CHAPTER II

REAL INTERVALS AS Γ-SUBSEMIGROUPS OF $\mathbb{R}$

We recall from Example 1.3 that $\mathbb{R}$ is a $\Gamma-$semigroup under usual addition and multiplication for any nonempty subset Γ of $\mathbb{R}$. In this chapter, we focus $\Gamma-$subsemigroups of $\mathbb{R}$ in two aspects. One hand, for each real interval I, we characterize which types of nonempty subsets Γ of $\mathbb{R}$ such that I is a $\Gamma-$subsemigroup of $\mathbb{R}$. On the other hand, for each real interval Γ, we describe types of real intervals which are $\Gamma-$subsemigroups of $\mathbb{R}$.

We demonstrate these notions in two sections. In the first section, $\Gamma-$subsemigroups of $\mathbb{R}$ under usual addition are considered. Next, $\Gamma-$subsemigroups of $\mathbb{R}$ under usual multiplication are studied.

2.1 $\Gamma-$Subsemigroups of $\mathbb{R}$ under Usual Addition

In this section, we consider only $\Gamma-$subsemigroups of $\mathbb{R}$ under usual addition. First, for each given real interval I, we look up all possibilities of nonempty subsets Γ of $\mathbb{R}$ such that I is a $\Gamma-$subsemigroup of $\mathbb{R}$. Later, we fix a real interval Γ in order to find all choices of real intervals I which I is a $\Gamma-$subsemigroup of $\mathbb{R}$.

The following proposition will play a major role in this section.

Proposition 2.1.1. *Let I and Γ be nonempty subsets of $\mathbb{R}$. Then*

$$I \text{ is a } \Gamma\text{-semigroup if and only if } -I \text{ is a } (-\Gamma)\text{-semigroup.}$$

Proof. First, assume that I is a $\Gamma-$semigroup. Let $x, y \in -I$ and $\alpha \in -\Gamma$. Then $-x, -y \in I$ and $-\alpha \in \Gamma$ which imply that $(-x) + (-\alpha) + (-y) \in I$. Hence

$x + \alpha + y \in -I$. This shows that $-I$ is a $(-\Gamma)-$semigroup.

Conversely, the result holds by changing I and Γ to $-I$ and $-\Gamma$, respectively.

□

The following example shows results obtained from Proposition 2.1.1.

Example 2.1.2. For any real number a, it is clear that $\{-a\}$ is a $\{a\}-$subsemigroup of $\mathbb{R}$ and (a, ∞) is a $(-a, \infty)-$subsemigroup of $\mathbb{R}$. By applying Proposition 2.1.1, we also obtain that $\{a\}$ is a $\{-a\}-$subsemigroup of $\mathbb{R}$ and $(-\infty, -a)$ is a $(-\infty, a)-$sub semigroup of $\mathbb{R}$ for any real number a.

Theorem 2.1.3. *Let $x \in \mathbb{R}$. For a nonempty subset Γ of $\mathbb{R}$,*

$$\{x\} \text{ is a } \Gamma-\text{subsemigroup of } \mathbb{R} \text{ if and only if } \Gamma = \{-x\}.$$

Proof. First, assume that $\{x\}$ is a $\Gamma-$subsemigroup of $\mathbb{R}$. Let $k \in \Gamma$. Then $x + k + x = x$, so $k = -x$. Hence $\Gamma = \{-x\}$. On the other hand, obviously, $\{x\}$ is a $\{-x\}-$subsemigroup of $\mathbb{R}$. □

Theorem 2.1.4. *Let $a, b \in \mathbb{R}$ and $a < b$. Then $(a, b), [a, b), (a, b]$ and $[a, b]$ are not $\Gamma-$subsemigroups of $\mathbb{R}$ for all nonempty subsets Γ of $\mathbb{R}$.*

Proof. Let Γ be a nonempty subset of $\mathbb{R}$. Without loss of generality, suppose that (a, b) is a $\Gamma-$subsemigroup of $\mathbb{R}$. Let $k \in \Gamma$. Then $(2a + k, 2b + k) = (a, b) + k + (a, b) \subseteq (a, b) + \Gamma + (a, b) \subseteq (a, b)$. Thus $a \leq 2a + k < 2b + k \leq b$. As a result, $-k \leq a < b \leq -k$ which is a contradiction. Hence (a, b) is not a $\Gamma-$subsemigroup. □

Theorem 2.1.5. *Let $b \in \mathbb{R}$ and Γ be a nonempty subset of $\mathbb{R}$. Then the followings are equivalent.*

(i) $(-\infty, b)$ *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$.

(ii) $(-\infty, b]$ *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$.

(iii) $\sup \Gamma \leq -b$.

Proof. First, by contrapositive, suppose that $\sup \Gamma > -b$. Let $k \in \Gamma$ be such that $\sup \Gamma \geq k > -b$. Then $\frac{b-k}{2} < b$. Let $m \in (\frac{b-k}{2}, b) \subseteq (-\infty, b)$. Thus $m + k + m > \frac{b-k}{2} + k + \frac{b-k}{2} = b$, so that $m + k + m \notin (-\infty, b)$. Therefore $(-\infty, b)$ is not a $\Gamma-$subsemigroup of $\mathbb{R}$, neither is $(-\infty, b]$ by using the same argument.

Conversely, assume that $\sup \Gamma \leq -b$. Then $(-\infty, b)+\Gamma+(-\infty, b) \subseteq (-\infty, b)$ and $(-\infty, b]+\Gamma+(-\infty, b] \subseteq (-\infty, b]$. The result follows. □

Consequently from Theorem 2.1.5 and Proposition 2.1.1, we have

Corollary 2.1.6. *Let* $a \in \mathbb{R}$ *and* Γ *be a nonempty subset of* $\mathbb{R}$. *Then the followings are equivalent.*

(i) (a, ∞) *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$.

(ii) $[a, \infty)$ *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$.

(iii) $\inf \Gamma \geq -a$.

Proof. We obtain from Theorem 2.1.5 and Proposition 2.1.1 that

(a, ∞) is a $\Gamma-$subsemigroup of $\mathbb{R}$

if and only if $(-\infty, -a)$ is a $(-\Gamma)-$subsemigroup of $\mathbb{R}$

if and only if $\sup(-\Gamma) \leq -(-a)$

if and only if $-\inf \Gamma \leq a$

if and only if $\inf \Gamma \geq -a$. □

For a fixed $\Gamma \subseteq \mathbb{R}$, we list intervals which are $\Gamma-$subsemigroups of $\mathbb{R}$. Before doing so, it is a good place to point out the following simple result.

Remark 2.1.7. *If I is a $\Gamma-$semigroup and Γ' is a nonempty subset of Γ, then I is a $\Gamma'-$semigroup.*

Toward the end of this section, for each given real interval Γ, we characterize all types of real intervals which are $\Gamma-$subsemigroups of $\mathbb{R}$.

Proposition 2.1.8. *Let I be a nonempty subset of $\mathbb{R}$. Then I is a $\{0\}-$subsemigroup of $\mathbb{R}$ if and only if I is a subsemigroup of $\mathbb{R}$ under the usual addition.*

Proof. First, assume that I is a $\{0\}-$subsemigroup of $\mathbb{R}$. Then $I+I = I+\{0\}+I \subseteq I$ which implies that I is a subsemigroup of $\mathbb{R}$ under the usual addition. The converse is obvious. □

We see from Proposition 2.1.8 that subsemigroups of $\mathbb{R}$ and $\{0\}-$subsemigroups of $\mathbb{R}$ (under usual addition) are identical.

Theorem 2.1.9. *Let $\Gamma = \{a\}$. Then a real interval I is a $\Gamma-$subsemigroup of $\mathbb{R}$ if and only if I is one of the following forms :*

(i) $\mathbb{R}$, *(ii)* $\{-a\}$,

(iii) $[y,\infty)$ *where* $y \geq -a$, *(iv)* (y,∞) *where* $y \geq -a$,

(v) $(-\infty,y]$ *where* $y \leq -a$, *(vi)* $(-\infty,y)$ *where* $y \leq -a$.

Proof. First, according to Proposition 2.1.8 and Proposition 1.1, it suffices to suppose that $a \neq 0$. Assume that I is a $\Gamma-$subsemigroup of $\mathbb{R}$. If I is not bounded above and bounded below, then $I = \mathbb{R}$. Thus, there are three more cases.

Case 1. $I = [y,\infty)$ or (y,∞) for some $y \in \mathbb{R}$.

If $I = [y,\infty)$, then $y + a + y \geq y$, so $y \geq -a$.

Now assume that $I = (y,\infty)$. Suppose $y < -a$, then $\frac{y-a}{2} \in I$. So $y = \frac{y-a}{2} + a + \frac{y-a}{2} \in I$ which is a contradiction. Hence $y \geq -a$.

Case 2. $I = (-\infty,y]$ or $(-\infty,y)$ for some $y \in \mathbb{R}$.

Since I is a $\Gamma-$semigroup, by Proposition 2.1.1, $-I$ is a $(-\Gamma)-$semigroup. Now, we obtain that $[-y,\infty)$ (or $(-y,\infty)$) is a $\{-a\}-$subsemigroup of $\mathbb{R}$. By Case 1, we have $-y \geq -(-a) = a$, i.e., $y \leq -a$.

Case 3. $I = [x,y], [x,y), (x,y]$ or (x,y) for some $x,y \in \mathbb{R}$.

If $x < y$, by Theorem 2.1.4, we obtain that I cannot be a $\Gamma-$subsemigroup of $\mathbb{R}$.

Now assume that $I = \{x\}$. Then $x + a + x = x$ which implies that $x = -a$.

Conversely, it is obvious that $\mathbb{R}$ and $\{-a\}$ are $\Gamma-$subsemigroups of $\mathbb{R}$. If $y \geq -a$, then $[y,\infty)+a+[y,\infty) \subseteq [y,\infty)$ and $(y,\infty)+a+(y,\infty) \subseteq (y,\infty)$. Next, suppose that $y \leq -a$. Thus $(-\infty,y]+a+(-\infty,y] \subseteq (-\infty,y]$ and $(-\infty,y)+a+(-\infty,y) \subseteq (-\infty,y)$. Hence (i)–(vi) are $\Gamma-$subsemigroups of $\mathbb{R}$. □

Theorem 2.1.10. *Let $\Gamma = (a,b)$, $[a,b)$, $(a,b]$ or $[a,b]$ where $a < b$. Then a real interval I is a $\Gamma-$subsemigroup of $\mathbb{R}$ if and only if I is one of the following forms:*

(i) $\mathbb{R}$, *(ii)* $[y,\infty)$ *where* $y \geq -a$,

(iii) (y,∞) *where* $y \geq -a$, *(iv)* $(-\infty,y]$ *where* $y \leq -b$,

(v) $(-\infty,y)$ *where* $y \leq -b$.

Proof. First, let I be a $\Gamma-$subsemigroup. Clearly, $I = \mathbb{R}$ if I is not bounded above or bounded below. Then there are only three cases to be considered.

Case 1. $I = [y,\infty)$ or (y,∞) for some $y \in \mathbb{R}$.

If $I = [y,\infty)$, then $y + x + y \geq y$ for all $x \in \Gamma$, i.e., $y \geq -x$ for all $x \in \Gamma$ which implies that $y \geq \sup(-\Gamma) = -a$.

Now, assume that $I = (y,\infty)$. Suppose that $y < -a$ and let $k \in (y,-a) \cap (-b,-a)$. Then $\frac{y+k}{2} \in I$ and $-k \in \Gamma$. So $y = \frac{y+k}{2} - k + \frac{y+k}{2} \in I$ which is a contradiction. Hence $y \geq -a$.

Case 2. $I = (-\infty,y]$ or $(-\infty,y)$ for some $y \in \mathbb{R}$.

By Proposition 2.1.1, $-I$ is a $(-\Gamma)-$semigroup. Now, $-I = [-y,\infty)$ or $(-y,\infty)$ and $-\Gamma = (-b,-a)$, $(-b,-a]$, $[-b,-a)$ or $[-b,-a]$. It follows from Case 1 that

$-y \geq -(-b) = b$, i.e., $y \leq -b$.

Case 3. $I = (x, y), [x, y), (x, y]$ or $[x, y]$ for some $x, y \in \mathbb{R}$.

If $x < y$, by Theorem 2.1.4, the real interval I cannot be a $\Gamma-$subsemigroup of $\mathbb{R}$. Thus $I = \{x\}$, by Theorem 2.1.3, $\Gamma = \{-x\}$ which is a contradiction. As a result, this case is impossible.

For the reverse direction, we see that

$[y, \infty) + \Gamma + [y, \infty) \subseteq [2y + a, \infty) \subseteq [y, \infty)$ where $y \geq -a$,

$(y, \infty) + \Gamma + (y, \infty) \subseteq (2y + a, \infty) \subseteq (y, \infty)$ where $y \geq -a$,

$(-\infty, y] + \Gamma + (-\infty, y] \subseteq (-\infty, 2y + b] \subseteq (-\infty, y]$ where $y \leq -b$,

$(-\infty, y) + \Gamma + (-\infty, y) \subseteq (-\infty, 2y + b) \subseteq (-\infty, y)$ where $y \leq -b$.

Therefore, the proof is complete. □

Theorem 2.1.11. *Let* $\Gamma = (-\infty, b]$ *or* $(-\infty, b)$ *where* $b \in \mathbb{R}$. *Then a real interval* I *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$ *if and only if* I *is one of the following forms:*

(i) $\mathbb{R}$,

(ii) $(-\infty, y]$ *where* $y \leq -b$,

(iii) $(-\infty, y)$ *where* $y \leq -b$.

Proof. Suppose first that I is a $\Gamma-$subsemigroup of $\mathbb{R}$. Note that I cannot be bounded below because Γ is not bounded below. Since $(b - 1, b] \subseteq (-\infty, b]$ and $(b - 1, b) \subseteq (-\infty, b)$, it follows from Remark 2.1.7 that I is also a $(b - 1, b]-$ or $(b - 1, b)-$subsemigroup of $\mathbb{R}$. By Theorem 2.1.10 and the fact that I is not bounded below, we have $I = \mathbb{R}$, $(-\infty, y]$ or $(-\infty, y)$ where $y \leq -b$.

For the converse, we see that

$$(-\infty, y] + \Gamma + (-\infty, y] \subseteq (-\infty, 2y + b] \subseteq (-\infty, y] \quad \text{where } y \leq -b.$$

Therefore, the theorem is completely proved. □

Corollary 2.1.12. *Let* $\Gamma = [a, \infty)$ *or* (a, ∞) *where* $a \in \mathbb{R}$. *Then a real interval* I *is a* Γ-*subsemigroup of* $\mathbb{R}$ *if and only if* I *is one of the following forms:*

(i) $\mathbb{R}$,

(ii) $[y, \infty)$ *where* $y \geq -a$,

(iii) (y, ∞) *where* $y \geq -a$.

Proof. The result follows by replacing I and Γ in Theorem 2.1.11 by $-I$ and $-\Gamma$, respectively, and applying Proposition 2.1.1. □

2.2 $\Gamma-$Subsemigroups of $\mathbb{R}$ under Usual Multiplication

Recall from Example 1.3 that $\mathbb{R}$ is a $\Gamma-$semigroup under usual multiplication for any nonempty subset Γ of $\mathbb{R}$. In the first part of this section, for each real interval subset I of $\mathbb{R}$, we find subsets Γ of $\mathbb{R}$ so that I is a $\Gamma-$subsemigroup of $\mathbb{R}$. The following results will be used variously in this section.

Proposition 2.2.1. *Let I and Γ be nonempty subsets of $\mathbb{R}$. Then*

$$I \text{ is a } \Gamma\text{-semigroup if and only if } -I \text{ is a } (-\Gamma)\text{-semigroup.}$$

Proof. The proof is similar to the proof of Proposition 2.1.1. □

Remark 2.2.2. *Let I and Γ be nonempty subsets of $\mathbb{R}$. If I is a $\Gamma-$subsemigroup of $\mathbb{R}$ and $0 \notin I$, then $0 \notin \Gamma$.*

Equivalently, Γ containing 0 implies that I must contain 0.

Proposition 2.2.3. *Let I and Γ be nonempty subsets of $\mathbb{R}$ such that I is a $\Gamma-$subsemigroup of $\mathbb{R}$ and $I \neq \{0\}$. If I is bounded, then Γ is bounded.*

Proof. Assume that I is bounded. Choose $x \in I\backslash\{0\}$ and let $\alpha \in \Gamma$. By the assumption, $\inf I$ and $\sup I$ exist. Since I is a $\Gamma-$subsemigroup of $\mathbb{R}$, $x\alpha x \in I$ so that $\inf I \leq x\alpha x \leq \sup I$. Thus $\frac{\inf I}{x^2} \leq \alpha \leq \frac{\sup I}{x^2}$. Since α is arbitrary, Γ is bounded. □

Now, we are ready to characterize subsets Γ of $\mathbb{R}$ which I is a $\Gamma-$subsemigroup of $\mathbb{R}$ for any real interval I. We consider first where I is a singleton set.

Remark 2.2.4. *Let Γ be a nonempty subset of $\mathbb{R}$ and $x \in \mathbb{R}\backslash\{0\}$. Then*
(i) $\{0\}$ is a $\Gamma-$subsemigroup of $\mathbb{R}$,
(ii) $\{x\}$ is a $\Gamma-$subsemigroup of $\mathbb{R}$ if and only if $\Gamma = \{\frac{1}{x}\}$.

Now, we study where I is a bounded interval. Remark 2.2.2 suggests us to consider in many cases depending on the existence of 0 in I.

Theorem 2.2.5. *Let $b \in \mathbb{R}$ with $b > 0$ and Γ be a nonempty subset of $\mathbb{R}$. Then the followings are equivalent :*
(i) $[0, b]$ is a $\Gamma-$subsemigroup of $\mathbb{R}$.
(ii) $[0, b)$ is a $\Gamma-$subsemigroup of $\mathbb{R}$.
(iii) $0 \leq \inf \Gamma \leq \sup \Gamma \leq \frac{1}{b}$.

Proof. First, assume that $[0, b)$ is a $\Gamma-$subsemigroup of $\mathbb{R}$. If there exists $k \in \Gamma$ and $k < 0$, then $(\frac{b}{2})k(\frac{b}{2}) = (\frac{b}{2})^2k < 0$ so $(\frac{b}{2})k(\frac{b}{2}) \notin [0, b)$ which is a contradiction. Thus $\inf \Gamma \geq 0$.

Suppose that $\sup \Gamma > \frac{1}{b}$. Let $k \in (\frac{1}{b}, \sup \Gamma] \cap \Gamma$, thus $b > \frac{1}{k}$, so $b^2 > \frac{b}{k}$ and it implies that $b > \sqrt{\frac{b}{k}}$. Let $x \in (\sqrt{\frac{b}{k}}, b) \subseteq [0, b)$. Then $b < x^2k$. Hence $xkx = x^2k \notin [0, b)$ which is a contradiction. Therefore $0 \leq \inf \Gamma \leq \sup \Gamma \leq \frac{1}{b}$.

Conversely, assume that $0 \leq \inf \Gamma \leq \sup \Gamma \leq \frac{1}{b}$. Let $k \in \Gamma$ and $m, n \in [0, b)$. Then $0 \leq k \leq \frac{1}{b}$ and $0 \leq mn < b^2$. Thus $0 \leq mkn = mnk < b^2k \leq b$, so $mkn \in [0, b)$. The proof of (ii) if and only if (iii) is complete.

The proof of (i) if and only if (iii) is obtained similarly. □

Immediately, from Theorem 2.2.5 and Proposition 2.2.1, we have

Corollary 2.2.6. *Let $a \in \mathbb{R}$ with $a < 0$ and Γ be a nonempty subset of $\mathbb{R}$. Then the followings are equivalent:*

(i) $[a, 0]$ *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$.

(ii) $(a, 0]$ *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$.

(iii) $\frac{1}{a} \leq \inf\Gamma \leq \sup\Gamma \leq 0$.

Proposition 2.2.7. *Let $a \in \mathbb{R}$ with $a < 0$ and Γ be a nonempty subset of $\mathbb{R}$. Then the followings are equivalent:*

(i) $[a, 0)$ *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$.

(ii) $(a, 0)$ *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$.

(iii) $\frac{1}{a} \leq \inf\Gamma \leq \sup\Gamma \leq 0$ *and* $0 \notin \Gamma$.

Proof. Without loss of generality, we prove (ii) if and only if (iii). First, assume that $(a, 0)$ is a $\Gamma-$subsemigroup of $\mathbb{R}$. Then $0 \notin \Gamma$ by Remark 2.2.2. If there exists $k \in \Gamma$ and $k > 0$, then $(\frac{a}{2})k(\frac{a}{2}) = (\frac{a}{2})^2k > 0$ so $(\frac{a}{2})k(\frac{a}{2}) \notin (a, 0)$ which is a contradiction. Thus $\sup\Gamma \leq 0$. Next, suppose that $\inf\Gamma < \frac{1}{a}$, let $k \in [\inf\Gamma, \frac{1}{a}) \cap \Gamma$. Then $a < \frac{1}{k}$, so $a^2 > \frac{a}{k}$ which implies that $a < -\sqrt{\frac{a}{k}}$. Let $x \in (a, -\sqrt{\frac{a}{k}}) \subseteq (a, 0)$. Thus $x^2k < a$. Hence $xkx = x^2k \notin (a, 0)$ which is a contradiction. Therefore, $\frac{1}{a} \leq \inf\Gamma \leq \sup\Gamma \leq 0$ and $0 \notin \Gamma$.

Conversely, assume that $\frac{1}{a} \leq \inf\Gamma \leq \sup\Gamma \leq 0$ and $0 \notin \Gamma$. Let $k \in \Gamma$ and $m, n \in (a, 0)$. Then $\frac{1}{a} \leq k < 0$ and $0 < mn < a^2$. Thus $a \leq a^2k < mnk = mkn < 0$. So $mkn \in (a, 0)$.

This proof is complete. □

Corollary 2.2.8 is obtained immediately from Proposition 2.2.7 and Proposition 2.2.1.

Corollary 2.2.8. *Let $b \in \mathbb{R}$ with $b > 0$ and Γ be a nonempty subset of $\mathbb{R}$. Then the followings are equivalent :*

(i) $(0,b]$ *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$.

(ii) $(0,b)$ *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$.

(iii) $0 \le \inf\Gamma \le \sup\Gamma \le \frac{1}{b}$ *and* $0 \notin \Gamma$.

The following results are the characterization of subsets Γ of $\mathbb{R}$ as I is a bounded interval such that 0 is not an endpoint of I.

Theorem 2.2.9. *Let* $a, b \in \mathbb{R}$ *with* $a < 0 < b$ *and* Γ *be a nonempty subset of* $\mathbb{R}$. *Then* (a,b), $[a,b)$, $(a,b]$ *or* $[a,b]$ *are* $\Gamma-$*subsemigroups of* $\mathbb{R}$ *if and only if* $\max\{\frac{1}{a}, \frac{a}{b^2}\} \le \inf\Gamma \le \sup\Gamma \le \min\{\frac{1}{b}, \frac{b}{a^2}\}$.

Proof. First, let I=(a,b), $[a,b)$, $(a,b]$ or $[a,b]$ be a $\Gamma-$subsemigroup of $\mathbb{R}$. Since I is bounded and from Proposition 2.2.3, Γ must be bounded. There are two cases.

Case 1. $\inf\Gamma \ge 0$.

If $\sup\Gamma > \frac{1}{b}$, let $k \in (\frac{1}{b}, \sup\Gamma] \cap \Gamma$, thus $b > \frac{1}{k}$, so $b^2 > \frac{b}{k}$ implies that $b > \sqrt{\frac{b}{k}}$. Let $x \in (\sqrt{\frac{b}{k}}, b) \subseteq (a,b)$. Then $b < x^2k$. Hence $xkx = x^2k \notin I$ which is a contradiction. Hence $\sup\Gamma \le \frac{1}{b}$. Suppose further that $\sup\Gamma > \frac{b}{a^2}$, let $k \in (\frac{b}{a^2}, \sup\Gamma] \cap \Gamma$, thus $a^2 > \frac{b}{k}$ implies that $a < -\sqrt{\frac{b}{k}}$. Let $x \in (a, -\sqrt{\frac{b}{k}}) \subseteq (a,b)$. Then $b < x^2k$. Hence $xkx = x^2k \notin I$, again, a contradiction occurs. Hence $\sup\Gamma \le \frac{b}{a^2}$. This shows that $\max\{\frac{1}{a}, \frac{a}{b^2}\} < 0 \le \inf\Gamma \le \sup\Gamma \le \min\{\frac{1}{b}, \frac{b}{a^2}\}$.

Case 2. $\inf\Gamma < 0$.

There are two subcases depending on $\sup\Gamma$.

Subcase 2.1. $\sup\Gamma \le 0$.

If $\inf\Gamma < \frac{1}{a}$, let $k \in [\inf\Gamma, \frac{1}{a}) \cap \Gamma$, thus $a^2 > \frac{a}{k}$ implies that $a < -\sqrt{\frac{a}{k}}$. Let $x \in (a, -\sqrt{\frac{a}{k}}) \subseteq (a,b)$. Then $xkx = x^2k < a$ so $xkx \notin I$ which is a contradiction. Hence $\inf\Gamma \ge \frac{1}{a}$. Suppose that $\inf\Gamma < \frac{a}{b^2}$, let $k \in [\inf\Gamma, \frac{a}{b^2}) \cap \Gamma$, thus $b^2 > \frac{a}{k}$ implies that $\sqrt{\frac{a}{k}} < b$. Let $x \in (\sqrt{\frac{a}{k}}, b) \subseteq (a,b)$. Then $xkx = x^2k < a$ so $xkx \notin I$ which is a contradiction. Hence $\inf\Gamma \ge \frac{a}{b^2}$. Therefore $\max\{\frac{1}{a}, \frac{a}{b^2}\} \le \inf\Gamma \le \sup\Gamma \le 0 < \min\{\frac{1}{b}, \frac{b}{a^2}\}$.

Subcase 2.2. $\sup \Gamma > 0$.

The same proof of Case 1 shows that $\sup \Gamma \leq \min\{\frac{1}{b}, \frac{b}{a^2}\}$. Moreover, $\max\{\frac{1}{a}, \frac{a}{b^2}\} \leq \inf \Gamma$ from the proof of Subcase 2.1. Then the result follows.

Conversely, assume that $\max\{\frac{1}{a}, \frac{a}{b^2}\} \leq \inf \Gamma \leq \sup \Gamma \leq \min\{\frac{1}{b}, \frac{b}{a^2}\}$ and let $k \in \Gamma$. Suppose that $I = [a, b]$. Note that the proofs for the other choices of I are obtained similarly. Firstly, assume that $k \geq 0$. Let $m, n \in [a, b]$. We will show that $mkn \in [a, b]$. Without loss of generality, there are three possibilities.

Case 1. $m, n < 0$.

Then $a < mkn$ and $mn \leq a^2$. Thus $mkn \leq a^2k$ so that $mkn \leq a^2k \leq b$ since $k \leq \frac{b}{a^2}$.

Case 2. $m, n > 0$.

Then $a < mkn$ and $mn \leq b^2$. Thus $mkn \leq b^2k$ and then $mkn \leq b^2k \leq b$ because $k \leq \frac{1}{b}$.

Case 3. $m < 0$ and $n > 0$.

Then $mkn < b$. Since $a \leq m < 0$ and $0 < n \leq b$, we have $akb \leq mkn$. Since $k \leq \frac{1}{b}$, it follows that $a \leq akb$.

From all cases we can conclude that $mkn \in [a, b]$.

Secondly, assume that $k < 0$ and $m, n \in [a, b]$ $(a < 0 < b)$. We will show that $mkn \in [a, b]$. There are three different choices for m and n.

Case 1. $m, n < 0$.

Then $mkn \leq 0 < b$. Since $mn \leq a^2$ and $\frac{1}{a} \leq k$, we have $a = \frac{a^2}{a} \leq a^2k \leq mkn$.

Case 2. $m, n > 0$.

Then $mkn \leq 0 < b$. Since $mn \leq b^2$ and $\frac{a}{b^2} \leq k$, we obtain that $a = \frac{ab^2}{b^2} \leq b^2k \leq mkn$.

Case 3. $m < 0$ and $n > 0$.

Then $a < mkn$. Since $a \leq m < 0$ and $0 < n \leq b$, $mkn \leq akb$. Since $k \geq \frac{1}{a}$, $akb \leq b$.

From all cases we can conclude that $mkn \in [a, b]$.

Therefore I is a $\Gamma-$subsemigroup of $\mathbb{R}$. $\square$

Theorem 2.2.10. *Let $a, b \in \mathbb{R}\backslash\{0\}$ and I=(a,b), $[a,b)$, $(a,b]$ or $[a,b]$ with $0 \notin I$. Then I is not a $\Gamma-$subsemigroup of $\mathbb{R}$ for any nonempty subset Γ of $\mathbb{R}$.*

Proof. Since $0 \notin I$ and $a, b \in \mathbb{R}\backslash\{0\}$, we have either $0 < a < b$ or $a < b < 0$. First, assume that $0 < a < b$. Suppose that there exists a nonempty subset Γ of $\mathbb{R}$ such that I is a $\Gamma-$subsemigroup of $\mathbb{R}$. Since $0 \notin I$ and I is bounded, $0 \notin \Gamma$ and Γ is bounded so that $\inf \Gamma$ and $\sup \Gamma$ exist. If there exists $k \in \Gamma$ with $k < 0$, then $(\frac{a+b}{2})^2 k < 0 < a$, thus $(\frac{a+b}{2})k(\frac{a+b}{2}) = (\frac{a+b}{2})^2 k \notin I$ which is a contradiction. Then $\inf \Gamma \geq 0$.

Suppose that $\sup \Gamma > \frac{1}{b}$, let $k \in (\frac{1}{b}, \sup \Gamma] \cap \Gamma$, thus $b > \frac{1}{k}$, so $b^2 > \frac{b}{k}$ implies that $b > \sqrt{\frac{b}{k}}$. Let $x \in (\sqrt{\frac{b}{k}}, b) \cap (a,b)$. Then $b < x^2 k$. Hence $xkx = x^2 k \notin I$ which is a contradiction. Thus $0 \leq \inf \Gamma \leq \sup \Gamma \leq \frac{1}{b} < \frac{1}{a}$. Since $\inf \Gamma < \frac{1}{a}$, there exists $k \in [\inf \Gamma, \frac{1}{a}) \cap \Gamma$. Thus $a^2 < \frac{a}{k}$ implies that $a < \sqrt{\frac{a}{k}}$. Let $y \in (a, \sqrt{\frac{a}{k}}) \cap (a,b)$. Then $yky = y^2 k < a$. So $yky \notin I$. A contradiction occurs.

Consequently, I is not a $\Gamma-$subsemigroup of $\mathbb{R}$ for any nonempty subset Γ of $\mathbb{R}$ and $0 < a < b$.

Finally, assume $a < b < 0$. Suppose that there exists a nonempty subset Γ of $\mathbb{R}$ such that I is a $\Gamma-$subsemigroup of $\mathbb{R}$. By Proposition 2.2.1, $-I$ is a $(-\Gamma)-$subsemigroup of $\mathbb{R}$ which contradicts the first case of the proof.

This proof is complete. □

The following results are the characterization of subsets Γ of $\mathbb{R}$ as I is an unbounded interval.

Theorem 2.2.11. *For any nonempty subset Γ of $\mathbb{R}$,*

$$(-\infty, 0] \text{ is a } \Gamma-\text{subsemigroup of } \mathbb{R} \text{ if and only if } \sup \Gamma \leq 0.$$

Proof. First, assume that $(-\infty, 0]$ is a $\Gamma-$subsemigroup of $\mathbb{R}$. If there exists $k \in \Gamma$ with $k > 0$, then $(-1)k(-1) = k \notin (-\infty, 0]$ which leads to a contradiction. Hence sup $\Gamma \leq 0$.

Conversely, it is clear that $(-\infty, 0]$ is a $\Gamma-$subsemigroup of $\mathbb{R}$ provided $\sup \Gamma \leq 0$.

□

The following corollary results from Theorem 2.2.11 and Proposition 2.2.1.

Corollary 2.2.12. *For any nonempty subset Γ of $\mathbb{R}$,*

$$[0, \infty) \text{ is a } \Gamma\text{−subsemigroup of } \mathbb{R} \text{ if and only if } \inf \Gamma \geq 0.$$

Proposition 2.2.13. *For any nonempty subset Γ of $\mathbb{R}$,*

$$(-\infty, 0) \text{ is a } \Gamma\text{−subsemigroup of } \mathbb{R} \text{ if and only if } \sup \Gamma \leq 0 \text{ and } 0 \notin \Gamma.$$

Proof. First, assume that $(-\infty, 0)$ is a $\Gamma-$subsemigroup of $\mathbb{R}$. From the argument in the proof of Theorem 2.2.11, we have $\sup \Gamma \leq 0$. By Remark 2.2.2, $0 \notin \Gamma$ since $0 \notin (-\infty, 0)$.

Conversely, it is clear that $(-\infty, 0)$ is a $\Gamma-$subsemigroup of $\mathbb{R}$ provided $\sup \Gamma \leq 0$ and $0 \notin \Gamma$. □

Applying Proposition 2.2.1 and Proposition 2.2.13, we obtain the following corollary.

Corollary 2.2.14. *For any nonempty subset Γ of $\mathbb{R}$,*

$$(0, \infty) \text{ is a } \Gamma\text{−subsemigroup of } \mathbb{R} \text{ if and only if } \inf \Gamma \geq 0 \text{ and } 0 \notin \Gamma.$$

The next results are the characterization of subsets Γ of $\mathbb{R}$ as I is an unbounded real interval which does not contain 0.

Theorem 2.2.15. *Let $b \in \mathbb{R}$ with $b < 0$ and Γ be a nonempty subset of $\mathbb{R}$. Then the followings are equivalent:*

(i) $(-\infty, b)$ *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$.

(ii) $(-\infty, b]$ *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$.

(iii) $\sup \Gamma \leq \frac{1}{b}$.

Proof. Let I be $(-\infty, b)$ or $(-\infty, b]$. First, assume that I is a $\Gamma-$subsemigroup of $\mathbb{R}$. Note that $0 \notin \Gamma$ because $0 \notin I$. If there exists $k \in \Gamma$ with $k > 0$, then $(b-1)^2 k > 0$ so that $(b-1)k(b-1) = (b-1)^2 k \notin I$ which is impossible. Hence $k < 0$ for all $k \in \Gamma$. Suppose that there exists $k \in \Gamma$ with $k > \frac{1}{b}$. Then $b^2 k > b$, so $b^2 < \frac{b}{k}$. Now $b > -\sqrt{\frac{b}{k}}$. Let $x \in (-\sqrt{\frac{b}{k}}, b) \subseteq I$, thus $x^2 k > b$. Hence $xkx = x^2 k \notin I$. A contradiction occurs. Therefore for all $k \in \Gamma$, $k \leq \frac{1}{b}$, i.e., $\sup \Gamma \leq \frac{1}{b}$.

Conversely, it is clear that I is a $\Gamma-$subsemigroup of $\mathbb{R}$ where $\sup \Gamma \leq \frac{1}{b}$. □

Immediately from Theorem 2.2.15 and Proposition 2.2.1, we have the following.

Corollary 2.2.16. *Let $a \in \mathbb{R}$ with $a > 0$ and Γ be a nonempty subset of $\mathbb{R}$. Then the followings are equivalent:*

(i) (a, ∞) *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$.

(ii) $[a, \infty)$ *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$.

(iii) $\inf \Gamma \geq \frac{1}{a}$.

The following results are the characterization of subsets Γ of $\mathbb{R}$ as I is an unbounded real interval containing 0.

Theorem 2.2.17. *Let $a \in \mathbb{R}\backslash\{0\}$ and I be $(-\infty, a)$, $(-\infty, a]$, (a, ∞) or $[a, \infty)$ with $0 \in I$. Then for any nonempty subset Γ of $\mathbb{R}$,*

$$I \text{ is a } \Gamma-\text{subsemigroup of } \mathbb{R} \text{ if and only if } \Gamma = \{0\}.$$

Proof. First, assume that $I=(-\infty,a)$. Then $a>0$. We prove by contrapositive. Let Γ be a nonempty subset of $\mathbb{R}$ such that $\Gamma\neq\{0\}$. Then there exists $k\in\Gamma\backslash\{0\}$.

Case 1. $k>0$.

Since $-\frac{1}{k},-2a\in I$, it follows that $(-\frac{1}{k})(k)(-2a)=2a\notin(-\infty,a)$.

Case 2. $k<0$.

Since $\frac{4}{k},\frac{a}{2}\in I$, it follows that $(\frac{a}{2})k(\frac{4}{k})=2a\notin(-\infty,a)$.

We can conclude from all cases that $(-\infty,a)$ is not a $\Gamma-$subsemigroup of $\mathbb{R}$.

The converse is obvious.

Similarly, $(-\infty,a]$ is a $\Gamma-$subsemigroup of $\mathbb{R}$ if and only if $\Gamma=\{0\}$. Besides, let $I=(a,\infty)$ or $[a,\infty)$. Then $a<0$ because $0\in I$ and $a\neq 0$. Applying Proposition 2.2.1, we obtain that (a,∞) or $[a,\infty)$ are $\Gamma-$subsemigroups of $\mathbb{R}$ if and only if $\Gamma=\{0\}$. □

From now on, for each real interval Γ we characterize all types of real intervals I such that I is a $\Gamma-$subsemigroup of $\mathbb{R}$. We consider the case that Γ is a singleton set.

Remark 2.2.18. *Let I be a nonempty subset of $\mathbb{R}$.*

$$I \text{ is a } \{0\}-\text{subsemigroup of } \mathbb{R} \text{ if and only if } 0\in I.$$

Theorem 2.2.19. *Let $a>0$. Then a real interval I is a $\{a\}-$subsemigroup of $\mathbb{R}$ if and only if I is one of the following forms:*

(i) $\mathbb{R}$, *(ii)* $\{0\}$, *(iii)* $\{\frac{1}{a}\}$,

(iv) $(0,\infty)$, *(v)* $[0,\infty)$,

(vi) (x,∞) *where* $x\geq\frac{1}{a}$ *(vii)* $[x,\infty)$ *where* $x\geq\frac{1}{a}$,

(viii) $[x,y]$ *where* $-\frac{1}{a}\leq x\leq 0\leq x^2a\leq y\leq\frac{1}{a}$,

(ix) $[x,y)$ *where* $-\frac{1}{a}\leq x\leq 0\leq x^2a<y\leq\frac{1}{a}$,

(x) $(x,y]$ *where* $-\frac{1}{a} \le x \le 0 \le x^2a \le y \le \frac{1}{a}$,

(xi) (x,y) *where* $-\frac{1}{a} \le x \le 0 \le x^2a \le y \le \frac{1}{a}$.

Proof. First, let a real interval I be a $\{a\}$–subsemigroup.

Case 1. I is not bounded above and below. Then $I = \mathbb{R}$.

Case 2. $I = [x,\infty)$ or (x,∞) for some $x \in \mathbb{R}$.

If $x < 0$, then $2x = (\frac{4}{a})(a)(\frac{x}{2}) \in I$ which is a contradiction. Thus $x \ge 0$. Suppose that $0 < x < \frac{1}{a}$. Then $x^2 < \frac{x}{a}$ so $x < \sqrt{\frac{x}{a}}$. Let $k \in (x, \sqrt{\frac{x}{a}}) \subseteq I$ so that $k^2a < x$. Thus $kak = k^2a \notin I$ which is a contradiction. Hence $x = 0$ or $x \ge \frac{1}{a}$.

Case 3. $I = (-\infty, x]$ or $(-\infty, x)$ for some $x \in \mathbb{R}$.

If $x > 0$, then $2x = (-1)(a)(-\frac{2x}{a}) \in I$ which is a contradiction. Thus $x \le 0$. Note that $a(x-1)^2 > 0$. Then $(x-1)a(x-1) = a(x-1)^2 \notin I$ which is a contradiction. Hence this case is impossible.

Case 4. $I = [x,y], [x,y), (x,y]$ or (x,y) for some $x, y \in \mathbb{R}$.

If $I = \{x\}$, then $xax = x$ implies that $x = 0$ or $x = \frac{1}{a}$.

Now, assume that $x < y$. If $y \le 0$, then $a(\frac{x+y}{2})^2 > 0$ so that $(\frac{x+y}{2})a(\frac{x+y}{2}) = a(\frac{x+y}{2})^2 \notin I$ which leads to a contradiction. Thus $y > 0$. Suppose that $y > \frac{1}{a}$. Then $y^2 > \frac{y}{a}$, so $y > \sqrt{\frac{y}{a}}$. Let $k \in (x,y) \cap (\sqrt{\frac{y}{a}}, y)$. Thus $y < k^2a$ and then $kak = k^2a \notin I$ which is a contradiction. Hence $0 < y \le \frac{1}{a}$.

Next, suppose that $x > 0$. Then $0 < x < y \le \frac{1}{a}$, so $x^2 < \frac{x}{a}$. Thus $x < \sqrt{\frac{x}{a}}$. Then there exists $k \in (x,y) \cap (x, \sqrt{\frac{x}{a}})$ and $k^2a < x$. As a result, $kak = k^2a \notin I$ which is a contradiction. Thus $x \le 0$. Suppose further that $x < -\frac{1}{a}$. Then let $k \in (x, -\frac{1}{a}) \subseteq (x,y)$. Thus $k^2 > \frac{1}{a^2}$ and $k^2a > \frac{1}{a} \ge y$. Hence $kak = k^2a \notin I$, again, a contradiction occurs. Thus $-\frac{1}{a} \le x \le 0$.

Finally, suppose that $x^2a > y$. Then $x^2 > \frac{y}{a}$ and $x < -\sqrt{\frac{y}{a}}$. Let $k \in (x, -\sqrt{\frac{y}{a}}) \subseteq (x,y)$. Therefore $k^2a > y$. This leads to a contradiction because $kak = k^2a \notin I$.

From the above argument, we can see that k was chosen with $k \in (x,y)$. Consequently, if $I = [x,y], [x,y), (x,y]$ or (x,y) (where $x < y$), then $-\frac{1}{a} \le x \le 0 <$

$x^2a \le y \le \frac{1}{a}$. Moreover, if $I = [x, y)$, then the inequality $x^2a \le y$ is, in fact, $x^2a < y$ since $xax \in I$.

Conversely, it is clear that (i)–(v) are $\{a\}-$subsemigroups of $\mathbb{R}$. Next, we show that (vii) is a $\{a\}-$subsemigroup of $\mathbb{R}$.

If $x \ge \frac{1}{a}$, then $x > 0$ and $[x, \infty)\{a\}[x, \infty) \subseteq [x^2a, \infty) \subseteq [x, \infty)$.

If $-\frac{1}{a} \le x \le 0 < x^2a \le y \le \frac{1}{a}$, then $xa \le 0 < ya \le 1$ so $x \le xya < y$ and then

$$\begin{aligned}
[x,y]\{a\}[x,y] &= [x,y][x,y]\{a\} \\
&\subseteq [xy,k]\{a\} \quad \text{where} \quad k = \max\{x^2, y^2\} \\
&\subseteq [xya, ka] \\
&\subseteq [x,y].
\end{aligned}$$

The other cases can be shown similarly to the above argument. □

Corollary 2.2.20. *Let I be a nonempty subset of $\mathbb{R}$. Then a real interval I is a $\{1\}-$subsemigroup of $\mathbb{R}$ if and only if I is a subsemigroup of $\mathbb{R}$.*

Proof. This follows from $I\{1\}I = II$. □

The following corollary is the immediate result from Theorem 2.2.19 and Proposition 2.2.1.

Corollary 2.2.21. *Let $a < 0$. Then a real interval I is a $\{a\}-$subsemigroup of $\mathbb{R}$ if and only if I is one of the following forms:*

(i) $\mathbb{R}$, *(ii)* $\{0\}$, *(iii)* $\{\frac{1}{a}\}$,

(iv) $(-\infty, 0)$, *(v)* $(-\infty, 0]$,

(vi) $(-\infty, x)$ *where* $x \le \frac{1}{a}$ *(vii)* $(-\infty, x]$ *where* $x \le \frac{1}{a}$,

(viii) $[x, y]$ *where* $\frac{1}{a} \le x \le y^2a \le 0 \le y \le -\frac{1}{a}$,

(ix) $[x, y)$ *where* $\frac{1}{a} \le x \le y^2a \le 0 \le y \le -\frac{1}{a}$,

(x) $(x, y]$ *where* $\frac{1}{a} \le x < y^2a \le 0 \le y \le -\frac{1}{a}$,

(xi) (x, y) *where* $\frac{1}{a} \le x \le y^2a \le 0 \le y \le -\frac{1}{a}$.

Next, we find all nonempty subsets I of $\mathbb{R}$ which are $\Gamma-$subsemigroups of $\mathbb{R}$ as Γ is an interval which is not bounded above.

Theorem 2.2.22. *Let* $\Gamma = [a, \infty)$ *and* $a > 0$. *Then a real interval* I *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$ *if and only if* I *is one of the following forms:*

(i) $\mathbb{R}$, *(ii)* $\{0\}$,

(iii) $[0, \infty)$, *(iv)* $(0, \infty)$,

(v) $[x, \infty)$ *where* $x \ge \frac{1}{a}$, *(vi)* (x, ∞) *where* $x \ge \frac{1}{a}$.

Proof. First, assume that a real interval I is a $\Gamma-$subsemigroup of $\mathbb{R}$. Then

$$I\{a\}I \subseteq I[a, \infty)I \subseteq I.$$

Thus I is a $\{a\}-$subsemigroup of $\mathbb{R}$. By Theorem 2.2.19, I is one of the following forms: (i) $\mathbb{R}$, (ii) $\{0\}$, (iii) $\{\frac{1}{a}\}$,

(iv) $(0, \infty)$, (v) $[0, \infty)$,

(vi) $[x, \infty)$ where $x \ge \frac{1}{a}$ (vii) (x, ∞) where $x \ge \frac{1}{a}$,

(viii) $[x, y]$ where $-\frac{1}{a} \le x \le 0 \le x^2a \le y \le \frac{1}{a}$,

(ix) $[x, y)$ where $-\frac{1}{a} \le x \le 0 \le x^2a < y \le \frac{1}{a}$,

(x) $(x, y]$ where $-\frac{1}{a} \le x \le 0 \le x^2a \le y \le \frac{1}{a}$,

(xi) (x, y) where $-\frac{1}{a} \le x \le 0 \le x^2a \le y \le \frac{1}{a}$.

If $I = \{\frac{1}{a}\}$, then $\frac{1}{a} + \frac{1}{a^2} = \frac{1}{a^2}(a+1) = \frac{1}{a}(a+1)\frac{1}{a} = \frac{1}{a}$ which is a contradiction. Thus (iii) is impossible. Furthermore, I cannot be (ix)–(xi) because Γ is unbounded and by Proposition 2.2.3.

Conversely, it is clear that (i)–(ii) and (iv)–(v) are $\Gamma-$subsemigroups of $\mathbb{R}$. Note that if $x \ge \frac{1}{a}$, then $[x, \infty)[a, \infty)[x, \infty) \subseteq [x^2a, \infty) \subseteq [x, \infty)$ and $(x, \infty)[a, \infty)(x, \infty) \subseteq (x^2a, \infty) \subseteq (x, \infty)$. Hence (vi) and (vii) are $\Gamma-$subsemigroups of $\mathbb{R}$. □

The following corollary is a result of Theorem 2.2.22 and Proposition 2.2.1.

Corollary 2.2.23. *Let* $\Gamma = (-\infty, a]$ *and* $a < 0$. *Then a real interval* I *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$ *if and only if* I *is one of the following forms:*

(i) $\mathbb{R}$, *(ii)* $\{0\}$,

(iii) $(-\infty, 0]$, *(iv)* $(-\infty, 0)$,

(v) $(-\infty, x]$ *where* $x \leq \frac{1}{a}$, *(vi)* $(-\infty, x)$ *where* $x \leq \frac{1}{a}$.

Theorem 2.2.24. *Let* $\Gamma = [0, \infty)$. *Then a real interval* I *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$ *if and only if* I *is one of the following forms:*

(i) $\mathbb{R}$, *(ii)* $\{0\}$, *(iii)* $[0, \infty)$.

Proof. First, let a real interval I be a $\Gamma-$subsemigroup of $\mathbb{R}$. Since $I\{1\}I \subseteq I[0, \infty)I \subseteq I$. This shows that I a is $\{1\}-$subsemigroup of $\mathbb{R}$. Note that Γ is unbounded. By Proposition 2.2.3 and $0 \in I$, I is one of the forms $\mathbb{R}$, $\{0\}$ and $[0, \infty)$.

The converse is obvious. □

Immediately from Theorem 2.2.24 and Proposition 2.2.1, we have

Corollary 2.2.25. *Let* $\Gamma = (-\infty, 0]$. *Then a real interval* I *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$ *if and only if* I *is one of the following forms:*

(i) $\mathbb{R}$, *(ii)* $\{0\}$, *(iii)* $(-\infty, 0]$.

Theorem 2.2.26. *Let* $\Gamma = [a, \infty)$ *and* $a < 0$. *Then a real interval* I *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$ *if and only if* $I = \mathbb{R}$ *or* $\{0\}$.

Proof. First, let a real interval I be a $\Gamma-$subsemigroup of $\mathbb{R}$. Then

$$I\{a\}I \subseteq I[a, \infty)I \subseteq I \quad \text{and} \quad I[0, \infty)I \subseteq I[a, \infty)I \subseteq I.$$

By Corollary 2.2.21 and Theorem 2.2.24, I must be either $\mathbb{R}$ or $\{0\}$.

The converse is obvious. □

As a consequence of Theorem 2.2.26 and Proposition 2.2.1, we have

Corollary 2.2.27. *Let $\Gamma = (-\infty, b]$ and $b > 0$. Then a real interval I is a $\Gamma-$subsemigroup of $\mathbb{R}$ if and only if $I = \mathbb{R}$ or $\{0\}$.*

Finally, we find all real intervals I of $\mathbb{R}$ which is a $\Gamma-$subsemigroup of $\mathbb{R}$ as Γ is a bounded interval.

Theorem 2.2.28. *Let $\Gamma = [a, 0]$. Then a real interval I is a $\Gamma-$subsemigroup of $\mathbb{R}$ if and only if I is one of the following forms:*

(i) $\mathbb{R}$, *(ii)* $\{0\}$, *(iii)* $(-\infty, 0]$,

(iv) $[x, y]$ *where* $\frac{1}{a} \leq x \leq y^2 a \leq 0 \leq y \leq -\frac{1}{a}$,

(v) $[x, y)$ *where* $\frac{1}{a} \leq x \leq y^2 a < 0 < y \leq -\frac{1}{a}$,

(vi) $(x, y]$ *where* $\frac{1}{a} \leq x < y^2 a \leq 0 \leq y \leq -\frac{1}{a}$,

(vii) (x, y) *where* $\frac{1}{a} \leq x \leq y^2 a < 0 < y \leq -\frac{1}{a}$.

Proof. First, let a real interval I be a $\Gamma-$subsemigroup. Since $I\{a\}I \subseteq I[a, 0]I \subseteq I$ and $0 \in I$, we obtain that I is a $\{a\}-$subsemigroup of $\mathbb{R}$. Applying Corollary 2.2.21, we have I is one of the mentioned forms.

Conversely, it is clear that $\mathbb{R}$, $\{0\}$ and $(-\infty, 0]$ are $\Gamma-$subsemigroups of $\mathbb{R}$. If $\frac{1}{a} \leq x \leq y^2 a < 0 \leq y \leq -\frac{1}{a}$, then $xa \leq 1$ so $x \leq 0 \leq xya \leq y$ and

$$\begin{aligned}
[x, y][a, 0][x, y] &= [x, y][x, y][a, 0] \\
&\subseteq [xy, k][a, 0] \text{ where } k = \max\{x^2, y^2\} \\
&\subseteq [ka, xya] \\
&\subseteq [x, y].
\end{aligned}$$

The other cases can be shown similarly to the above argument. □

The immediate result from Theorem 2.2.28 and Proposition 2.2.1 is

Corollary 2.2.29. *Let* $\Gamma = [0, a]$. *Then a real interval* I *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$ *if and only if* I *is one of the following forms:*

(i) $\mathbb{R}$, *(ii)* $\{0\}$, *(iii)* $[0, \infty)$,

(iv) $[x, y]$ *where* $-\frac{1}{a} \leq x \leq 0 \leq x^2a \leq y \leq \frac{1}{a}$,

(v) $[x, y)$ *where* $-\frac{1}{a} \leq x \leq 0 \leq x^2a < y \leq \frac{1}{a}$,

(vi) $(x, y]$ *where* $-\frac{1}{a} \leq x < 0 < x^2a \leq y \leq \frac{1}{a}$,

(vii) (x, y) *where* $-\frac{1}{a} \leq x < 0 < x^2a \leq y \leq \frac{1}{a}$.

Theorem 2.2.30. *Let* $\Gamma = [a, b]$ *and* $0 < a < b$. *Then a real interval* I *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$ *if and only if* I *is one of the following forms:*

(i) $\mathbb{R}$, *(ii)* $\{0\}$, *(iii)* $[0, \infty)$, *(iv)* $(0, \infty)$,

(v) $[x, \infty)$ *where* $x \geq \frac{1}{a}$,

(vi) (x, ∞) *where* $x \geq \frac{1}{a}$,

(vii) $[x, y]$ *where* $x \leq 0 \leq x^2b \leq y \leq \frac{1}{b}$,

(viii) $[x, y)$ *where* $x \leq 0 \leq x^2b < y \leq \frac{1}{b}$,

(ix) $(x, y]$ *where* $x \leq 0 \leq x^2b \leq y \leq \frac{1}{b}$,

(x) (x, y) *where* $x \leq 0 \leq x^2b \leq y \leq \frac{1}{b}$.

Proof. First, assume that I is a $\Gamma-$subsemigroup of $\mathbb{R}$. Then $I\{a\}I \subseteq I\Gamma I \subseteq I$ and $I\{b\}I \subseteq I\Gamma I \subseteq I$. Thus I is both a $\{a\}-$subsemigroup of $\mathbb{R}$ and a $\{b\}-$subsemigroup of $\mathbb{R}$. By Theorem 2.2.19 and $\frac{1}{a} > \frac{1}{b}$, the interval I must be one of the mentioned forms.

Conversely, it is obvious that (i)–(iv) are $\Gamma-$subsemigroups of $\mathbb{R}$. If $x \geq \frac{1}{a}$, then $[x, \infty)[a, b][x, \infty) \subseteq [xa, \infty)[x, \infty) \subseteq [x^2a, \infty) \subseteq [x, \infty)$, i.e., $[x, \infty)$ is a $\Gamma-$subsemigroup of $\mathbb{R}$. Next, assume that $x \leq 0 \leq x^2b \leq y \leq \frac{1}{b}$. Let $n = \max\{x^2b, y^2b\}$. Then $[x, y][a, b][x, y] \subseteq [xb, yb][x, y] \subseteq [xby, n] \subseteq [x, y]$. Again, this shows that $[x, y]$ is a $\Gamma-$subsemigroup of $\mathbb{R}$.

The other cases can be shown similarly to the above argument.

This proof is complete. $\square$

Applying Theorem 2.2.30 and Proposition 2.2.1, we obtain the following result.

Corollary 2.2.31. *Let* Γ=$[a,b]$ *and* $a < b < 0$. *Then a real interval* I *of is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$ *if and only if* I *is one of the following forms:*

(i) $\mathbb{R}$, *(ii)* $\{0\}$, *(iii)* $(-\infty, 0]$, *(iv)* $(-\infty, 0)$,

(v) $(-\infty, x]$ *where* $x \leq \frac{1}{b}$,

(vi) $(-\infty, x)$ *where* $x \leq \frac{1}{b}$,

(vii) $[x, y]$ *where* $\frac{1}{a} \leq x \leq y^2a \leq 0 \leq y$,

(viii) $[x, y)$ *where* $\frac{1}{a} \leq x \leq y^2a \leq 0 \leq y$,

(ix) $(x, y]$ *where* $\frac{1}{a} \leq x < y^2a \leq 0 \leq y$,

(x) (x, y) *where* $\frac{1}{a} \leq x \leq y^2a \leq 0 \leq y$.

Finally, the remaining type of Γ is $[a,b]$ where $a < 0 < b$. We look up all possibilities of intervals I such that I is a $\Gamma-$subsemigroup of $\mathbb{R}$.

Theorem 2.2.32. *Let* $\Gamma = [a,b]$ *and* $a < 0 < b$. *Then a real interval* I *is a* $\Gamma-$*subsemigroup of* $\mathbb{R}$ *if and only if* I *is one of the following forms:*

(i) $\mathbb{R}$, *(ii)* $\{0\}$,

(iii) $[x, y]$ *where* $\max\{\frac{1}{a}, -\frac{1}{b}\} \leq x \leq y^2a < 0 < x^2b \leq y \leq \min\{\frac{1}{b}, -\frac{1}{a}\}$,

(iv) $(x, y]$ *where* $\max\{\frac{1}{a}, -\frac{1}{b}\} \leq x < y^2a < 0 < x^2b \leq y \leq \min\{\frac{1}{b}, -\frac{1}{a}\}$,

(v) $[x, y)$ *where* $\max\{\frac{1}{a}, -\frac{1}{b}\} \leq x \leq y^2a < 0 < x^2b < y \leq \min\{\frac{1}{b}, -\frac{1}{a}\}$,

(vi) (x, y) *where* $\max\{\frac{1}{a}, -\frac{1}{b}\} \leq x \leq y^2a < 0 < x^2b \leq y \leq \min\{\frac{1}{b}, -\frac{1}{a}\}$.

Proof. First, let a real interval I be a $\Gamma-$subsemigroup of $\mathbb{R}$. Then $I[a,0]I \subseteq I\Gamma I \subseteq I$ and $I[0,b]I \subseteq I\Gamma I \subseteq I$. Thus I is both a $[a,0]-$subsemigroup of $\mathbb{R}$ and a $[0,b]-$subsemigroup of $\mathbb{R}$. Suppose that $I = [x,y], (x,y], [x,y)$ or (x,y) where

$x < y$. If $x \geq 0$, then $(\frac{x+y}{2})a(\frac{x+y}{2}) = (\frac{x+y}{2})^2a < 0$. So $(\frac{x+y}{2})^2b \notin I$ which is a contradiction. Thus $x < 0$. Suppose $y \leq 0$, then $(\frac{x+y}{2})b(\frac{x+y}{2}) = (\frac{x+y}{2})^2b > 0$. So $(\frac{x+y}{2})^2a \notin I$ which is a contradiction. Hence $x < 0 < y$. By Theorem 2.2.28, Proposition 2.2.1 and Corollary 2.2.29, we obtain that I is one of the mentioned forms.

Conversely, it is obvious that $\mathbb{R}$ and $\{0\}$ are $\Gamma-$subsemigroups of $\mathbb{R}$. Assume that $I = [x, y]$ and $\max\{\frac{1}{a}, -\frac{1}{b}\} \leq x < y^2a < 0 < x^2b \leq y \leq \min\{\frac{1}{b}, -\frac{1}{a}\}$. Then $\frac{1}{a} \leq x \leq y^2a < 0 < x^2b \leq y \leq \frac{1}{b}$. Let $m = \min\{xb, ya\}$ and $n = \max\{xa, yb\}$. Thus

$$[x, y][a, b][x, y] \subseteq [m, n][x, y] \subseteq [c, d],$$

where $c = \min\{my, nx\}$ and $d = \max\{mx, ny\}$. Claim that $x \leq c$ and $d \leq y$, i.e., $x \leq xby$, $x \leq y^2a$, $x \leq x^2a$, $x^2b \leq y$, $yax \leq y$ and $y^2b \leq y$. Since $yb \leq 1$ and $xa \leq 1$, we see that $x \leq xby$, $y^2b \leq y$, $yax \leq y$ and $x \leq x^2a$. Thus $[x, y]$, where $\max\{\frac{1}{a}, -\frac{1}{b}\} \leq x \leq y^2a < 0 < x^2b \leq y \leq \min\{\frac{1}{b}, -\frac{1}{a}\}$, is a $\Gamma-$subsemigroup of $\mathbb{R}$.

The other cases can be shown similarly to the above argument.

$\square$