ผลเฉลยของระบบสมการฟังก์ชันนัล

$f(x \circ y) = f(x)g(y) + g(x)f(y)$ กับ $g(x \circ y) = g(x)g(y) - f(x)f(y)$

บนกึ่งกลุ่ม

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SOLUTIONS OF THE SYSTEM OF FUNCTIONAL EQUATIONS $f(x \circ y) = f(x)g(y) + g(x)f(y)$ AND $g(x \circ y) = g(x)g(y) - f(x)f(y)$ ON SEMIGROUPS

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ศูนย์วิทยทรัพยากร
จุฬาลงกรณ์มหาวิทยาลัย

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ให้ $(S, \circ)$ เป็นกึ่งกลุ่ม $(F, +, \cdot)$ เป็นสนามซึ่งมีแคแรกเทอริสติคต่างจาก 2 เราพิจารณาหา $f, g : S \to F$ ซึ่งทำให้

$$\left.\begin{aligned} f(x \circ y) &= f(x)g(y) + g(x)f(y), \\ g(x \circ y) &= g(x)g(y) - f(x)f(y) \end{aligned}\right\} \quad (*)$$

สำหรับทุก ๆ x, y ใน S ถ้า F มีสมาชิก i ซึ่ง $i^2 = -1$ ดังนั้น f, g เป็นฟังก์ชันในรูปของ

$$\left.\begin{aligned} f(x) &= \frac{i}{2}(\varphi_1(x) - \varphi_2(x)), \\ g(x) &= \frac{1}{2}(\varphi_1(x) + \varphi_2(x)), \end{aligned}\right\} \quad (**)$$

โดยที่ φ_1, φ_2 เป็นฮอโมมอร์ฟิสม์ จาก S ไปสู่ $(F, \cdot)$ ในกรณีที่ F ไม่มีสมาชิก i ดังกล่าว เราสามารถขยาย F ไปเป็น $\overline{F}$ ซึ่งมีสมาชิก i และจะได้ว่า f, g เป็นฟังก์ชันในรูปของ

$$\left.\begin{aligned} f(x) &= \frac{i}{2}(\varphi(x) - \overline{\varphi}(x)), \\ g(x) &= \frac{1}{2}(\varphi(x) + \overline{\varphi}(x)), \end{aligned}\right\} \quad (***)$$

โดยที่ φ เป็นฮอโมมอร์ฟิสม์ จาก $(S, \circ)$ ไปสู่ $(\overline{F}, \cdot)$ และ $\overline{\varphi}$ กำหนดโดย $\overline{\varphi}(x) = \overline{\varphi(x)}$ ซึ่งคือ สังยุคของ $\varphi(x)$

ในกรณีที่ $(S, \circ)$ เป็นกึ่งกลุ่มเชิงทอพอโลยี $(F, +, \cdot)$ เป็นสนามเชิงทอพอโลยีซึ่งมีแคแรกเทอริสติคต่างจาก 2 เราพิจารณาหาฟังก์ชันที่ต่อเนื่อง $f, g : S \to F$ ซึ่งทำให้ (*) เป็นจริงสำหรับทุก ๆ x, y ใน S ถ้า F มีสมาชิก i ซึ่ง $i^2 = -1$ ดังนั้น f, g เป็นฟังก์ชันในรูปของ (**) โดยที่ φ_1, φ_2 เป็นฮอโมมอร์ฟิสม์ ที่ต่อเนื่องจาก $(S, \circ)$ ไปสู่ $(F, \cdot)$ ในกรณีที่ F ไม่มีสมาชิก i ดังกล่าว เราสามารถขยาย F ไปเป็น $\overline{F}$ ซึ่งมีสมาชิก i และในกรณีนี้ f, g เป็นฟังก์ชันในรูป (***) โดยที่ φ เป็นฮอโมมอร์ฟิสม์ ที่ต่อเนื่องจาก $(S, \circ)$ ไปสู่ $(\overline{F}, \cdot)$ และ $\overline{\varphi}$ นิยามโดย $\overline{\varphi}(x) = \overline{\varphi(x)}$ ซึ่งคือสังยุคของ $\varphi(x)$

3970003023 MATHEMATIC
SYSTEM OF FUNCTIONAL EQUATIONS / HOMOMORPHISM

KANOKPORN PALASRI : SOLUTIONS OF THE SYSTEM OF FUNCTIONAL EQUATIONS $f(x \circ y) = f(x)g(y) + g(x)f(y)$ AND $g(x \circ y) = g(x)g(y) - f(x)f(y)$ ON SEMIGROUPS : THESIS ADVISOR : ASSO. PROF. VIROOL BOONYASOMBAT, Ph.D. 30 pp.
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Let $(S, \circ)$ be any semigroup, $(F, +, \cdot)$ be a field of characteristic different from 2. We determine all pairs of $f, g : S \to F$ such that

$$\left.\begin{aligned} f(x \circ y) &= f(x)g(y) + g(x)f(y), \\ g(x \circ y) &= g(x)g(y) - f(x)f(y) \end{aligned}\right\} \quad (*)$$

for all x, y in S. It is proved that if F contains an element i such that $i^2 = -1$, then f, g are of the form

$$\left.\begin{aligned} f(x) &= \frac{i}{2}(\varphi_1(x) - \varphi_2(x)), \\ g(x) &= \frac{1}{2}(\varphi_1(x) + \varphi_2(x)), \end{aligned}\right\} \quad (**)$$

where φ_1, φ_2 are any homomorphisms from $(S, \circ)$ into $(F, \cdot)$. In the case that F does not contain such element i, we can extend it to $\overline{F}$ that contains such element. In this case f, g are of the form

$$\left.\begin{aligned} f(x) &= \frac{i}{2}(\varphi(x) - \overline{\varphi}(x)), \\ g(x) &= \frac{1}{2}(\varphi(x) + \overline{\varphi}(x)), \end{aligned}\right\} \quad (***)$$

where φ is a homomorphism from $(S, \circ)$ into $(\overline{F}, \cdot)$ and $\overline{\varphi}$ is defined by $\overline{\varphi}(x) = \overline{\varphi(x)}$, the conjugate of $\varphi(x)$.

In case $(S, \circ)$ is a topological semigroup, $(F, +, \cdot)$ is a topological field of characteristic different from 2, we determine all pairs of continuous functions $f, g : S \to F$ such that (*) holds for all x, y in S. It is proved that if F contains an element i such that $i^2 = -1$, then f, g are of the form (**) where φ_1, φ_2 are any continuous homomorphisms from $(S, \circ)$ into $(F, \cdot)$. In the case that F does not contain such element i, we can extend it to $\overline{F}$ that contains such element. In this case f, g are of the form (***) where φ is a continuous homomorphism from $(S, \circ)$ into $(\overline{F}, \cdot)$ and $\overline{\varphi}$ is defined by $\overline{\varphi}(x) = \overline{\varphi(x)}$, the conjugate of $\varphi(x)$.

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CONTENTS

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CHAPTER I

INTRODUCTION

It is well known that $f(x) = \sin x$ and $g(x) = \cos x$ satisfy

$$f(x+y) = f(x)g(y) + g(x)f(y). \tag{1}$$

Such a pair (f, g) will be called a *solution* of (1).

However, sine and cosine are not the only solution of (1).

Srichan Layraman [10] solved (1) in the case where the domain is any cyclic monoid and the codomain is any algebraically closed field of characteristic different from 2. In [10], (1) was also solved in the case where the domain is a dense submonoid of $(\mathbf{R}^+ \cup \{0\}, +)$ which in turn solved the problem of finding all continuous solutions of (1) from $(\mathbf{R}^+ \cup \{0\}, +)$ into $\mathbf{C}$; the field of complex numbers.

Somporn Malangpoo [9] found all solutions of the equation

$$g(xy^{-1}) = g(x)g(y) + f(x)f(y), \tag{2}$$

where the domain can be any abelian group and the codomain can be any field of characteristic different from 2. In [9], all continuous solutions of (2) were also found in the case where the domain is any abelian topological group and the codomain is any T_1- topological field of characteristic different from 2.

Up to now, only partial solutions of (1) appeared in the literature, for example, [1] and [10].

Solutions of the system of functional equations

$$f(x+y) = f(x)g(y) + g(x)f(y),$$
$$g(x+y) = g(x)g(y) - f(x)f(y),$$

have been also considered, for example, [6], [13], [14] and [15]. Most of these problems were considered in the case where f and g are functions from $\mathbf{R}$ into $\mathbf{C}$.

The purpose of this study is to characterize all the solutions of the system of functional equations

$$\left.\begin{aligned} f(x\circ y) &= f(x)g(y)+g(x)f(y) \\ g(x\circ y) &= g(x)g(y)-f(x)f(y) \end{aligned}\right\} \qquad (*)$$

on a semigroup into a field of characteristic different from 2.

In Chapter III, we show that all the solutions of (*) can be expressed in term of homomorphisms. In the case where the domain is a topological semigroup and the codomain is a topological field with characteristic different from 2 , we show that all the continuous solutions of (*) can be expressed in term of continuous homomorphisms.

In Chapter IV, all additive interval semigroups in **R** are determined. We apply our results in Chapter III to give examples of continuous and discontinuous solutions of (*) from these interval semigroups into the field **C** of complex numbers.

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CHAPTER II

PRELIMINARIES

In this chapter we shall collect some definitions which will be used in our investigation. Some algebraic concepts are described in Section 2.1, while in Section 2.2 we describe some topological concepts. Some properties of complex numbers are described in Section 2.3, while in Section 2.4 we describe the functions Exp and Log .

The following notations will be used.

$\mathbf{Z}$ = the set of integers,

$\mathbf{R}$ = the set of real numbers,

$\mathbf{Q}$ = the set of rational numbers,

$\mathbf{Q}^* = \mathbf{Q} \setminus \{0\}$,

$\mathbb{C}$ = the set of complex numbers,

$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$,

and in general for any field F we denote $F \setminus \{0\}$ by F^*.

2.1. Algebraic Concepts

Let $(T,\circ)$ and $(S,*)$ be semigroups. A mapping φ from T into S is said to be a *homomorphism* if for any $x, y \in T$,

$$\varphi(x \circ y) = \varphi(x) * \varphi(y).$$

Any field F in which $x^2 + 1 \neq 0$ for any $x \in F$ can be extended into a field $\overline{F}$ such that $x^2 + 1 = 0$ for some $x \in \overline{F}$. An easy way to do this is as following . Let

$$\overline{F} = \{(a,b) / a,b \in F\}.$$

Define addition and multiplication on $\overline{F}$ as follows:

$$(a, b) + (c, d) = (a+c, b+d) \quad \text{and} \quad (a, b) \cdot (c, d) = (ac-bd, ad+bc).$$

It can be shown that $\overline{F}$ under the above addition and multiplication forms a field. This field contains $\{ (a,0) \ / \ a \in F \}$ as a subfield isomorphic to F. Hence we may view F as a subfield of $\overline{F}$. In the sequel, we shall denote the element $(a, 0)$ of $\overline{F}$ by a and denote $(0, 1)$ by i, so that each element (a, b) of $\overline{F}$ can be expressed as

$$(a, b) = (a, 0) + (b,0)(0, 1) = a + bi.$$

Note that from the definition of i, we have $i^2 = -1$. Hence $\overline{F}$ is an extension field of F such that $\overline{F}$ contains an element i such that $i^2 + 1 = 0$. We call i the *imaginary unit*. It can be shown that the mapping $\theta : \overline{F} \to \overline{F}$ given by

$$\theta(a + bi) = a - bi$$

is the unique automorphism of $\overline{F}$ fixing all elements of F taking i into $-i$. We call $a-bi$ the *conjugate* of $a+bi$, denoted by $\overline{a+bi}$.

Remark 2.1.1. For any $a+bi \in \overline{F}$,

1) $i((a+bi) - \overline{(a+bi)}) = i(a+bi-a+bi) = i(2bi) = -2b \in F$.

2) $(a+bi) + \overline{(a+bi)} = (a+bi+a-bi) = 2a \in F$.

3) For any $\alpha, \beta \in \overline{F}$, $\overline{\alpha}\overline{\beta} = \overline{\alpha\beta}$.

2.2. Topological Concepts

If $(S, \circ)$ is a semigroup and $\Im$ is a topology on S such that the binary opreation $\circ$, considered as a function from the product space $S \times S$ into S, is continuous, then we say that $(S, \circ, \Im)$ is a *topological semigroup*.

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A triple $(G, \circ, \Im)$ is a *topological group* if and only if $(G, \circ)$ is a group, $(G, \Im)$ is a topological space, $(G, \circ)$ is a topological semigroup and a mapping x into x^{-1} is continuous from G into itself.

A *topological field* is a quadruple $(F, +, \cdot, \Im)$ such that $(F, +, \cdot)$ is a field, $(F, +, \Im)$ and $(F^*, \cdot, \Im_{F^*})$ are topological groups where $\Im_{F^*}$ is the topology induced by $\Im$ on F^*. We sometimes say "F is a topological field". If F is a topological field in which $a^2+1 \neq 0$ for all $a \in F$, then $\overline{F}$ endowed with the product topology is a topological field.

Let $(X, \Im)$ be a topological space and ρ an equivalence relation on X. For each $x \in X$ we denote the equivalence class containing x by $[x]_\rho$, denote the set of all equivalence classes by X/ρ and denote the quotient topology by $\Im_\rho$. The function $\gamma_\rho : X \to X/\rho$ which maps each point x in X to its equivalence class $[x]_\rho$ in X is called the *quotient map* of ρ. It is continuous.

2.3. Some Properties of Complex Numbers

We adapt terminology of [8] and [11] with some minor changes. Now, we shall collect some definitions and properties as follows : For any complex number z, we can write

$$z = |z|(\cos\theta + i\sin\theta). \tag{1}$$

Any such θ is called an *argument* of z. Among such θ there is exactly one which is in the interval $[0, 2\pi)$. Such θ will be denoted by $Arg\, z$.

Let $z \in \mathbf{C}^*$. An n^{th} roots of z is any complex number w such that $w^n = z$. For each $k = 0,1,2,\ldots,n-1$, let

$$w_k = |z|^{\frac{1}{n}}(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}).$$

It can be verified that w_k, $k = 0,1,2,\ldots,n-1$, are the n distinct n^{th} roots of z.

For any complex number $z = x + iy$, $\exp(z)$ is defined by

$$\exp z = e^x(\cos y + i\sin y).$$

It can be seen that

$$\exp(z + 2k\pi i) = \exp(z) \tag{1}$$

for all $k \in \mathbf{Z}$. So that exp is a many-to-one function. Note that $\exp(z) \neq 0$ for any $z \in \mathbf{C}$. For any $w \in \mathbf{C}^*$, if

$$z = \ln|w| + i\text{Arg}\, w + 2k\pi i \quad ,$$

where $k \in \mathbf{Z}$, we have

$$\exp(z) = w.$$

Since the function e^x, $\cos y$, and $\sin y$ are continuous functions of x and y . Consequently,

$$\text{Re}(\exp z) = e^x \cos y \quad \text{and} \quad \text{Im}(\exp z) = e^x \sin y$$

are both continuous, so $\exp z$ is continuous at all points of the plane .

Logarithm , i.e. log , is defined to be the inverse of the exponential function. It is many-valued function. For any complex number z, we may write $z = r\exp(i\theta)$. The values of $\log z$ are given by

$$\log z = \log(re^{i\theta}) = \ln r + i\theta + 2k\pi i ,$$

where $k \in \mathbf{Z}$.

In the next section we shall defined Exp and Log in such a way that they are inverses of each other and such that Log is one-to-one.

2.4. The Functions Exp and Log

In our work we need to do some calculation of logarithm and exponential function. In doing so, we need our logarithm to be single valued. This can be done by considering the range of logarithm to be a quotient of $\mathbf{C}$ over an equivalence relations. This is done as follows. We define ρ by saying that

$$x+iy \;\rho\; x'+iy' \text{ if and only if } x=x' \text{ and } y-y'=2k\pi \text{ for some } k\in\mathbf{Z}.$$

It can be verified that ρ is an equivalence relation. Let $\mathbf{C}_\rho = \mathbf{C}/\rho$. For any z_1, z_2, z_1', z_2' such that

$$z_1 \;\rho\; z_1' \quad \text{and} \quad z_2 \;\rho\; z_2',$$

it can be verified that $z_1+z_2 \;\rho\; z_1'+z_2'$. So that $\oplus$ defined by

$$[z_1]_\rho \oplus [z_2]_\rho = [z_1+z_2]_\rho$$

is well-defined.

For any $z\in\mathbf{C}$, there exists a unique $z_0\in\mathbf{C}$ such that $0\le \text{Arg } z_0 < 2\pi$ and $z \;\rho\; z_0$. We shall denote such z_0 by $\gamma(z)$. Observe that for any $z, z'\in\mathbf{C}$ we have $[z]_\rho=[z']_\rho$ if and only if $\text{Re}(z)=\text{Re}(z')$ and $\gamma(z)=\gamma(z')$. For any $[z]\in\mathbf{C}_\rho$, $r\in\mathbf{R}$, defined $\otimes$ by

$$r\otimes[z]_\rho = [\gamma(r\gamma(z))]_\rho .$$

However, when it is not ambiguous we shall denote $r\otimes[z]_\rho$ simply by $r[z]_\rho$.

Lemma 2.4.1. For any $z_1,z_2\in\mathbf{C}$, $[\gamma(z_1+z_2)]_\rho = [\gamma(z_1)]_\rho \oplus [\gamma(z_2)]_\rho$.

Proof. Let $z_1=x_1+iy_1$, $z_2=x_2+iy_2\in\mathbf{C}$. Therefore ,

$$\gamma(z_1)=x_1+i(y_1+2k_1\pi) \quad\text{and}\quad \gamma(z_2)=x_2+i(y_2+2k_2\pi)$$

for some $k_1,k_2\in\mathbf{Z}$ such that $0\le y_1+2k_1\pi<2\pi$ and $0\le y_2+2k_2\pi<2\pi$.

We also have

$$\gamma(z_1+z_2)=(x_1+x_2)+i(y_1+y_2+2k_{12}\pi)$$

for some $k_{12}\in\mathbf{Z}$ such that $0\le (y_1+y_2)+2k_{12}\pi<2\pi$.

Observe that $\gamma(z_1)+\gamma(z_2)=(x_1+x_2)+i(y_1+y_2+2(k_1+k_2)\pi)$.

Hence $\gamma(z_1+z_2)\;\rho\;\gamma(z_1)+\gamma(z_2)$. This implies that

$$[\gamma(z_1+z_2)]_\rho = [\gamma(z_1)+\gamma(z_2)]_\rho \quad \#$$

Lemma 2.4.2. Let $r_1, r_2 \in \mathbf{R}$ and $[z]_\rho \in \mathbf{C}_\rho$. Then

$$(r_1 + r_2) \otimes [z]_\rho = (r_1 \otimes [z]_\rho) \oplus (r_2 \otimes [z]_\rho).$$

Proof. Let $r_1, r_2 \in \mathbf{R}$ and $[z]_\rho \in \mathbf{C}_\rho$. Therefore

$$\begin{aligned}(r_1 + r_2) \otimes [z]_\rho &= [\gamma((r_1 + r_2)\gamma(z))]_\rho \\ &= [\gamma(r_1\gamma(z))]_\rho \oplus [\gamma(r_2\gamma(z))]_\rho \\ &= (r_1 \otimes [z]_\rho) \oplus (r_2 \otimes [z]_\rho). \qquad \#\end{aligned}$$

Observe that the exponential function exp defined in Section 2.3 is a many-to-one function such that for any $z_1, z_2 \in \mathbf{C}$,

$$[z_1]_\rho = [z_2]_\rho \quad \text{implies} \quad \exp z_1 = \exp z_2 .$$

So that Exp : $\mathbf{C}_\rho \to \mathbf{C}^*$ defined by

$$\text{Exp}([z]_\rho) = \exp(z)$$

is well-defined. Since the range of exp is $\mathbf{C}^*$. The range of Exp is also $\mathbf{C}^*$. We now show that Exp is one-to-one . Assume that $[z_1]_\rho, [z_2]_\rho \in \mathbf{C}_\rho$ are such that

$$\text{Exp}([z_1]_\rho) = \text{Exp}([z_2]_\rho).$$

Then $\exp(z_1) = \exp(z_2)$ .So, there exists $k \in \mathbf{Z}$ such that $z_1 = z_2 + 2k\pi i$. This implies that $z_1 \,\rho\, z_2$. Hence $[z_1]_\rho = [z_2]_\rho$. Therefore Exp is a one-to-one function from $\mathbf{C}_\rho$ onto $\mathbf{C}^*$.

We shall denote the inverse function of Exp by Log . Thus Log is also a one-to-one function from $\mathbf{C}^*$ onto $\mathbf{C}_\rho$. For any $z \in \mathbf{C}^*$ and $r \in \mathbf{R}$, we define z^r by

$$z^r = \text{Exp}(r \text{ Log } z).$$

Therefore

$$\begin{aligned}\text{Log } z^r &= \text{Log}(\text{Exp}(r \text{ Log } z)) \\ &= r \text{ Log } z.\end{aligned}$$

CHAPTER III

GENERAL SOLUTIONS OF

$f(x \circ y) = f(x)g(y) + g(x)f(y)$ AND $g(x \circ y) = g(x)g(y) - f(x)f(y)$

ON SEMIGROUPS

In this chapter we explain what are meant by solutions and continuous solutions of the system of functional equations

$$\left.\begin{aligned} f(x \circ y) &= f(x)g(y) + g(x)f(y), \\ g(x \circ y) &= g(x)g(y) - f(x)f(y). \end{aligned}\right\} \qquad (*)$$

Section 3.1 gives all the continuous solutions of (*) in the case where the domain of f and g is any topological semigroup and their codomain is any topological field. Section 3.2 gives all the solutions of (*) in the most general setting.

3.1. Continuous solutions of (*) on a topological semigroup

Definition 3.1.1. Let $(S,\circ)$ be any semigroup, $(F,+,\cdot)$ be any field. By a *solution* of the system of functional equations (*) on S into F, we mean an ordered pair (f, g) where f and g are functions from S into F such that (*) holds for all $x, y \in S$.

Remark 3.1.2. Let $(S,\circ)$ be a semigroup, $(F,+,\cdot)$ be a field. Let f be identically zero. Then the ordered pair (f, g) is a solution of (*) on S into F if and only if g is a homomorphism.

Definition 3.1.3. Any solution (f, g) of (*) is said to be the *trivial solution* if f is identically zero, any other solution will be called a *non-trivial solution* of (*).

Definition 3.1.4. Let S be any topological semigroup. Let F be any topological field. By a *continuous solution* of the system of functional equations (*) on S into F, we mean a solution (f, g) such that f and g are continuous. If either f or g is not continuous, then (f, g) will be called a *discontinuous solution.*

Theorem 3.1.5. Let $(S,\circ)$ be a topological semigroup. Let $(F,+,\cdot)$ be a topological field of characteristic different from 2 such that there exists $i \in F$, $i^2+1=0$. Then (f, g) is a continuous solution of (*) on S into F if and only if there exist continuous homomorphisms φ_1 and φ_2 from $(S,\circ)$ into $(F,\cdot)$ such that f and g are functions of the form

$$f(x)=\frac{i}{2}(\varphi_1(x)-\varphi_2(x)),$$

$$g(x)=\frac{1}{2}(\varphi_1(x)+\varphi_2(x)),$$

for all $x \in S$.

Proof. Let (f, g) be any continuous solution of (*) from S into F.

Define $\varphi_1 : S \to F$ by $\varphi_1(x)=g(x)-if(x)$ for all $x \in S$,

$\varphi_2 : S \to F$ by $\varphi_2(x)=g(x)+if(x)$ for all $x \in S$.

For any x, y in S, we have

$$\begin{aligned}
\varphi_1(x)\varphi_1(y) &= [g(x)-if(x)][g(y)-if(y)] \\
&= g(x)g(y)-ig(x)f(y)-if(x)g(y)+i^2 f(x)f(y) \\
&= [g(x)g(y)-f(x)f(y)]-i[f(x)g(y)+g(x)f(y)] \\
&= g(x\circ y)-if(x\circ y) \\
&= \varphi_1(x\circ y),
\end{aligned}$$

and

$$\begin{aligned}
\varphi_2(x)\varphi_2(y) &= [g(x)+if(x)][g(y)+if(y)] \\
&= g(x)g(y)+ig(x)f(y)+if(x)g(y)+i^2 f(x)f(y) \\
&= [g(x)g(y)-f(x)f(y)]+i[f(x)g(y)+g(x)f(y)] \\
&= g(x\circ y)+if(x\circ y) \\
&= \varphi_2(x\circ y).
\end{aligned}$$

So φ_1 and φ_2 are homomorphisms. Since f and g are continuous, by definitions of φ_1 and φ_2, φ_1 and φ_2 are continuous. And it follows that

$$f(x) = \frac{i}{2}(\varphi_1(x) - \varphi_2(x)),$$
$$g(x) = \frac{1}{2}(\varphi_1(x) + \varphi_2(x)).$$

Now, let φ_1 and φ_2 be any continuous homomorphisms from S into F. Let f, g be functions from $(S,\circ)$ into $(F,\cdot)$ defined by

$$f(x) = \frac{i}{2}(\varphi_1(x) - \varphi_2(x)),$$
$$g(x) = \frac{1}{2}(\varphi_1(x) + \varphi_2(x)),$$

for all $x \in S$. Since φ_1 and φ_2 are continuous, so are f and g. Then

$$\begin{aligned} f(x)g(y)+g(x)f(y) &= \left[\frac{i}{2}(\varphi_1(x) - \varphi_2(x))\right]\left[\frac{1}{2}(\varphi_1(y) + \varphi_2(y))\right] \\ &\quad + \left[\frac{1}{2}(\varphi_1(x) + \varphi_2(x))\right]\left[\frac{i}{2}(\varphi_1(y) - \varphi_2(y))\right] \\ &= \frac{i}{4}[\varphi_1(x)\varphi_1(y)+\varphi_1(x)\varphi_2(y)-\varphi_2(x)\varphi_1(y)-\varphi_2(x)\varphi_2(y)] \\ &\quad + \frac{i}{4}[\varphi_1(x)\varphi_1(y)-\varphi_1(x)\varphi_2(y)+\varphi_2(x)\varphi_1(y)-\varphi_2(x)\varphi_2(y)] \\ &= \frac{i}{4}[2\varphi_1(x)\varphi_1(y)-2\varphi_2(x)\varphi_2(y)] \\ &= \frac{i}{2}[\varphi_1(x)\varphi_1(y)-\varphi_2(x)\varphi_2(y)] \\ &= \frac{i}{2}[\varphi_1(x\circ y)-\varphi_2(x\circ y)] \\ &= f(x\circ y), \end{aligned}$$

and

$$\begin{aligned} g(x)g(y)-f(x)f(y) &= \left[\frac{1}{2}(\varphi_1(x) + \varphi_2(x))\right]\left[\frac{1}{2}(\varphi_1(y) + \varphi_2(y))\right] \\ &\quad - \left[\frac{i}{2}(\varphi_1(x) - \varphi_2(x))\right]\left[\frac{i}{2}(\varphi_1(y) - \varphi_2(y))\right] \\ &= \frac{1}{4}[\varphi_1(x)\varphi_1(y)+\varphi_1(x)\varphi_2(y)+\varphi_2(x)\varphi_1(y)+\varphi_2(x)\varphi_2(y)] \\ &\quad - \frac{i^2}{4}[\varphi_1(x)\varphi_1(y)-\varphi_1(x)\varphi_2(y)-\varphi_2(x)\varphi_1(y)+\varphi_2(x)\varphi_2(y)] \\ &= \frac{1}{4}[2\varphi_1(x)\varphi_1(y)+2\varphi_2(x)\varphi_2(y)] \end{aligned}$$

$$= \frac{1}{2}[\varphi_1(x)\varphi_1(y)+\varphi_2(x)\varphi_2(y)]$$
$$= \frac{1}{2}[\varphi_1(x\circ y)+\varphi_2(x\circ y)]$$
$$= g(x\circ y).$$

#

Remark 3.1.6. It can be seen from Theorem 3.1.5, that (f, g) is a continuous trivial solution of (*) if and only if φ_1 and φ_2 are identical.

Theorem 3.1.7. Let $(S,\circ)$ be a topological semigroup. Let $(F,+,\cdot)$ be a topological field of characteristic different from 2 such that F has no an imaginary unit. Let $\overline{F}$ be a topological extension field of F such that $\overline{F}$ has an imaginary unit, says i. Then (f, g) is a continuous solution of (*) on S into F if and only if there exists a continuous homomorphism φ from $(S,\circ)$ into $(\overline{F},\cdot)$ such that f and g are functions of the form

$$f(x) = \frac{i}{2}\left(\varphi(x) - \overline{\varphi(x)}\right),$$
$$g(x) = \frac{1}{2}\left(\varphi(x) + \overline{\varphi(x)}\right),$$

for all $x \in S$.

Proof. Let (f, g) be any continuous solution of (*) from S into F. Define $\varphi : S \to \overline{F}$ by $\varphi(x)=g(x)-if(x)$ for all $x \in S$. Then, for any x, y in S,

$$\begin{aligned}\varphi(x)\varphi(y) &= [g(x)-if(x)][g(y)-if(y)]\\ &= g(x)g(y)-ig(x)f(y)-if(x)g(y)-f(x)f(y)\\ &= [g(x)g(y)-f(x)f(y)]-i[g(x)f(y)+f(x)g(y)]\\ &= g(x\circ y)-if(x\circ y)\\ &= \varphi(x\circ y).\end{aligned}$$

This proved that φ is a homomorphism. Since f and g are continuous, so is φ. And from the definition of φ, it follows that

$$f(x)=\frac{i}{2}\left(\varphi(x)-\overline{\varphi(x)}\right),$$
$$g(x)=\frac{1}{2}\left(\varphi(x)+\overline{\varphi(x)}\right).$$

Now, let φ be a continuous homomorphism from $(S,\circ)$ into $(\overline{F},\cdot)$. Then it is clear that $f(x)=\frac{i}{2}\left(\varphi(x)-\overline{\varphi(x)}\right)$ and $g(x)=\frac{1}{2}\left(\varphi(x)+\overline{\varphi(x)}\right)$ are continuous. Observe that $i(\varphi(x)-\overline{\varphi(x)})$ and $\varphi(x)+\overline{\varphi(x)}$ are in F. Therefore f and g are continuous functions from S into F. For any x, y in S, we see that

$$\begin{aligned}
f(x)g(y)+g(x)f(y) &= \left[\frac{i}{2}\left(\varphi(x)-\overline{\varphi(x)}\right)\right]\left[\frac{1}{2}\left(\varphi(y)+\overline{\varphi(y)}\right)\right] \\
&\quad+\left[\frac{1}{2}\left(\varphi(x)+\overline{\varphi(x)}\right)\right]\left[\frac{i}{2}\left(\varphi(y)-\overline{\varphi(y)}\right)\right] \\
&= \frac{i}{4}\left[\varphi(x)\varphi(y)+\varphi(x)\overline{\varphi(y)}-\overline{\varphi(x)}\varphi(y)-\overline{\varphi(x)}\ \overline{\varphi(y)}\right] \\
&\quad+\frac{i}{4}\left[\varphi(x)\varphi(y)-\varphi(x)\overline{\varphi(y)}+\overline{\varphi(x)}\varphi(y)-\overline{\varphi(x)}\ \overline{\varphi(y)}\right] \\
&= \frac{i}{4}\left[2\varphi(x)\varphi(y)-2\overline{\varphi(x)}\ \overline{\varphi(y)}\right] \\
&= \frac{i}{2}\left[\varphi(x)\varphi(y)-\overline{\varphi(x)\varphi(y)}\right] \\
&= \frac{i}{2}\left[\varphi(x\circ y)-\overline{\varphi(x\circ y)}\right] \\
&= f(x\circ y),
\end{aligned}$$

and

$$\begin{aligned}
g(x)g(y)-f(x)f(y) &= \left[\frac{1}{2}\left(\varphi(x)+\overline{\varphi(x)}\right)\right]\left[\frac{1}{2}\left(\varphi(y)+\overline{\varphi(y)}\right)\right] \\
&\quad-\left[\frac{i}{2}\left(\varphi(x)-\overline{\varphi(x)}\right)\right]\left[\frac{i}{2}\left(\varphi(y)-\overline{\varphi(y)}\right)\right] \\
&= \frac{1}{4}\left[\varphi(x)\varphi(y)+\varphi(x)\overline{\varphi(y)}+\overline{\varphi(x)}\varphi(y)+\overline{\varphi(x)}\ \overline{\varphi(y)}\right] \\
&\quad+\frac{1}{4}\left[\varphi(x)\varphi(y)-\varphi(x)\overline{\varphi(y)}-\overline{\varphi(x)}\varphi(y)+\overline{\varphi(x)}\ \overline{\varphi(y)}\right] \\
&= \frac{1}{4}\left[2\varphi(x)\varphi(y)+2\overline{\varphi(x)}\ \overline{\varphi(y)}\right] \\
&= \frac{1}{2}\left[\varphi(x)\varphi(y)+\overline{\varphi(x)\varphi(y)}\right]
\end{aligned}$$

$$= \frac{1}{2}\left[\varphi(x \circ y) + \overline{\varphi(x \circ y)}\right]$$
$$= g(x \circ y).$$

Hence f and g satisfy (*). #

Remark 3.1.8. It can be seen from Theorem 3.1.7, that (f, g) is a trivial solution of (*) if and only if $\varphi[S] \subseteq F$.

3.2. General solutions of (*) on a semigroup

Theorem 3.2.1. Let S be a semigroup. Let $(F,+,\cdot)$ be a field of characteristic different from 2 such that F has an imaginary unit, says i. Then (f, g) is a solution of (*) on S into F if and only if there exist homomorphisms φ_1 and φ_2 from $(S,\circ)$ into $(F,\cdot)$ such that

$$\begin{aligned} f(x) &= \frac{i}{2}(\varphi_1(x) - \varphi_2(x)), \\ g(x) &= \frac{1}{2}(\varphi_1(x) + \varphi_2(x)), \end{aligned} \tag{1}$$

for all $x \in S$.

Proof. Assume that (f, g) is any solution of (*) from S into F.

Define $\varphi_1 : S \to F$ by $\varphi_1(x) = g(x) - if(x)$ for all $x \in S$,

$\varphi_2 : S \to F$ by $\varphi_2(x) = g(x) + if(x)$ for all $x \in S$.

Then φ_1 and φ_2 can be shown as in the proof of Theorem 3.1.5 that φ_1 and φ_2 are homomorphisms and satisfy (1). Then by definition of φ_1 and φ_2, f and g can be written in the form (1).

Now, let φ_1 and φ_2 be homomorphisms from $(S,\circ)$ into $(F,\cdot)$. Then it can be shown by using the same argument in the proof of Theorem 3.1.5 that f and g satisfy (*). #

Theorem 3.2.2. Let $(S,\circ)$ be a semigroup. Let $(F,+,\cdot)$ be a field of characteristic different from 2 such that F has no an imaginary unit. Let $\overline{F}$ be an extension field of F such that $\overline{F}$ has an imaginary unit, says i. Then (f, g) is a solution of (*) on S into F if and only if there exists a homomorphism φ from $(S,\circ)$ into $(\overline{F},\cdot)$ such that f and g are functions of the form

$$\begin{aligned} f(x) &= \frac{i}{2}\left(\varphi(x) - \overline{\varphi(x)}\right), \\ g(x) &= \frac{1}{2}\left(\varphi(x) + \overline{\varphi(x)}\right), \end{aligned} \tag{2}$$

for all $x \in S$.

Proof. Assume that (f, g) is a solution of (*) from S into F.

Define $\varphi : S \to \overline{F}$ by $\varphi(x)=g(x)-if(x)$ for all $x \in S$.

Then φ can be shown as in the proof of Theorem 3.1.7 that φ is a homomorphism and satisfy (2). Then by definition of φ, f and g can be written in the form (2).

Now, let φ be a homomorphism from $(S,\circ)$ into $(\overline{F},\cdot)$. Then it can be shown as in the proof of Theorem 3.1.7 that f and g satisfy (*). #

CHAPTER IV

CONTINUOUS AND DISCONTINUOUS SOLUTIONS OF $f(x \circ y) = f(x)g(y) + g(x)f(y)$ AND $g(x \circ y) = g(x)g(y) - f(x)f(y)$ ON ADDITIVE INTERVAL SEMIGROUPS IN R

By an *additive interval semigroup* in **R**, we mean an interval in **R** which is closed under the usual addition. In case the interval semigroup consists of a single element, we say that it is *degenerate* otherwise it is *non-degenerate.*

In the sequel, by a *homomorphism* from an additive interval semigruop T into the fiele **C** of complex numbers we mean that is a homomorphism from $(T,+)$ into $(\mathbf{C},\cdot)$.

Observe that $\varphi : T \to \mathbf{C}$, defined by

$$\varphi(x) = 0$$

for all $x \in T$, is a homomorphism. Such a homomorphism will be called the *trivial homomorphism.* Any other homomorphisms will be called *non-trivial homomorphisms.*

In this chapter we characterize the continuous and discontinuous solutions of the system of functional equations

$$\left.\begin{aligned} f(x \circ y) &= f(x)g(y) + g(x)f(y), \\ g(x \circ y) &= g(x)g(y) - f(x)f(y), \end{aligned}\right\} \qquad (*)$$

where f, g are functions from any additive interval semigroup T in **R** into the field **C** under the usual topologies. This will be done by using the results in Chapter III. All additive interval semigroups in **R** are described in Section 4.1, In Section 4.2 we determined all the continuous solutions of (*) on degenerate additive interval semigroup $\{0\}$, while in Section 4.3 we characterize the continuous solutions of (*) on each of the non-degenerate additive interval semigroups T in $\mathbf{R}$ into $\mathbf{C}$.

Moreover we shall find some discontinuous solutions of the system of functional equations (*) from any additive interval semigroup T in $\mathbf{R}$ into $\mathbf{C}$. Our method of construction of discontinuous solutions of (*) will make uses of the fact that any non-degenerate additive interval semigroup T contains a Hamel basis. This fact is established in Section 4.4. Section 4.5 deals with construction of discontinuous solutions of (*) on T into $\mathbf{C}$.

4.1. Additive interval semigroups in R

Proposition 4.1.1. A subset T of $\mathbf{R}$ is an additive interval semigroup in $\mathbf{R}$ if and only if T is one of the following types:

(1) $\{0\}$, (2) $\mathbf{R}$,

(3) (a,∞) where $a \geq 0$, (4) $[a,\infty)$ where $a \geq 0$,

(5) $(-\infty, b)$ where $b \leq 0$, (6) $(-\infty, b]$ where $b \leq 0$.

Proof. Let T be an additive interval semigroup in $\mathbf{R}$. Let

$$a = \begin{cases} \inf T & \text{if } T \text{ is bounded below,} \\ -\infty & \text{if } T \text{ is not bounded below,} \end{cases}$$

$$b = \begin{cases} \sup T & \text{if } T \text{ is bounded above,} \\ \infty & \text{if } T \text{ is not bounded above.} \end{cases}$$

Since T is an interval in $\mathbf{R}$, $(a, b) \subseteq T$.

Case 1. $a = -\infty$ and $b = \infty$. Then $(a, b) = \mathbf{R}$, so $T = \mathbf{R}$. Thus T is of type (2).

Case 2. $-\infty < a$ and $b = \infty$. Then $(a, b) = (a,\infty)$, so

$$T = \begin{cases} (a,\infty) & \text{if } a \notin T, \\ [a,\infty) & \text{if } a \in T. \end{cases}$$

Suppose that $a<0$. Since $\frac{2a}{3}\in T$, $\frac{4a}{3}=\frac{2a}{3}+\frac{2a}{3}\in T$. But $\frac{4a}{3}<a$. Thus $\frac{4a}{3}\notin T$. This is a contradiction. Hence $a\geq 0$. Then T is of type (3), (4).

Case 3. $a=-\infty$ and $b<\infty$. Then $(a, b)=(-\infty, b)$, so

$$T=\begin{cases}(-\infty,b) & if\ \ b\notin T,\\ (-\infty,b] & if\ \ b\in T.\end{cases}$$

Suppose that $b>0$. Since $\frac{2b}{3}\in T$, $\frac{4b}{3}=\frac{2b}{3}+\frac{2b}{3}\in T$. But $\frac{4b}{3}>b$. Thus $\frac{4b}{3}\notin T$. This is a contradiction. Hence $b\leq 0$. Then T is of type (5), (6).

Case 4. $-\infty<a$ and $b<\infty$. Then

$$T=\begin{cases}(a,b) & if\ \ a\notin T\ \ and\ \ b\notin T,\\ (a,b] & if\ \ a\notin T\ \ and\ \ b\in T,\\ [a,b) & if\ \ a\in T\ \ and\ \ b\notin T,\\ [a,b] & if\ \ a\in T\ \ and\ \ b\in T.\end{cases}$$

Subcace 4.1 : $a=b$. Then $T=\{a\}$. Suppose $a\neq 0$. Therefore $2a\neq a$ but $2a=a+a\in T$. Hence $T\neq\{a\}$. This is a contradiction. Thus $a=0$, so T is type (1).

Subcace 4.2 : $a<b\leq 0$. Since $a<\frac{3a+b}{4}<b$, $\frac{3a+b}{4}\in T$. Therefore $\frac{3a+b}{2}=\frac{3a+b}{4}+\frac{3a+b}{4}\in T$. But $\frac{3a+b}{2}<a$. Thus $\frac{3a+b}{2}\notin T$. This is a contradiction.

Subcace 4.3 : $0\leq a<b$. Since $a<\frac{a+3b}{4}<b$, $\frac{a+3b}{4}\in T$. Therefore $\frac{a+3b}{2}=\frac{a+3b}{4}+\frac{a+3b}{4}\in T$. But $\frac{a+3b}{2}>b$. Thus $\frac{a+3b}{2}\notin T$. This is a contradiction.

Subcace 4.4 : $a<0$ and $b>0$. Since $\frac{2b}{3}\in T$, $\frac{4b}{3}=\frac{2b}{3}+\frac{2b}{3}\in T$. But $\frac{4b}{3}>b$. Thus $\frac{4b}{3}\notin T$. This is a contradiction.

The converse follows easily from the property of addition in **R**. #

4.2. Solutions of (*) on degenerate additive interval semigroup {0}

Proposition 4.2.1. A function φ from $\{0\}$ into $\mathbf{C}$ is a homomorphism if and only if $\varphi(0)=0$ or $\varphi(0)=1$.

Proof. It is clear that φ with $\varphi(0)=0$ or $\varphi(0)=1$ is a homomorphism.

To show the converse, assume that $\varphi:\{0\}\to\mathbf{C}$ is a homomorphism. Then $\varphi(0)=\varphi(0+0)=(\varphi(0))^2$. Thus $\varphi(0)(1-\varphi(0))=0$.
Therefore $\varphi(0)=0$ or $\varphi(0)=1$. #

Corollary 4.2.2. (f,g) is a solution of

$$\left.\begin{aligned} f(x\circ y)&=f(x)g(y)+g(x)f(y),\\ g(x\circ y)&=g(x)g(y)-f(x)f(y),\end{aligned}\right\}\qquad (*)$$

on $\{0\}$ into $\mathbf{C}$ if and only if f and g are of the form

(I) $\qquad f(0)=0, \qquad g(0)=0$,

or

(II) $\qquad f(0)=-\frac{i}{2}, \qquad g(0)=\frac{1}{2}$,

or

(III) $\qquad f(0)=\frac{i}{2}, \qquad g(0)=\frac{1}{2}$,

or

(IV) $\qquad f(0)=0, \qquad g(0)=1$.

Proof. By Theorem 3.1.5, (f,g) is a solution of (*) on $\{0\}$ into $\mathbf{C}$ if and only if there exist homomorphisms φ_1 and φ_2 from $\{0\}$ into $\mathbf{C}$ such that f and g are of the form

$$\begin{aligned} f(0)&=\frac{i}{2}(\varphi_1(0)-\varphi_2(0)),\\ g(0)&=\frac{1}{2}(\varphi_1(0)+\varphi_2(0)).\end{aligned}\qquad (**)$$

By Proposition 4.2.1, φ_1 and φ_2 are homomorphisms from $\{0\}$ into $\mathbf{C}$ if and only if

(1.1) $\varphi_1(0) = 0$,

or

(1.2) $\varphi_1(0) = 1$,

and

(2.1) $\varphi_2(0) = 0$,

or

(2.2) $\varphi_2(0) = 1$.

By substituting φ_1 of the form (1.1) or (1.2) and φ_2 of the form (2.1) or (2.2) in (**) we have (I)-(IV). #

Remark 4.2.3. Since any constant function is continuous. So φ_1 and φ_2 in Corollary 4.2.2 are continuous. This implies that f and g are also continuous solutions of (*).

4.3. Continuous solutions of (*) on non-degenerate additive interval semigroups

In this section we shall construct a class of homomorphisms from the non-degenerate additive interval semigroups T into $\mathbf{C}$. By applying Theorem 3.1.5. We obtain a class of continuous solutions of (*).

Proposition 4.3.1. Let T be any non-degenerate additive interval semigroup in $\mathbf{R}$. Let $\varphi : T \to \mathbf{C}$. If there exists $c \in \mathbf{C}^*$ such that

$$\varphi(x) = c^x,$$

for all $x \in T$, then φ is a continuous non-trivial homomorphism.

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Proof. Let $c \in \mathbf{C}^*$ be arbitrary and any function $\varphi : T \to \mathbf{C}$ defined by

$$\varphi(x) = c^x ,$$

for all $x \in T$. Since c^x = Exp (x Log c) and Exp is continuous, so is c^x. We shall show that φ is a homomorphism. Let $x, y \in T$. Then

$$\begin{aligned}
\varphi(x+y) &= c^{x+y} \\
&= \operatorname{Exp}((x+y)\operatorname{Log} c) \\
&= \operatorname{Exp}(x\operatorname{Log} c + y\operatorname{Log} c) \\
&= \operatorname{Exp}(x\operatorname{Log} c)\operatorname{Exp}(y\operatorname{Log} c) \\
&= c^x \cdot c^y \\
&= \varphi(x)\varphi(y).
\end{aligned}$$

Therefore φ is a continuous non-trivial homomorphism. #

Corollary 4.3.2. Let T be any non-degenerate additive interval semigroup in $\mathbf{R}$. Then the following (f, g) given (I) - (IV) are continuous solutions of

$$\left.\begin{aligned}
f(x \circ y) &= f(x)g(y) + g(x)f(y), \\
g(x \circ y) &= g(x)g(y) - f(x)f(y),
\end{aligned}\right\} \qquad (*)$$

on T into $\mathbf{C}$.

(I) $$f(x) = 0, \qquad g(x) = 0,$$

for all $x \in T$, or

(II) there exists $c \in \mathbf{C}^*$ such that

$$f(x) = -\frac{i}{2}c^x, \quad g(x) = \frac{1}{2}c^x,$$

for all $x \in T$, or

(III) there exists $c \in \mathbf{C}^*$ such that

$$f(x) = \frac{i}{2}c^x, \quad g(x) = \frac{1}{2}c^x,$$

for all $x \in T$, or

(IV) there exist $c_1, c_2 \in \mathbf{C}^*$ such that

$$f(x) = \frac{i}{2}(c_1^{\ x} - c_2^{\ x}), \qquad g(x) = \frac{1}{2}(c_1^{\ x} + c_2^{\ x}),$$

for all $x \in T$.

Proof. Let φ_1 and φ_2 be functions from T into $\mathbf{C}$ such that

(1.1) $\varphi_1(x) = 0$,

for all $x \in T$, or

(1.2) there exists $c_1 \in \mathbf{C}^*$ such that

$$\varphi_1(x) = {c_1}^x,$$

for all $x \in T$. And

(2.1) $\varphi_2(x) = 0$,

for all $x \in T$, or

(2.2) there exists $c_2 \in \mathbf{C}^*$ such that

$$\varphi_2(x) = {c_2}^x,$$

for all $x \in T$.

Then φ_1 and φ_2 of the form (1.1) and (2.1) are continuous trivial homomorphisms. By proposition 4.3.1, φ_1 and φ_2 of the form (1.2) and (2.2) are continuous non-trivial homomorphisms. By Theorem 3.1.7, (f, g) is a continuous solution of (*) on T into $\mathbf{C}$ if and only if

$$\begin{aligned} f(x) &= \frac{i}{2}(\varphi_1(x) - \varphi_2(x)), \\ g(x) &= \frac{1}{2}(\varphi_1(x) + \varphi_2(x)), \end{aligned} \qquad (**)$$

where φ_1 and φ_2 are continuous homomorphisms from T into $\mathbf{C}$. By substituting φ_1 of the form (1.1) or (1.2) and φ_2 of the form (2.1) or (2.2) in (**) we have (I) - (IV) are continuous solutions of (*). #

4.4. Existence of Hamel basis in an interval

It is a well-known fact that every vector space has a basis. A proof can be found in [12]. Since $\mathbf{R}$ is a vector space over $\mathbf{Q}$, $\mathbf{R}$ has a basis. Such a basis is known as a *Hamel basis* for $\mathbf{R}$.

Proposition 4.4.1. Let H be any Hamel basis for $\mathbf{R}$. Let (a,b) be any non-empty open interval. Then,

(1) For each $h \in H$ there exists a non-zero rational number s_h such that

$$a < s_h h < b.$$

(2) The set $H^* = \{s_h h \ / \ h \in H\}$ is a Hamel basis for $\mathbf{R}$ and $H^* \subseteq (a,b)$.

Proof. Let (a,b) be any non-empty open interval in $\mathbf{R}$. Let H be a Hamel basis for $\mathbf{R}$. Let h be any element of H. So that $|h| > 0$. Hence

$$\frac{a}{|h|} < \frac{b}{|h|}.$$

Since $\mathbf{Q}^*$ is a dense subset of $\mathbf{R}$, there exists a non-zero rational number r_h such that

$$\frac{a}{|h|} < r_h < \frac{b}{|h|}.$$

For each $h \in H$, let

$$s_h = \begin{cases} r_h & \text{if} \quad h > 0, \\ -r_h & \text{if} \quad h < 0. \end{cases}$$

It follows that $a < s_h h < b$. Let $H^* = \{s_h h \ / \ h \in H\}$. Observe that

$$H^* = \{s_h h \ / \ h \in H\} \subseteq (a,b).$$

Now we shall show that H^* is a Hamel basis for $\mathbf{R}$. Assume that for any $r_1, r_2, \ldots, r_n \in \mathbf{Q}$ and $s_{h_{\alpha_1}} h_{\alpha_1}, s_{h_{\alpha_2}} h_{\alpha_2}, \ldots, s_{h_{\alpha_n}} h_{\alpha_n} \in H^*$ are such that

$$0 = r_1(s_{h_{\alpha_1}} h_{\alpha_1}) + r_2(s_{h_{\alpha_2}} h_{\alpha_2}) + \ldots + r_n(s_{h_{\alpha_n}} h_{\alpha_n}).$$

Then

$$0 = (r_1 s_{h_{\alpha_1}}) h_{\alpha_1} + (r_2 s_{h_{\alpha_2}}) h_{\alpha_2} + \ldots + (r_n s_{h_{\alpha_n}}) h_{\alpha_n}.$$

Since H is a Hamel basis for $\mathbf{R}$, it follows that $r_i s_{h_{\alpha_i}} = 0$ for all $i = 1,2,\ldots,n$. Since $s_{h_{\alpha_i}} \neq 0$ for all $i = 1,2,\ldots,n$, $r_i = 0$ for all $i = 1,2,\ldots,n$. This implies that H^* is linearly independent. Let $x \in \mathbf{R}$. Then

$$x = r_1 h_{\alpha_1} + r_2 h_{\alpha_2} + \ldots + r_n h_{\alpha_n},$$

for some $r_1, r_2, \ldots, r_n \in \mathbf{Q}$ and $h_{\alpha_1}, h_{\alpha_2}, \ldots, h_{\alpha_n} \in H$, where n is positive integer. Then

$$x = \frac{r_1}{s_{h_{\alpha_1}}}(s_{h_{\alpha_1}} h_{\alpha_1}) + \frac{r_2}{s_{h_{\alpha_2}}}(s_{h_{\alpha_2}} h_{\alpha_2}) + \ldots + \frac{r_n}{s_{h_{\alpha_n}}}(s_{h_{\alpha_n}} h_{\alpha_n}).$$

Since $s_{h_{\alpha_i}} h_{\alpha_i} \in H^*$ and $\frac{r_i}{s_{h_{\alpha_i}}} \in \mathbf{Q}$ for all $i = 1, 2, \ldots, n$, x is a linear combination of elements of H^*. This proves that H^* is a Hamel basis for $\mathbf{R}$. #

Corollary 4.4.2. Any interval I in $\mathbf{R}$ with $|I| > 1$ contains a Hamel basis.

Proof. Let I be any interval in $\mathbf{R}$ with $|I| > 1$. Let $a, b \in \mathbf{R}$ be such that $(a, b) \subseteq I$. By Proposition 4.4.1, contains a Hamel basis. Therefore any interval I in $\mathbf{R}$ contained a Hamel basis. #

4.5. Discontinuous solutions of (*) on non-degenerate additive interval semigroups

In this section we shall provide a general method for constructing discontinuous solutions of (*) on any non-degenerate additive interval semigroup T in $\mathbf{R}$ into $\mathbf{C}$.

Proposition 4.5.1. Let T be any non-degenerate additive interval semigroup in $\mathbf{R}$. Let H be a Hamel basis contained in T. Let $\widetilde{\varphi} : H \to \mathbf{C}^*$ be arbitrary. Let $\varphi : T \to \mathbf{C}$ be defined, for any $x = \sum_{i=1}^{n} r_i h_{\alpha_i}$ in T, where $r_i \in \mathbf{Q}$, $h_{\alpha_i} \in H$, by

$$\varphi(x) = \mathrm{Exp}\,(\sum_{i=1}^{n} r_i \;\mathrm{Log}\, \widetilde{\varphi}(h_{\alpha_i}))\,. \tag{1}$$

Then φ is a non-trivial homomorphism.

Proof. To show that φ is well-defined, assume that $x \in T$ has two representations

(1) $$x = \sum_{i=1}^{n} r_i h_{\alpha_i} ,$$

(2) $$x = \sum_{i=1}^{n} r_i' h_{\alpha_i} ,$$

for some $r_i, r_i' \in \mathbf{Q}$, $h_{\alpha_i} \in H$, $i = 1,2,...,n$. Here some of r_i and r_i' may be zero. By definition of φ we have

(3) $$\varphi(x)=\text{Exp}\,(\sum_{i=1}^{n} r_i \text{ Log}\,\widetilde{\varphi}(h_{\alpha_i}))$$

according to (1) , and

(4) $$\varphi(x)=\text{Exp}\,(\sum_{i=1}^{n} r_i' \text{Log}\,\widetilde{\varphi}(h_{\alpha_i}))$$

according to (2). However, from (1) and (2) we have

$$0 = x - x = \sum_{i=1}^{n} (r_i - r_i') h_{\alpha_i} ,$$

which implies that $r_i = r_i'$ for all $i = 1,2,...,n$. So that the right hand sides of (3) and (4) are equal. Hence $\varphi(x)$ is well-defined.

Let x, y be any elements in T. Therefore,

$$x = \sum_{i=1}^{n} r_i h_{\alpha_i} , \quad y = \sum_{i=1}^{n} r_i' h_{\alpha_i} ,$$

where $r_i, r_i' \in \mathbf{Q}$, $h_{\alpha_i} \in H$, $i = 1,2,...,n$. Then $x + y = \sum_{i=1}^{n} (r_i + r_i') h_{\alpha_i}$, so

$$\begin{aligned}\varphi(x+y) &= \text{Exp}\,(\sum_{i=1}^{n} (r_i + r_i') \text{ Log}\,\widetilde{\varphi}(h_{\alpha_i})) \\ &= \text{Exp}\,(\sum_{i=1}^{n} r_i \text{ Log}\,\widetilde{\varphi}(h_{\alpha_i}) + \sum_{i=1}^{n} r_i' \text{ Log}\,\widetilde{\varphi}(h_{\alpha_i})) \\ &= \text{Exp}\,(\sum_{i=1}^{n} r_i \text{ Log}\,\widetilde{\varphi}(h_{\alpha_i}))\,\text{Exp}\,(\sum_{i=1}^{n} r_i' \text{ Log}\,\widetilde{\varphi}(h_{\alpha_i})) \\ &= \varphi(x)\varphi(y).\end{aligned}$$

Hence φ is a homomorphism. Since $\widetilde{\varphi} : H \to \mathbf{C}^*$, it follows that φ is non-trivial. #

Remark 4.5.2. From Proposition 4.3.1 , if $\varphi : T \to \mathbf{C}$ is defined by

$$\varphi(x) = c^x$$

for all $x \in T$, then φ is a continuous non-trivial homomorphism . Observe that such φ can be obtained from Proposition 4.5.1 by choosing $\widetilde{\varphi} : H \to \mathbf{C}^*$ to be

$$\widetilde{\varphi}(h) = c^h$$

for each $h \in H$, where $c \in \mathbf{C}^*$. By using this $\widetilde{\varphi}$, φ defined by (1) of Proposition 4.5.1 has the value at any $x = \sum_{i=1}^{n} r_i h_{\alpha_i}$, where $r_i \in \mathbf{Q}$, $h_{\alpha_i} \in H$, $i = 1,2,\ldots,n$ given by

$$\begin{aligned}\varphi(x) &= \text{Exp } (\sum_{i=1}^{n} r_i \text{ Log } c^{h_{\alpha_i}}) \\ &= \text{Exp } (\sum_{i=1}^{n} r_i h_{\alpha_i} \text{ Log } c) \\ &= \text{Exp } ((\sum_{i=1}^{n} r_i h_{\alpha_i}) \text{Log } c) \\ &= \text{Exp } (x \text{Log } c) \\ &= c^x .\end{aligned}$$

Discontinuous non-trivial homomorphisms can be constructed from (1) of Proposition 4.5.1 by choosing $\widetilde{\varphi}$ different from above . For example, let T be one of the following types:

(1) (a, ∞) where $a \geq 0$, (2) $[a, \infty)$ where $a \geq 0$, (3) $\mathbf{R}$.

Let $c_1, c_2 \in \mathbf{C}^*$ be such that $c_1 \neq c_2$. Let $h_1, h_2 \in H$ be such that $h_1 \neq h_2$.

Define

$$\begin{aligned}&\widetilde{\varphi}(h_1) = c_1^{h_1} , \\ &\widetilde{\varphi}(h_2) = c_2^{h_2} , \\ &\widetilde{\varphi}(h) = 1 \quad \text{for all } h \neq h_1, h_2 .\end{aligned}$$

Then φ defined from $\widetilde{\varphi}$ is a homomorphism. Let $x_0 \in T$ be such that

$$x_0 = \sum_{i=1}^{n} r_i h_{\alpha_i} ,$$

$h_{\alpha_i} \neq h_1, h_2$ for all $i = 1,2,\ldots,n$. So that $x_0 + 1 \in T$. Let $(r_n^{(1)})$, $(r_n^{(2)})$ be sequences converging to $\frac{1}{h_1}$ and $\frac{1}{h_2}$, respectively. Let

$$x_n^{(1)} = x_0 + r_n^{(1)} h_1 \qquad \text{and} \qquad x_n^{(2)} = x_0 + r_n^{(2)} h_2 .$$

So that $(x_n^{(1)})$ converges to x_0+1 and $(x_n^{(2)})$ converges to x_0+1. Therefore

$$\begin{aligned}
\varphi(x_n^{(1)}) &= \varphi(x_0 + r_n^{(1)}h_1) \\
&= \varphi(x_0)\varphi(r_n^{(1)}h_1) \\
&= \operatorname{Exp}(\sum_{i=1}^{n} r_i \operatorname{Log}\widetilde{\varphi}(h_{\alpha_i}))\ \operatorname{Exp}(\sum_{i=1}^{n} r_n^{(1)} \operatorname{Log}\widetilde{\varphi}(h_1)) \\
&= \operatorname{Exp}(\sum_{i=1}^{n} r_n^{(1)} \operatorname{Log} c_1^{h_1}) \\
&= \operatorname{Exp}(\sum_{i=1}^{n} r_n^{(1)} h_1 \operatorname{Log}\ c_1) \\
&= c_1^{r_n^{(1)}h_1}.
\end{aligned}$$

Since $(r_n^{(1)}h_1)$ converges to 1, $\left(c_1^{r_n^{(1)}h_1}\right)$ converges to c_1. Thus $(\varphi(x_n^{(1)}))$ converges to c_1. Similarly $(\varphi(x_n^{(2)}))$ converges to c_2. Therefore $\lim\limits_{x\to x_0+1}\varphi(x)$ does not exist. Hence φ is discontinuous. For the case T is one of the types:

(1) $(-\infty, b)$ where $b \le 0$, (2) $(-\infty, b]$ where $b \le 0$,

we can construct discontinuous non-trivial homomorphism in similarly way.

Remark 4.5.3. Let T and H be given as Proposition 4.5.1. From Theorem 3.1.7, $f, g : T \to \mathbf{C}$ is a continuous solution of

$$\left.\begin{aligned} f(x \circ y) &= f(x)g(y) + g(x)f(y), \\ g(x \circ y) &= g(x)g(y) - f(x)f(y), \end{aligned}\right\} \qquad (*)$$

if and only if

$$\begin{aligned} f(x) &= \frac{i}{2}(\varphi_1(x) - \varphi_2(x)), \\ g(x) &= \frac{1}{2}(\varphi_1(x) + \varphi_2(x)), \end{aligned} \qquad (**)$$

where φ_1 and φ_2 are continuous homomorphisms from T into $\mathbf{C}$. By choosing one of the φ_1, φ_2 to be continuous and the other to be discontinuous we obtain discontinuous f, g which form a discontinuous solution of $(*)$. Such choices of φ_1, φ_2 can be done by using Remark 4.5.2.

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