$ if $(x, y) \neq (j, i)$ and $b_{ji} = \frac{1}{\lambda}$, i.e., $[b_{xy}] \in M$. This shows that $\Gamma \subseteq M$. Since M is a singleton set and Γ must be a nonempty subset of M, it must follow that $\Gamma = M$.

Conversely, assume that $\Gamma = \{ \, [b_{pq}] \in M_{nm}(\mathbb{R}) \mid b_{ji} = \frac{1}{\lambda}$ and 0 otherwise $\}$. Let $[a_{r\beta}], [c_{\alpha v}] \in T$ and $[b_{pq}] \in \Gamma$. Then

$$\sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} = \sum_{\beta=1}^{n} a_{i\beta} b_{\beta i} c_{ij} + \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} = a_{ij} b_{ji} c_{ij} = \lambda.$$

Thus, $[a_{r\beta}][b_{\beta\alpha}][c_{\alpha v}] \in T$. Therefore, T is a Γ−subsemigroup of $M_{mn}(\mathbb{R})$. □

Theorem 3.7. *Let $\lambda \in \mathbb{R} \backslash \{0\}$, $i \in N_m$, $j, q \in N_n$ and $j \neq q$. Let $T = \{ \, [a_{rs}] \in M_{mn}(\mathbb{R}) \mid a_{ij} = \lambda$ and $a_{iq} = 0 \, \}$. Then T is a Γ−subsemigroup of $M_{mn}(\mathbb{R})$ if and only if*

$$\Gamma \subseteq \{\ [b_{\alpha\beta}] \in M_{nm}(\mathbb{R}) \mid b_{j\beta} = \tfrac{1}{\lambda}\delta_{\beta i} \textit{ and } b_{\alpha\beta} = 0 \textit{ for all } \alpha \neq j, q \textit{ and for all } \beta\ \}.$$

Proof. Let $M = \{\ [b_{\alpha\beta}] \in M_{nm}(\mathbb{R}) \mid b_{j\beta} = \frac{1}{\lambda}\delta_{\beta i}$ and $b_{\alpha\beta}$ for all $\alpha \neq j, q$ and for all $\beta\ \}$.

First, assume that T is a Γ-subsemigroup of $M_{mn}(\mathbb{R})$. Let $[b_{\alpha\beta}] \in \Gamma$. Let $[c_{\alpha v}] \in T$ be such that $c_{\alpha j} = 0$ where $\alpha \neq i$. For each $[a_{r\beta}] \in T$, similarly to the proof of Theorem 3.6, we obtain that

$$\sum_{\beta=1}^{n} a_{i\beta} b_{\beta i} = 1. \tag{3.5}$$

Suppose there exists $t \in N_n \backslash \{j, q\}$ such that $b_{ti} \neq 0$. Choose $[a_{r\beta}] \in T$ such that

$$a_{i\beta} = \begin{cases} \dfrac{2 - \lambda b_{ji}}{b_{ti}} & ,\text{if } \beta = t, \\ 0 & ,\text{if } \beta \in N_n \backslash \{j, t, q\}. \end{cases} \tag{3.6}$$

From (3.5) and (3.6), we see that $1 = a_{ij}b_{ji} + a_{it}b_{ti} + a_{iq}b_{qi} = 2$ which is a contradiction. Hence $b_{ti} = 0$ for all $t \in N_n \backslash \{j, q\}$. From (3.5) and the fact that $a_{ij} = \lambda$ and $a_{iq} = 0$, we have $b_{ji} = \dfrac{1}{\lambda}$. Thus, for each $[a_{r\beta}], [c_{\alpha v}] \in T$,

$$\begin{aligned} \lambda = \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} &= \sum_{\substack{\alpha=1 \\ \alpha \neq i}}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} + \sum_{\beta=1}^{n} a_{i\beta} b_{\beta i} c_{ij} \\ &= \sum_{\substack{\alpha=1 \\ \alpha \neq i}}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} + a_{ij} b_{ji} c_{ij} + a_{iq} b_{qi} c_{ij} \\ &= \sum_{\substack{\alpha=1 \\ \alpha \neq i}}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} + \lambda, \end{aligned}$$

so that

$$0 = \sum_{\substack{\alpha=1 \\ \alpha \neq i}}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j}.$$

Next, we show that $b_{j\alpha} = 0$ for all $\alpha \neq i$. Suppose that there exists $k \in N_m \backslash \{i\}$

such that $b_{jk} \neq 0$. Choosing $[a_{r\beta}], [c_{\alpha v}] \in T$ such that

$$a_{i\beta} = 0 \quad \text{for all } \beta \in N_n \backslash \{j, q\} \quad \text{and} \quad c_{\alpha j} = \begin{cases} \dfrac{1}{\lambda b_{jk}} & , \text{ if } \alpha = k, \\ 0 & , \text{ if } \alpha \in N_m \backslash \{i, k\}. \end{cases}$$

Then

$$0 = \sum_{\substack{\alpha=1 \\ \alpha \neq i}}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} = \sum_{\beta=1}^{n} a_{i\beta} b_{\beta k} c_{kj} = a_{ij} b_{jk} c_{kj} + a_{iq} b_{qk} c_{kj} = a_{ij} b_{jk} c_{kj} = 1$$

which is a contradiction. Hence $b_{j\alpha} = 0$ for all $\alpha \neq i$ as desired. As a result,

$$0 = \sum_{\substack{\alpha=1 \\ \alpha \neq i}}^{m} \sum_{\substack{\beta=1 \\ \beta \neq j}}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j}.$$

Next, suppose for the contradiction that there exist $t \in N_n \backslash \{j, q\}$ and $k \in N_m \backslash \{i\}$ such that $b_{tk} \neq 0$. Set

$$a_{i\beta} = \begin{cases} \dfrac{1}{b_{tk}} & , \text{ if } \beta = t, \\ 0 & , \text{ if } \beta \in N_n \backslash \{j, t, q\}, \end{cases} \quad \text{and} \quad c_{\alpha j} = \begin{cases} 1 & , \text{ if } \alpha = k, \\ 0 & , \text{ if } \alpha \in N_m \backslash \{i, k\}. \end{cases}$$

Then

$$0 = \sum_{\substack{\alpha=1 \\ \alpha \neq i}}^{m} \sum_{\substack{\beta=1 \\ \beta \neq j}}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} = \sum_{\substack{\beta=1 \\ \beta \neq j}}^{n} a_{i\beta} b_{\beta k} c_{kj} = a_{it} b_{tk} c_{kj} = 1$$

which is impossible. Consequently, $[b_{\alpha\beta}] \in M$. This shows that $\Gamma \subseteq M$.

Conversely, assume that $\Gamma \subseteq M$. Let $[a_{r\beta}], [c_{\alpha v}] \in T$ and $[b_{\beta\alpha}] \in \Gamma$. We need to show that

$$\sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} = \lambda \quad \text{and} \quad \sum_{\alpha=1}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha q} = 0.$$

Note that since $b_{ji} = \frac{1}{\lambda}$ and $b_{\beta\alpha} = 0$ for all $\alpha \neq i$ and $\beta \neq q$ while $a_{iq} = 0$ and $a_{ij} = c_{ij} = \lambda$, we have

$$\begin{aligned}
\sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} &= \sum_{\substack{\alpha=1\\ \alpha\neq i}}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} + \sum_{\beta=1}^{n} a_{i\beta}b_{\beta i}c_{ij} \\
&= \sum_{\substack{\alpha=1\\ \alpha\neq i}}^{m}\sum_{\substack{\beta=1\\ \beta\neq q}}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} + \sum_{\substack{\alpha=1\\ \alpha\neq i}}^{m} a_{iq}b_{q\alpha}c_{\alpha j} + \lambda\sum_{\beta=1}^{n} a_{i\beta}b_{\beta i} \\
&= \lambda\sum_{\beta=1}^{n} a_{i\beta}b_{\beta i} \\
&= \lambda(a_{ij}b_{ji} + a_{iq}b_{qi}) = \lambda,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha q} &= \sum_{\substack{\alpha=1\\ \alpha\neq i}}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha q} + \sum_{\beta=1}^{n} a_{i\beta}b_{\beta i}c_{iq} \\
&= \sum_{\substack{\alpha=1\\ \alpha\neq i}}^{m}\sum_{\substack{\beta=1\\ \beta\neq q}}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha q} + \sum_{\substack{\alpha=1\\ \alpha\neq i}}^{m} a_{iq}b_{q\alpha}c_{\alpha q} \\
&= 0.
\end{aligned}$$

Thus $[a_{r\beta}][b_{\beta\alpha}][c_{\alpha v}] \in T$. Therefore, T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$. □

Corollary 3.8 is an immediate result from Theorem 3.7 and Proposition 3.4.

Corollary 3.8. *Let $\lambda \in \mathbb{R}\backslash\{0\}$, $i, p \in N_m$, $j \in N_n$ and $i \neq p$. Let $T = \{\ [a_{rs}] \in M_{mn}(\mathbb{R}) \mid a_{ij} = \lambda$ and $a_{pj} = 0\ \}$. Then T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$ if and only if*

$$\Gamma \subseteq \{\ [b_{\alpha\beta}] \in M_{nm}(\mathbb{R}) \mid b_{\alpha i} = \tfrac{1}{\lambda}\delta_{j\alpha} \text{ and } b_{\alpha\beta} = 0 \text{ for all } \beta \neq i, p \text{ and for all } \alpha\ \}.$$

Theorem 3.9. *Let $\lambda \in \mathbb{R}\backslash\{0\}$, $i, j \in N_m$ and $p, q \in N_n$ be such that $p \neq i$ and*

$q \neq j$. Let $T = \{ [a_{rs}] \in M_{mn}(\mathbb{R}) \mid a_{ij} = 0$ and $a_{pq} = \lambda \}$. Then T is not a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$ for any nonempty subset Γ of $M_{nm}(\mathbb{R})$.

Proof. Suppose that Γ is a nonempty subset of $M_{nm}(R)$ and T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$. Let $[b_{tk}] \in \Gamma$. Fix $t \neq j$ and $k \neq i$. Let $[a_{r\beta}], [c_{\alpha v}] \in T$ be such that

$$a_{i\beta} = \delta_{t\beta} \qquad \text{and} \qquad c_{\alpha j} = \delta_{\alpha k} \qquad \text{for all } \beta \neq j \text{ and } \alpha \neq i.$$

Then $[a_{r\beta}][b_{pq}][c_{\alpha v}] \in T$ so that

$$0 = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} = \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{m}\sum_{\substack{\beta=1 \\ \beta\neq j}}^{n} \delta_{t\beta}b_{\beta\alpha}\delta_{\alpha k} = b_{tk}.$$

This shows that $b_{tk} = 0$ for all $t \neq j$ and $k \neq i$. Next, choose $[a_{r\beta}], [c_{\alpha v}] \in T$ such that

$$a_{pj} = 0 \quad \text{and} \quad c_{\alpha q} = \begin{cases} 0 & , \text{ if } \alpha \neq p, \\ \lambda & , \text{ if } \alpha = p. \end{cases}$$

Then

$$\lambda = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{p\beta}b_{\beta\alpha}c_{\alpha q} = \sum_{\beta=1}^{n} a_{p\beta}b_{\beta p}c_{pq} = \sum_{\substack{\beta=1 \\ \beta\neq j}}^{n} a_{p\beta}b_{\beta p}c_{pq} + a_{pj}b_{jp}c_{pq} = 0$$

which is a contradiction. Hence T is not a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$ for all nonempty subsets Γ of $M_{nm}(\mathbb{R})$. □

Theorem 3.10. *Let $\lambda, \mu \in \mathbb{R}\backslash\{0\}$, $i, p \in N_m$ and $j, q \in N_n$ be such that $p \neq i$ and $q \neq j$. Let $T = \{ [a_{rs}] \in M_{mn}(\mathbb{R}) \mid a_{ij} = \lambda$ and $a_{pq} = \mu \}$. Then T is not a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$ for any nonempty subset Γ of $M_{nm}(\mathbb{R})$.*

Proof. Let Γ be a nonempty subset of $M_{nm}(R)$ and $[b_{tk}] \in \Gamma$. Suppose that T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$. Since $a_{ij} = \lambda$, $a_{pq} = \mu$ and from the proof of Theorem 3.6, we obtain that $\{ [b_{pq}] \in M_{nm}(\mathbb{R}) \mid b_{ji} = \frac{1}{\lambda}$ and 0 otherwise $\} = \Gamma = \{ [b_{pq}] \in M_{nm}(\mathbb{R}) \mid b_{qp} = \frac{1}{\mu}$ and 0 otherwise $\}$ which is impossible. Therefore, T is not a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$ for all nonempty subsets Γ of $M_{nm}(\mathbb{R})$. □

Theorem 3.11. *Let $\lambda, \mu \in \mathbb{R}\backslash\{0\}$, $i \in N_m$ and $q, j \in N_n$ be such that $q \neq j$. Let $T = \{ [a_{rs}] \in M_{mn}(\mathbb{R}) \mid a_{ij} = \lambda$ and $a_{iq} = \mu \}$. Then T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$ if and only if $\Gamma \subseteq \{ [b_{xy}] \in M_{nm}(\mathbb{R}) \mid 1 = \lambda b_{ji} + \mu b_{qi}$ and $\forall t \in N_m\backslash\{i\}$, $\lambda b_{jt} + \mu b_{qt} = 0$, and 0 otherwise $\}$.*

Proof. Let $M = \{ [b_{xy}] \in M_{nm}(\mathbb{R}) \mid 1 = \lambda b_{ji} + \mu b_{qi}$ and $\forall t \in N_m\backslash\{i\}$, $\lambda b_{jt} + \mu b_{qt} = 0$, and 0 otherwise $\}$.

First, assume that T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$. Let $[b_{\beta\alpha}] \in \Gamma$. Suppose that there exists $p \in N_n\backslash\{j, q\}$ such that $b_{pi} \neq 0$. Choose $[a_{r\beta}], [c_{\alpha v}] \in T$ such that

$$a_{i\beta} = \begin{cases} \dfrac{\lambda + 1 - \lambda^2 b_{ji} - \lambda\mu b_{qi}}{\lambda b_{pi}} & \text{, if } \beta = p, \\ 0 & \text{, if } \beta \in N_n\backslash\{j, p, q\}, \end{cases}$$

and

$$c_{\alpha j} = 0 \text{ for all } \alpha \neq i.$$

Then

$$\begin{aligned} \lambda = \sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} &= \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha j} + \sum_{\beta=1}^{n} a_{i\beta}b_{\beta i}c_{ij} \\ &= a_{ij}b_{ji}c_{ij} + a_{ip}b_{pi}c_{ij} + a_{iq}b_{qi}c_{ij} \\ &= \lambda^2 b_{ji} + (\lambda + 1 - \lambda^2 b_{ji} - \lambda\mu b_{qi}) + \lambda\mu b_{qi} \\ &= \lambda + 1 \end{aligned}$$

which is a contradiction. Hence $b_{pi} = 0$ for all $p \in N_n \backslash \{j, q\}$. Note that, for all $[a_{r\beta}], [c_{\alpha v}] \in T$,

$$\lambda = \sum_{\substack{\alpha=1 \\ \alpha \neq i}}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} + \sum_{\beta=1}^{n} a_{i\beta} b_{\beta i} c_{ij} \tag{3.7}$$

$$= \sum_{\substack{\alpha=1 \\ \alpha \neq i}}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} + a_{ij} b_{ji} c_{ij} + a_{iq} b_{qi} c_{ij}$$

$$= \sum_{\substack{\alpha=1 \\ \alpha \neq i}}^{m} \sum_{\beta=1}^{n} a_{i\beta} b_{\beta\alpha} c_{\alpha j} + \lambda^2 b_{ji} + \lambda\mu b_{qi}. \tag{3.8}$$

Next, suppose that there exist $p \in N_n \backslash \{j, q\}$ and $t \in N_m \backslash \{i\}$ such that $b_{pt} \neq 0$. Putting $[a_{r\beta}], [c_{\alpha v}] \in T$ in (3.8) where

$$a_{i\beta} = \begin{cases} \dfrac{\lambda + 1 - \lambda b_{jt} - \mu b_{qt} - \lambda^2 b_{ji} - \lambda\mu b_{qi}}{b_{pt}} & \text{, if } \beta = p, \\ 0 & \text{, if } \beta \in N_n \backslash \{j, p, q\}, \end{cases}$$

and

$$c_{\alpha j} = \begin{cases} 1 & \text{, if } \alpha = t, \\ 0 & \text{, if } \alpha \in N_m \backslash \{i, t\}, \end{cases}$$

yields $\lambda = \lambda + 1$ which is a contradiction. Hence $b_{pt} = 0$ for all $p \in N_n \backslash \{j, q\}$ and $t \in N_m \backslash \{i\}$. Moreover, from (3.8) for all $[a_{r\beta}], [c_{\alpha v}] \in T$,

$$\lambda = \sum_{\alpha=1}^{m} (\lambda b_{j\alpha} + \mu b_{q\alpha}) c_{\alpha j}. \tag{3.9}$$

Suppose that there exists $t \in N_m \backslash \{i\}$ such that $\lambda b_{jt} + \mu b_{qt} \neq 0$. Replacing $[c_{\alpha v}] \in T$ such that

$$c_{\alpha j} = \begin{cases} \dfrac{\lambda + 1 - \lambda^2 b_{ji} - \lambda\mu b_{qi}}{\lambda b_{jt} + \mu b_{qt}} & \text{, if } \alpha = t, \\ 0 & \text{, if } \alpha \in N_m \backslash \{i, t\}, \end{cases}$$

in (3.9) gives another contradiction because $\lambda = \lambda + 1$. Thus $\lambda b_{jt} + \mu b_{qt} = 0$ for all $t \in N_m \backslash \{i\}$.

Finally, from (3.9), we obtain that $\lambda = (\lambda b_{ji} + \mu b_{qi})c_{ij}$ and since $c_{ij} = \lambda$, we have $\lambda b_{ji} + \mu b_{qi} = 1$. Therefore $\Gamma \subseteq M$.

Conversely, assume that $\Gamma \subseteq M$. Let $[a_{r\beta}], [c_{\alpha v}] \in T$ and $[b_{\beta\alpha}] \in \Gamma$. Then

$$\begin{aligned}\sum_{\alpha=1}^{m}\sum_{\beta=1}^{n} a_{i\beta}b_{\beta\alpha}c_{\alpha v} &= \sum_{\alpha=1}^{m}(a_{ij}b_{j\alpha} + a_{iq}b_{q\alpha})c_{\alpha v} \\ &= \sum_{\alpha=1}^{m}(\lambda b_{j\alpha} + \mu b_{q\alpha})c_{\alpha v} \\ &= (\lambda b_{ji} + \mu b_{qi})c_{iv} \\ &= c_{iv} = \begin{cases} \lambda \text{ , if } v = j, \\ \mu \text{ , if } v = q. \end{cases}\end{aligned}$$

Therefore, T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$.

$\square$

Immediately from Theorem 3.11 and Proposition 3.4, we obtain

Corollary 3.12. *Let $\lambda, \mu \in \mathbb{R}\backslash\{0\}$, $i, p \in N_m$ and $j \in N_n$ be such that $i \neq p$. Let $T = \{ \ [a_{rs}] \in M_{mn}(\mathbb{R}) \mid a_{ij} = \lambda$ and $a_{pj} = \mu \ \}$. Then T is a $\Gamma-$subsemigroup of $M_{mn}(\mathbb{R})$ if and only if $\Gamma \subseteq \{ \ [b_{xy}] \in M_{nm}(\mathbb{R}) \mid 1 = \lambda b_{ji} + \mu b_{jp}$ and $\forall t \in N_n\backslash\{j\}$, $\lambda b_{ti} + \mu b_{tp} = 0$, and 0 otherwise $\}$.*