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CLIQUE-COLORINGS OF GRAPHS

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ธนวัฒน์ วิเชียรไพศาล : การระบายสีแบบคลีกของกราฟ (CLIQUE-COLORINGS OF GRAPHS) อ. ที่ปรึกษาวิทยานิพนธ์หลัก : ผศ.ดร.จริยา อุ่ยยะเสถียร, 40 หน้า

ในวิทยานิพนธ์ฉบับนี้ เราได้ศึกษารงคเลขแบบคลีกของกราฟ เราได้ค่าของรงคเลขแบบคลีกของกราฟเส้นของกราฟแบบบริบูรณ์ และยังได้ลักษณะเฉพาะของกราฟเส้นของกราฟที่ไม่มีสามเหลี่ยมเป็นกราฟย่อยที่ให้ค่ารงคเลขแบบคลีกของกราฟ เราได้ทำการปรับปรุงค่าขอบเขตของรงคเลขแบบคลีกของวงศ์ของกราฟที่ไม่มีกราฟย่อยก่อกำเนิดเป็น F เมื่อ $F = P_2 + kP_1$, $P_3 + kP_1$ และ $P_k + P_m$ นอกจากนี้ยังได้ศึกษารงคเลขแบบคลีกของวงศ์ย่อยบางชนิดของกราฟที่ไม่มีกราฟย่อยก่อกำเนิดเป็นกราฟคลอ

ภาควิชา คณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต ....

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In this dissertation, we study the clique-chromatic numbers of graphs. We obtain the exact values of the clique-chromatic numbers of the line graphs of complete graphs and a characterization of the clique-chromatic numbers of the line graphs of triangle-free graphs. We improve bounds of the clique-chromatic number of the family of F-free graphs when $F = P_2 + kP_1, P_3 + kP_1$ and $P_k + P_m$. Furthermore, the clique-chromatic numbers of some subfamilies of claw-free graphs are investigated.

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INTRODUCTION

1.1 Definitions and Notations

We use terminologies from West's textbook [19]. A *simple graph* is a graph having no loops and multiple edges. For our purpose, all graphs are simple and undirected graphs. $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G, respectively. A subgraph H of a graph G is said to be *induced* if, for any pair of vertices x and y of H, xy is an edge of H if and only if xy is an edge of G. If an induced subgraph H is chosen based on a vertex subset S of $V(G)$, then H can be written as $G[S]$ and is said to be *induced by S*. A *component* of a graph G is a maximal connected subgraph of G. The *neighborhood* of a vertex x in a graph G is the set of vertices adjacent to x, and is denoted by $N_G(x)$. For $S \subseteq V(G)$, $N_S(x)$ stands for the neighborhood of a vertex x in S, that is, $N_S(x) = N_G(x) \cap S$. The *degree* of a vertex x in a graph G is the size of the neighborhood of x in G. An *independent set* in a graph is a set of pairwise nonadjacent vertices. A *maximum independent set* of a graph G is a largest independent set of G and its size is denoted by $\alpha(G)$.

Given a graph G, let $M \subseteq E(G)$ and $e \in E(G)$. We write $G-M$, and $G-e$, for the subgraph of G obtained by deleting all edges of M, and an edge e, respectively. Let $A \subseteq V(G)$ and $v \in V(G)$. We write $G - A$, and $G - v$, for the subgraph of G obtained by deleting all vertices of A, and a vertex v, respectively. The *join* of graphs G and H, written $G \vee H$, is the graph obtained from the disjoint union of G

and H by adding the edges $\{xy \mid x \in V(G),\ y \in V(H)\}$. For the case $V(H) = \{v\}$, we write $G \vee v$ for $G \vee \{v\}$. The *union* of graphs $G_1, G_2, \ldots, G_k$ is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$, denoted by $G_1 \cup G_2 \cup \ldots \cup G_k$. A union of graphs $G_1, G_2, \ldots, G_k$ is called *disjoint union* if $G_1, G_2, \ldots, G_k$ have pairwise disjoint vertex sets, and is denoted by $G_1 + G_2 + \cdots + G_k$. For $k \in \mathbb{N}$, kG is the disjoint union of k pairwise disjoint copies of a graph G.

A *path* is a graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list; the path with n vertices is denoted by P_n. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle; the cycle with n vertices is denoted by C_n. A *complete graph* is a graph whose vertices are pairwise adjacent; the complete graph with n vertices is denoted by K_n. A *triangle* is the complete graph K_3. A *triangle-free graph* is a graph which contains no triangle as a subgraph. A *diamond* is the complete graph K_4 minus an edge. A *hole* in a graph is an induced cycle with at least four vertices. An *odd* (*even*) *hole* is a hole with an odd (even, respectively) number of vertices. A *tree* is a connected graph with no cycle. A *forest* is a disjoint union of trees. A *star* is a tree consisting of one vertex adjacent to all the others. The star with $k+1$ vertices is denoted by $K_{1,k}$. A *claw* is the star $K_{1,3}$. A *paw* is the claw plus an edge. A graph G is *bipartite* if $V(G)$ is the union of two disjoint independent sets. Equivalently, a bipartite graph is a graph which contains no odd cycle. A graph G is *complete multipartite* if $V(G)$ is the union of disjoint independent sets where any two vertices in different independent sets are adjacent.

A subset Q of $V(G)$ is a *clique* of a graph G if any two vertices of Q are adjacent. A clique is *maximal* if it is not properly contained in another clique. A *maximum clique* of a graph G is a clique of the largest possible cardinality in G, and its cardinality is denoted by $\omega(G)$. A *k-coloring* of a graph G is a function $f : V(G) \to X$, where $|X| = k$. A *proper k-coloring* of a graph G is a k-coloring of G such that adjacent vertices have different colors. A graph G is *k-colorable* if G has a proper k-coloring. The *chromatic number* of a graph G is the smallest positive integer k such that G has a proper k-coloring, denoted by $\chi(G)$. Given a k-coloring of a graph G, a clique Q of G is said to be *monocolored* if all vertices of Q are labeled by the same color. A *proper k-clique-coloring* of a graph G is a k-coloring of G without a monocolored maximal clique of G of size at least two. A graph G is *k-clique-colorable* if G has a proper k-clique-coloring. The *clique-chromatic number* of a graph G is the smallest positive integer k such that G has a proper k-clique-coloring, denoted by $\chi_c(G)$.

Example 1.1. Figure 1.1 illustrates a graph with a proper 3-clique-coloring and a proper 2-clique-coloring.

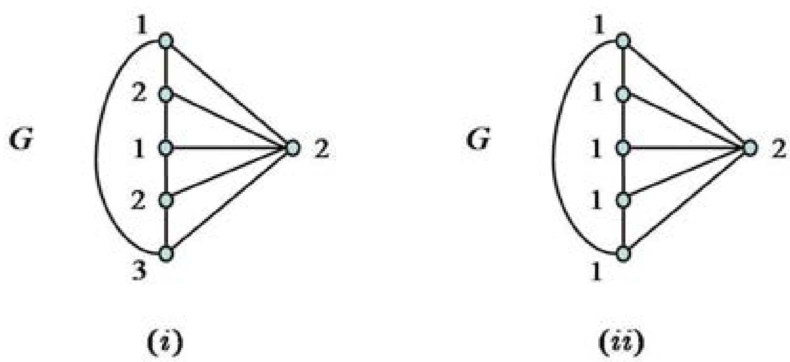

Figure 1.1: A graph G with a proper 3-clique-coloring (i) and a proper 2-clique-coloring (ii)

Remark 1.2. Let G be a graph.

(i) $\chi_c(G) = 1$ if and only if G is an edgeless graph.

(ii) $\chi_c(G) \leq \chi(G)$ because a proper k-coloring of a graph G is a proper k-clique-coloring of G.

(iii) If G is triangle-free, then all maximal cliques of G, which is not an isolated vertex, have size two; so $\chi_c(G) = \chi(G)$. For example, $\chi_c(C_{2n+1}) = \chi(C_{2n+1}) = 3$ where $n \geq 2$, and $\chi_c(G) = \chi(G) = 2$ if G is bipartite.

Remark 1.3. Some properties of the chromatic number of a graph do not belong to the clique-chromatic number of a graph:

(i) It is well known that $\chi(G) \geq \omega(G)$ for any graph G. But it is possible that $\chi_c(G) < \omega(G)$ for some graph G; for example, $\chi_c(K_n) = 2$ while $\omega(K_n) \to \infty$ as $n \to \infty$.

(ii) For a subgraph H of a graph G, it is always true that $\chi(H) \leq \chi(G)$. This analogous statement is not necessary true for the clique-chromatic number; see the graph G in Example 1.1. We have $\chi_c(G) = 2$ but G has a subgraph C_5 with the clique-chromatic number three.

1.2 History and Overview

The clique-coloring problem was originally defined in terms of the vertex-colorings of clique-hypergraphs. The authors of the first few papers studied this problem by finding the chromatic numbers of the clique-hypergraphs of perfect graphs, planar graphs, and circulant graphs (see more in [1, 2, 6, 9, 10, 13, 14]). Until 2008, Cerioli and Petito [4] restated the problem in terms of the clique-chromatic numbers of graphs which we use throughout this dissertation.

Like the chromatic numbers of graphs, some families of graphs have a bounded clique-chromatic number, while some families of graphs do not have. In 1955, Mycielski [15] showed that there is no constant C such that every triangle-free graph is C-colorable. That is, the family of triangle-free graphs has no bounded chromatic number. Consequently, it has no bounded clique-chromatic number, either. However, some families of graphs containing a triangle may have no bounded clique-chromatic number. Bacsó et al. [1] showed in 2004 that the family of line graphs has no bounded clique-chromatic number. Later, Cerioli and Petito [4] proved that the family of UE graphs also has no bounded clique-chromatic number (a *UE graph* is the edge intersection graph of a family of paths in a tree).

On the other hand, many families of graphs have bounded clique-chromatic numbers. Recall that a graph G is *perfect* if $\chi(G) = \omega(H)$ for every induced subgraph H of G. In 1991, the following question was proposed by Duffus et al. [8]: Does there exist a constant C such that each perfect graph is C-clique-colorable? This question has a positive answer for two subclasses of perfect graphs; every comparability graph is 2-clique-colorable, and every cocomparability graph is 3-clique-colorable (see more in [7, 8]). However, it is open in general case. Bacsó et al. [1] proved further that almost all perfect graphs are 3-clique-colorable and conjectured that all perfect graphs are 3-clique-colorable.

For a given graph F, a graph G is *F-free* if it does not contain F as an induced subgraph. A graph G is *$(F_1, F_2, \ldots, F_k)$-free* if it is F_i-free for all $1 \leq i \leq k$. In 2003, Gravier, Hoáng and Maffray [9] wrote an interesting paper on the clique-chromatic numbers of F-free graphs. They showed that, for any graph F, the family of all F-free graphs has a bounded clique-chromatic number if and only if F is a vertex-disjoint union of paths. Many families of $(F_1, F_2, \ldots, F_k)$-free graphs are 2-clique-colorable. For example, $(P_3 + P_1)$-free graphs unless it is C_5, (P_5, C_5)-free graphs, (claw, odd hole)-free graphs, and (bull, odd hole)-free graphs

[1, 6, 10]. Recently, Defossez in 2006 [6] also proved that (diamond, odd hole)-free graphs are 4-clique-colorable.

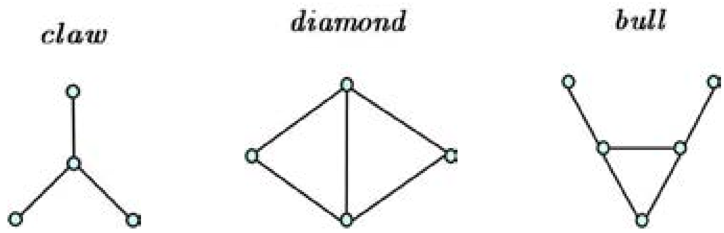

Figure 1.2: A claw, a diamond, and a bull

Furthermore, Mohar and Škrekovski in 1999 [14] proved that all planar graphs are 3-clique-colorable. In 2008, Campos, Dantas and Mello [2] proved that the clique-chromatic number of the power of a cycle, unless it is an odd cycle of size at least five, is two. Later, Cerioli and Korenchendler [3] showed in 2009 that every circular-arc graph, that is the intersection graph of a family of arcs on the circle, is 3-clique-colorable.

In this dissertation, we study two main problems, one in Chapter II, and the other in Chapter III. We obtain the values of the clique-chromatic numbers of the line graphs of complete graphs, and characterize the clique-chromatic numbers of the line graphs of triangle-free graphs in Chapter II. In Chapter III, we focus on the clique-chromatic numbers of F-free graphs where F is a vertex-disjoint union of paths, namely, (P_2+kP_1)-free graphs, (P_3+kP_1)-free graphs, and (P_k+P_m)-free graphs. Moreover, we study the clique-chromatic numbers of some subclasses of claw-free graphs. Lastly, the conclusion and some open problems for future work are located in Chapter IV.

CHAPTER II

CLIQUE-COLORINGS OF LINE GRAPHS

In 2004, Bacsó et al. [1] showed that there is no constant C such that all line graphs are C-clique-colorable, that is, the family of line graphs has no bounded clique-chromatic number. In particular, the Ramsey numbers recalled in Section 2.2 provide a sequence of the line graphs of complete graphs with no bounded clique-chromatic number. In this chapter, we give the exact values of the clique-chromatic numbers of the line graphs of complete graphs in terms of Ramsey numbers. Furthermore, the clique-chromatic numbers of the line graphs of triangle-free graphs are characterized.

2.1 Line Graphs

The *line graph* of a graph G, written $L(G)$, is the graph whose vertices are the edges of G; and for any edges e and f in G, ef is an edge in $L(G)$ if and only if e and f share a common vertex in G.

Example 2.1.

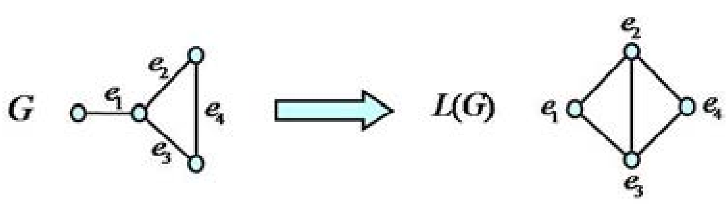

Figure 2.1: The line graph $L(G)$ of a graph G

A graph G is a *line graph* if there is a simple graph H such that $L(H) = G$.

Example 2.2. *Not all graphs are line graphs.* To see this, let G be a claw. Suppose that G is a line graph. Then there is a simple graph H such that $L(H) = G$. Let x be a vertex in G with neighbors a, b and c. Note that each pair of vertices a, b and c are not adjacent. The edges a, b and c in H must be incident to the edge x in H but not share a common vertex. However, the edge x in H has only two endpoints, a contradiction. Hence G is not a line graph.

A star in a graph G is called *maximal* if it is not properly contained in another star or a triangle in a graph G.

Proposition 2.3. *Let G be a graph. Then a maximal clique in $L(G)$ corresponds to a triangle or a maximal star in G.*

Proof. A clique in $L(G)$ corresponds to a triangle or a star in G [19, pp.275]. A triangle in G induces a maximal clique of size three in $L(G)$ because no edge of G is incident to all three edges of a triangle in G. Besides, a maximal star in G induces a maximal clique in $L(G)$. Therefore, a maximal clique in $L(G)$ corresponds to a triangle or a maximal star in G. □

To study a vertex-coloring of $L(G)$, we could study an edge-coloring of G instead. Recall that a *k-edge-coloring* of a graph G is a function $f : E(G) \to X$, where $|X| = k$. Given an edge-coloring of a graph G, a subgraph H of G is said to be *monocolored* if all edges of H are labeled by the same color. Since edges of G correspond to vertices of $L(G)$, by Proposition 2.3, a k-edge-coloring of G without a monocolored triangle and a monocolored maximal star corresponds to a k-coloring of $L(G)$ without a monocolored maximal clique, which is in fact a proper k-clique-coloring of $L(G)$.

2.2 The Line Graphs of the Complete Graphs

Ramsey numbers play a major role in Section 2.2. We recall its concept here.

2.2.1 Ramsey Numbers

The *Ramsey number* $R(k_1, k_2, \ldots, k_m)$ is the smallest positive integer such that every m-edge-coloring of $K_{R(k_1,k_2,\ldots,k_m)}$ gives a monocolored complete subgraph on k_i vertices for some $i \in \{1, 2, \ldots, m\}$. We denote the Ramsey number $R(\underbrace{3, 3, \ldots, 3}_{m})$ by $R(m)$.

In 1930, it was proved that the Ramsey numbers $R(k_1, k_2, \ldots, k_m)$ for any $k_1, k_2, \ldots, k_m \in \mathbb{N}$ always exist [17]. The next proposition shows that the sequence $\{R(m)\}_{m=1}^{\infty}$ is strictly increasing.

Proposition 2.4. *For $m \in \mathbb{N}, R(m) < R(m + 1)$.*

Proof. Suppose that $R(m + 1) \leq R(m)$. Then $R(m + 1) - 1 < R(m)$. By the definition of the Ramsey number $R(m)$, there is an m-edge-coloring g of $K_{R(m+1)-1}$ without a monocolored triangle. Let v be a vertex that is not in $V(K_{R(m+1)-1})$. Consider $K_{R(m+1)-1} \vee v$. Extend $g : E(K_{R(m+1)-1}) \to \{1, 2, \ldots, m\}$ to $\bar{g} : E(K_{R(m+1)-1} \vee v) \to \{1, 2, \ldots, m + 1\}$ by $\bar{g}(uv) = m + 1$ for all $u \in V(K_{R(m+1)-1})$. We have that $\bar{g}$ is an $(m + 1)$-edge-coloring of $K_{R(m+1)-1} \vee v$ without a monocolored triangle. But $K_{R(m+1)-1} \vee v$ is the complete graph with $R(m+1)$ vertices, this contradicts the definition of the Ramsey number $R(m+1)$. Hence $R(m) < R(m + 1)$. $\square$

It is well-known that $R(1) = 3$ and $R(2) = 6$. In 1955, Greenwood and Gleason [11] proved that $R(3) = 17$. We still do not know the exact value of the Ramsey numbers $R(m)$ where $m \geq 4$. Even for case $m = 4$, the recent best bounds provided by Chung [5] and Kramer [12]; $51 \leq R(4) \leq 62$.

2.2.2 The Clique-chromatic Number of $L(K_n)$

In [1], Bacsó et al. proved that the family of line graphs has no bounded clique-chromatic number. In particular, the family of the line graphs of complete graphs on Ramsey numbers of vertices has no bounded clique-chromatic number, that is, $\chi_c(L(K_{R(m)})) > m$ where $m \in \mathbb{N}$. In this section, we sharpen this bound by showing that $\chi_c(L(K_{R(m)})) = m+1$. Furthermore, we extend the result to the exact values of the clique-chromatic numbers of the line graphs of all complete graphs.

Lemma 2.5. *Let $m \in \mathbb{N}$. If a graph G has an m-edge-coloring without a monocolored triangle, then the line graph $L(G \vee v)$ has a proper $(m+1)$-clique-coloring, where v is a vertex that is not in G.*

Proof. For case $m = 1$, let $x \in V(G)$ be fixed. By assumption, G has no triangle. Note that all triangles and maximal stars in $G \vee v$ contain v. Then we define $f : E(G \vee v) \to \{1, 2\}$ by

$$f(ab) = \begin{cases} 2, & \text{if } \ (a \neq x \text{ and } b = v) \text{ or } (a = x \text{ and } b \neq v) \\ 1, & \text{otherwise.} \end{cases}$$

This function f is a 2-edge-coloring of $G \vee v$ without a monocolored triangle and a monocolored maximal star. Thus f corresponds to a proper 2-clique-coloring of $L(G \vee v)$.

Now, assume $m \geq 2$. Let ϕ be an m-edge-coloring of G without a monocolored triangle. Choose a vertex w in G such that $|N_G(w)| \neq 0$. Let i be a color of an edge incident to w in G. Extend $\phi : E(G) \to \{0, 1, \ldots, m-1\}$ to $\bar{\phi} : E(G \vee v) \to \{0, 1, \ldots, m\}$ by

$$\bar{\phi}(e) = \begin{cases} (i+1)(\text{mod } m), & \text{if } \ e = wv \\ m, & \text{if } \ e = uv \text{ for some } u \in V(G) \setminus \{w\} \\ \phi(e), & \text{otherwise.} \end{cases}$$

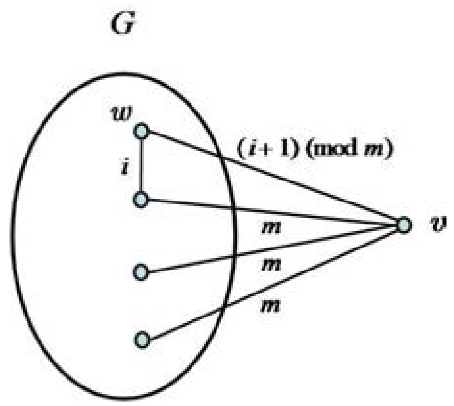

Figure 2.2: An $(m+1)$-edge-coloring $\bar{\phi}$ in Lemma 2.5

We have that $\bar{\phi}$ is an $(m+1)$-edge-coloring of $G \vee v$ without a monocolored triangle and a monocolored maximal star, and hence $\bar{\phi}$ corresponds to a proper $(m+1)$-clique-coloring of $L(G \vee v)$. □

Proposition 2.6. *Let G be a graph and $m \in \mathbb{N}$. If G contains $K_{R(m)}$ as a subgraph, then $\chi_c(L(G)) \geq m+1$.*

Proof. Suppose that $L(G)$ has a proper m-clique-coloring. Then G has an m-edge-coloring which contains no monocolored triangle, say f. Thus $f|_{K_{R(m)}}$ is an m-edge-coloring without a monocolored triangle. This contradicts the definition of the Ramsey number $R(m)$. Hence $L(G)$ has no proper m-clique-coloring, and so $\chi_c(L(G)) > m$. □

In the next theorem, we sharpen the bound in Bacsó's result by giving the exact values of the clique-chromatic numbers of $L(K_{R(m)})$ for all $m \in \mathbb{N}$.

Theorem 2.7. *For $m \in \mathbb{N}$, $\chi_c(L(K_{R(m)})) = m + 1$.*

Proof. By Proposition 2.6, we have $\chi_c(L(K_{R(m)})) \geq m+1$. The definition of $R(m)$ implies that $K_{R(m)-1}$ has an m-edge-coloring without a monocolored triangle. By Lemma 2.5, $L(K_{R(m)})$ has a proper $(m+1)$-clique-coloring. Hence $\chi_c(L(K_{R(m)})) \leq m + 1$. □

Note that $L(K_1)$ is the null graph (that is, the graph whose vertex set and edge set are empty), and $\chi_c(L(K_2)) = \chi_c(K_1) = 1$. Now, let $n \geq 3$. We have that there always exists a positive integer m such that $R(m) \leq n < R(m + 1)$ because the Ramsey numbers always exist and $\{R(m)\}_{m=1}^{\infty}$ is a strictly increasing sequence. The next theorem is the main theorem of this section. It gives the value of the clique-chromatic number of $L(K_n)$ where $n \geq 3$.

Theorem 2.8. *For $n \geq 3$, $\chi_c(L(K_n)) = m + 1$ where $R(m) \leq n < R(m + 1)$ for some positive integer m.*

Proof. Since $n \geq R(m)$, K_n contains $K_{R(m)}$ as a subgraph. By Proposition 2.6, $\chi_c(L(K_n)) \geq m + 1$. If $n = R(m)$, then $\chi_c(L(K_{R(m)})) = m + 1$ by Theorem 2.7. Assume that $R(m) < n < R(m + 1)$. The definition of $R(m + 1)$ implies that K_n has an $(m + 1)$-edge-coloring without a monocolored triangle, say ϕ. Suppose that ϕ gives a monocolored maximal star S, say labeled all edges in S by color 1. If there is an edge in K_n outside S colored by 1, then K_n contains a monocolored triangle, a contradiction. Thus $E(K_n) \setminus E(S)$ uses m colors, moreover they form a complete graph K_{n-1}. Since $n - 1 \geq R(m)$, for every m-edge-coloring of K_{n-1} gives a monocolored triangle, a contradiction. Thus ϕ is an $(m + 1)$-edge-coloring of G without a monocolored maximal star. Therefore ϕ corresponds to a proper $(m + 1)$-clique-coloring of $L(K_n)$, and hence $\chi_c(L(K_n)) \leq m + 1$. □

Given a positive integer m, Theorem 2.8 provides $R(m+1)-R(m)$ line graphs with clique-chromatic number $m+1$. In fact, there are infinitely many line graphs with the same clique-chromatic number as shown below.

Example 2.9. *Let $n \in \mathbb{N}$ where $R(m) \leq n < R(m+1)$ for some positive integer m. Let $G = kK_{n-t} \vee pK_t$ where $k, p \in \mathbb{N}$ and $1 \leq t \leq n-1$. Then $\chi_c(L(G)) = m+1$.*

Proof. Since K_n is a subgraph of G and $n \geq R(m)$, G contains $K_{R(m)}$ as a subgraph. By Proposition 2.6, $\chi_c(L(G)) \geq m+1$. Let $kK_{n-t} = K_{n-t}^{(1)} + K_{n-t}^{(2)} + \cdots + K_{n-t}^{(k)}$ and $pK_t = K_t^{(1)} + K_t^{(2)} + \cdots + K_t^{(p)}$where $K_{n-t}^{(i)}$ and $K_t^{(j)}$are copies of K_{n-t} and K_t, respectively.

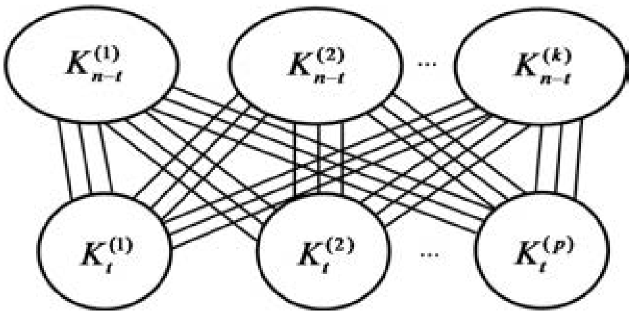

Figure 2.3: The graph $kK_{n-t} \vee pK_t$

Note that for $i = 1, 2, \ldots, k$ and $j = 1, 2, \ldots, p$, $K_{n-t}^{(i)} \vee K_t^{(j)} = K_n$. Since $R(m) \leq n < R(m+1)$, we have $\chi_c(L(K_{n-t}^{(1)} \vee K_t^{(1)})) = m+1$ by Theorem 2.8. Thus there is an $(m+1)$-edge-coloring ϕ of $K_{n-t}^{(1)} \vee K_t^{(1)}$ without a monocolored triangle and a monocolored maximal star, and then ϕ can be easily extended to an $(m+1)$-edge-coloring of G without a monocolored triangle and a monocolored maximal star. So $L(G)$ has a proper $(m+1)$-clique-coloring. Hence $\chi_c(L(G)) \leq m+1$. $\square$

We next extend the previous result to consider the line graphs of complete graphs in which some edges are removed.

Theorem 2.10. *Let $n \in \mathbb{N}$ where $R(m) \leq n < R(m+1)$ for some positive integer m. Let G be a graph with $n+1$ vertices and $E(G) = E(K_{n+1}) - E(K_{1,r})$ where $1 \leq r \leq n-1$. Then $L(G)$ is $(m+2)$-clique-colorable. In particular, if $r = 1$ or $n - m - 1 \leq r \leq n - 1$, then $\chi_c(L(G)) = m + 1$.*

Proof. Let x be the center of $K_{1,r}$ and $w \in N_G(x)$. Since $R(m) \leq n < R(m+1)$ and $G - x = K_n$, by Theorem 2.8, $\chi_c(L(G-x)) = m+1$. Thus there is a proper $(m+1)$-clique-coloring of $L(G-x)$, so there is an $(m+1)$-edge-coloring of $G - x$ without a monocolored triangle and a monocolored maximal star, say ϕ.

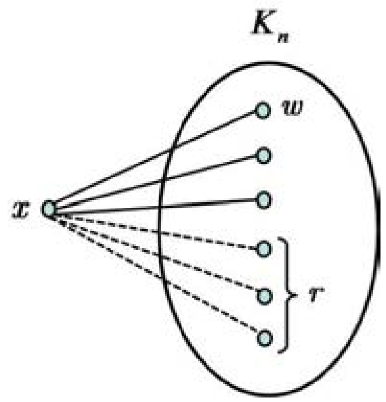

Figure 2.4: The graph G with $n+1$ vertices and the edge set $E(K_{n+1}) - E(K_{1,r})$

Extend $\phi : E(G-x) \to \{1, 2, \ldots, m+1\}$ to $\bar{\phi} : E(G) \to \{1, 2, \ldots, m+2\}$ by

$$\bar{\phi}(e) = \begin{cases} 1, & \text{if } \ e = xw \\ m+2, & \text{if } \ e = xu \text{ for some } u \in V(G) \setminus \{w\} \\ \phi(e), & \text{otherwise.} \end{cases}$$

Then $\bar{\phi}$ is an $(m+2)$-edge-coloring of G without a monocolored triangle and a monocolored maximal star. Thus $\bar{\phi}$ corresponds to a proper $(m+2)$-clique-coloring of $L(G)$, and hence $\chi_c(L(G)) \leq m+2$.

If $r = 1$, then $\chi_c(L(G)) = m+1$ by Example 2.9 when $k = 1$, $p = 2$ and $t = 1$. For case $n - m - 1 \leq r \leq n - 1$, let $N_G(x) = \{v_1, v_2, \ldots, v_l\}$ where $l = n - r$. Then $1 \leq l \leq m + 1$. Extend ϕ to an $(m+1)$-edge-coloring f of G by $f(xv_i) = i$ for all $1 \leq i \leq l$. Then f is an $(m+1)$-edge-coloring of G without a monocolored triangle and a monocolored maximal star. Thus $\chi_c(L(G)) \leq m + 1$. Since K_n is a subgraph of G and $n \geq R(m)$, by Proposition 2.6, $\chi_c(L(G)) \geq m + 1$. Hence $\chi_c(L(G)) = m + 1$. □

2.3 The Line Graphs of Triangle-free Graphs

In this section, we characterize the clique-chromatic numbers of the line graphs of triangle-free graphs. Given a triangle-free graph G, by Proposition 2.3, a maximal clique in $L(G)$ corresponds to a maximal star in G. Thus if f is a k-edge-coloring of G without a monocolored maximal star, then f corresponds to a proper k-clique-coloring of $L(G)$.

Theorem 2.11. *If G is a triangle-free graph, then $\chi_c(L(G)) \leq 3$.*

Proof. Let G be a triangle-free graph. Without lost of generality, assume that G is connected. Let $x \in V(G)$. Define $A_0 = \{x\}$, $A_1 = N_G(x)$, and $A_i = N_G(A_{i-1}) \smallsetminus (A_{i-1} \cup A_{i-2})$ for all $i \geq 2$. We refer to a vertex having distance i from x as a vertex of distance i. Then A_i contains all vertices with distance i. Each edge in G joins either two vertices of the same distance or vertices of distance $i - 1$ and i, for some i. If in the later case, we call such edge *a (distance i)-edge*. We first label all (distance 1)-edges by color 1 or color 2, at least one edge for each color. (If $|N_G(x)| = 1$, label the unique edge by color 1.) Then for each i^{th} step, $i = 2, 3, \ldots$, label a (distance i)-edge by color 1 if it is incident to a (distance $i - 1$)-edge of color 2, and by color 2, otherwise. Finally, label all edges joining two vertices of the same distance by color 3. This process garantees that each vertex is incident to edges of at least two colors except the end vertices (vertices incident to a (distance $i-1$)-edge but not to any (distance i)-edge). If an end vertex is incident to all edges of the same color, we can relabel one edge of them by color 3.

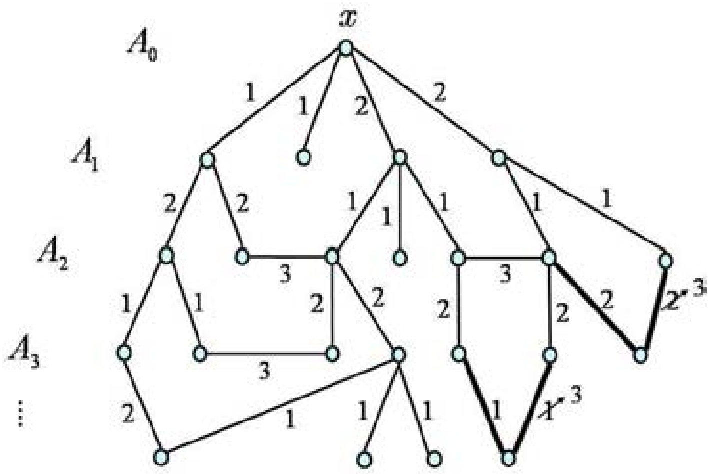

Figure 2.5: A 3-edge-coloring of G in Theorem 2.11

Therefore, we have a 3-edge-coloring of G without a monocolored maximal star. So the coloring corresponds to a proper 3-clique-coloring of $L(G)$, and hence $\chi_c(L(G)) \leq 3$. □

The upper bound in Theorem 2.11 is sharp by the odd cycle C_{2n+1} ($n \geq 2$) because C_{2n+1} is triangle-free and $\chi_c(L(C_{2n+1})) = 3$.

In our purpose, a graph is called *trivial* if it is the complete graph K_1 or K_2. Note that if a graph G has a nontrivial component, then $\chi_c(L(G)) \geq 2$.

Lemma 2.12. *If G is a forest having a nontrivial component, then $\chi_c(L(G)) = 2$.*

Proof. Use the same coloring in the proof of Theorem 2.11. Since G has no cycle, all end vertices have degree 1 and there is no edge incident to vertices of the same distance. Thus color 3 is not used in the coloring. Besides G has a nontrivial component, $\chi_c(L(G)) = 2$. $\square$

Lemma 2.13. *If G is a bipartite graph having a nontrivial component, then $\chi_c(L(G)) = 2$.*

Proof. If G contains no cycle, then G is a forest, it is done by Lemma 2.12. Now, assume that O_1 is any cycle of G. Since G is bipartite, O_1 is an even cycle. Label edges of O_1 alternately around the cycle by 1,2,1,2,..., then this is a 2-edge-coloring of O_1 without a monocolored maximal star. If $G - E(O_1)$ has a cycle, say O_2, then we color edges of O_2 similarly to O_1. Then similarly consider $G - (E(O_1) \cup E(O_2))$. Continue this process until the resulting graph contains no cycle. Label this resulting graph by the coloring in Lemma 2.12. Therefore, G has a 2-edge-coloring without a monocolored maximal star, and hence $\chi_c(L(G)) = 2$. $\square$

The next theorem is the main theorem of this section. It contains a characterization of the clique-chromatic numbers of the line graphs of triangle-free graphs.

Theorem 2.14. *Let G be a triangle-free graph with at least one edge. Then*

$$\chi_c(L(G)) = \begin{cases} 1, & \textit{if all components of } G \textit{ are trivial} \\ 3, & \textit{if } G \textit{ has an odd hole component} \\ 2, & \textit{otherwise.} \end{cases}$$

Proof. If all components of G are trivial, then $\chi_c(L(G)) = 1$. Assume that G has a nontrivial component. If G has an odd hole component, say O, then $\chi_c(L(O)) = 3$. Thus $\chi_c(L(G)) \geq 3$. By Theorem 2.11, $\chi_c(L(G)) = 3$. Assume that G has no odd hole component. Without lost of generality, assume that G is connected. Let H be the union of all odd holes of G. Then $G-E(H)$ is a bipartite graph. By Lemma 2.13, $G-E(H)$ has a 2-edge-coloring without a monocolored maximal star, say f. To label edges of H, assume that H is connected. We can write $H=\bigcup_{i=1}^{n} O_i$ for some $n \in \mathbb{N}$ where each O_i is an odd hole of G, and $V(O_j) \cap V(\bigcup_{i=1}^{j-1} O_i) \neq \varnothing$ for each $2 \leq j \leq n$.

Claim that G has a 2-edge-coloring (extend from f) without a monocolored maximal star by induction on n. If $n=1$, then H is an odd hole of G. Since G is connected, there is a vertex $x \in V(H)$ having an incident edge which is colored by f, say color 1. Label two incident edges of x in H by color 2 and label other edges of H alternately around the cycle by 1,2,1,2,.... Since the number of edges of H is odd, every vertex of H has two incident edges with different colors. Now, assume that $n \geq 2$ and $(G-E(H))+E(\bigcup_{i=1}^{n-1} O_i)$ has a 2-edge-coloring without a monocolored maximal star, say f'. Thus every vertex of $(G-E(H))+E(\bigcup_{i=1}^{n-1} O_i)$ has two incident edges with different colors by f'. If O_n and $\bigcup_{i=1}^{n-1} O_i$ have the only one common vertex, say y, then y has two incident edges in $\bigcup_{i=1}^{n-1} O_i$ with different colors. Label edges of O_n alternately around the cycle by 1,2,1,2,.... If $|O_n \cap \bigcup_{i=1}^{n-1} O_i)| \geq 2$, then consider each path in O_n such that each edge of a path is not contained in $\bigcup_{i=1}^{n-1} O_i$. Label edges of each such path alternately by 1,2,1,2,.... Then every vertex of O_n has two incident edges with different colors. So G has a 2-edge-coloring without a monocolored maximal star. Hence $\chi_c(L(G))=2$. $\square$

CHAPTER III

CLIQUE-COLORINGS OF F-FREE GRAPHS

In 2003, Gravier, Hoáng and Maffray [9] gave a significant result that, for any graph F, the family of all F-free graphs has a bounded clique-chromatic number if and only if F is a vertex-disjoint union of paths, and they give an upper bound for all such cases. In this chapter, we show better bounds for $F = P_2 + kP_1$, $F = P_3 + kP_1$ with $k \geq 3$, and $F = P_k + P_m$ with $\max\{k, m\} \geq 4$, and sharp bounds are given for some subclasses. In the last section of this chapter, we investigate the clique-chromatic numbers of claw-free graphs.

Let $f(F) = \max\{\chi_c(G) \mid G \text{ is an } F\text{-free graph}\}$.

Remark 3.1. Let F_1 be an induced subgraph of a graph F_2. If a graph G is F_1-free then G is also F_2-free, it follows that $f(F_1) \leq f(F_2)$.

In [9], Gravier, Hoáng and Maffray showed the following result.

Theorem 3.2. [9] *Let F be a graph. Then $f(F)$ exists if and only if F is a vertex-disjoint union of paths. Moreover,*

- if $|V(F)| \leq 2$ or $F = P_3$ then $f(F) \leq 2$,

- else $f(F) \leq cc(F) + |V(F)| - 3$ where $cc(F)$ is the number of components of a graph F.

Furthermore, they proved that (P_2+2P_1)-free graphs and (P_3+2P_1)-free graphs are 3-clique-colorable. Since the cycle C_5 is both (P_2+2P_1)-free and (P_3+2P_1)-free with $\chi_c(C_5) = 3$, this bound is sharp.

3.1 $(P_2 + kP_1)$-free Graphs

It follows from Theorem 3.2 that every $(P_2 + kP_1)$-free graph is $(2k)$-clique-colorable. We improve this bound for $k \geq 3$. In [1], Bacsó et al. stated the relationship between the clique-chromatic number and the size of a maximum independent set of a graph, as follows.

Theorem 3.3. [1] *Let G be a graph. If $G \neq C_5$ and G is not a complete graph, then $\chi_c(G) \leq \alpha(G)$.*

Theorem 3.4. *For $k \geq 3$, a $(P_2 + kP_1)$-free graph is $(k+1)$-clique-colorable.*

Proof. Let G be a $(P_2 + kP_1)$-free graph. Let $S = \{s_0, s_1, \ldots, s_{\alpha(G)-1}\}$ be a maximum independent set of G. If $\alpha(G) \leq k$, then $\chi_c(G) \leq k$ by Theorem 3.3. Assume $\alpha(G) \geq k+1$. Let $M(s_0) = V(G) \smallsetminus (S \cup N_G(s_0))$ and $A = \{v \in M(s_0) \mid N_S(v) = S \smallsetminus \{s_0\}\}$. For $\varnothing \neq R \subseteq S \smallsetminus \{s_0\}$, define $Y_R = \{v \in M(s_0) \mid N_S(v) = S \smallsetminus (\{s_0\} \cup R)\}$ and $\min(R) = \min\{i \in \mathbb{N} \mid s_i \notin R\}$. Note that $V(G)$ is the disjoint union of S, $N_G(s_0)$, A, and Y_R where $\varnothing \neq R \subseteq S \smallsetminus \{s_0\}$. Let f be the coloring of G defined by

$$f(v) = \begin{cases} 1, & \text{if } v \in S \\ 2, & \text{if } v \in N_G(s_0) \\ 3, & \text{if } v \in A \\ \min(R) + 2, & \text{if } v \in Y_R \text{ where } R = S \smallsetminus (\{s_0\} \cup N_S(v)). \end{cases}$$

To claim that f is a $(k+1)$-coloring of G, let $R \subseteq S \smallsetminus \{s_0\}$ where $Y_R \neq \varnothing$, and let $y \in Y_R$. If $R = S \smallsetminus \{s_0\}$, then $N_S(y) = \varnothing$; so $S \cup \{y\}$ is an independent set of G. This contradicts the maximality of S. Thus $R \neq S \smallsetminus \{s_0\}$. If $|R| \geq k-1$, then the subgraph of G induced by $S \cup \{y\}$ contains an induced subgraph $P_2 + kP_1$, a contradiction. Thus $|R| \leq k-2$, and it follows that $\min(R) \leq k-1$. Therefore, f is a $(k+1)$-coloring of G.

Suppose that G has a monocolored maximal clique Q of size at least two, say colored by m. Since S is an independent set, $m \neq 1$. Thus $Q \cap S = \varnothing$. Note that $s_{\min(R)}$ is adjacent to all vertices of Y_R. Thus s_{m-2} is adjacent to all vertices of Q. Then $Q \cup \{s_{m-2}\}$ is a clique of G. It contradicts the maximality of Q. Hence f is a proper $(k+1)$-clique-coloring of G, and so $\chi_c(G) \leq k+1$. □

Theorem 3.4 ensures that every $(P_2 + kP_1)$-free graph where $k \geq 3$ is $(k+1)$-clique-colorable but we have found no graph guaranteeing this sharpness yet. However, when $k = 3$ and 4, there is a $(P_2 + kP_1)$-free graph which is k-clique-colorable, namely, the cycle C_5 is $(P_2 + 3P_1)$-free and $\chi_c(C_5) = 3$, and the 4-chromatic Mycielski's graph G_4 is $(P_2 + 4P_1)$-free and $\chi_c(G_4) = 4$. Notice that both of them are diamond-free, this suggests the result in Theorem 3.5.

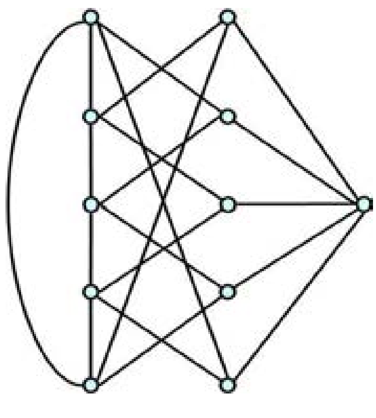

Figure 3.1: The 4-chromatic Mycielski's graph G_4

Theorem 3.5. *For $k \geq 3$, a $(P_2 + kP_1$,diamond$)$-free graph is k-clique-colorable.*

Proof. Let G be a $(P_2+kP_1$,diamond$)$-free graph. If $\alpha(G) \leq k$, then $\chi_c(G) \leq k$ by Theorem 3.3. Assume $\alpha(G) \geq k+1$. Let $S = \{s_0, s_1, \ldots, s_{\alpha(G)-1}\}$ be a maximum independent set of G.

Let $M(s_0) = V(G) \smallsetminus (S \cup N_G(s_0))$ and $A = \{v \in M(s_0) \mid N_S(v) = S \smallsetminus \{s_0\}\}$. For $\varnothing \neq R \subseteq S \smallsetminus \{s_0\}$, define $Y_R = \{v \in M(s_0) \mid N_S(v) = S \smallsetminus (\{s_0\} \cup R)\}$ and $\min(R) = \min\{i \in \mathbb{N} \mid s_i \notin R\}$. Note that $V(G)$ is the disjoint union of S, $N_G(s_0)$, A, and Y_R where $\varnothing \neq R \subseteq S \smallsetminus \{s_0\}$. By the same argument as in the proof of Theorem 3.4, we have that $\min(R) \leq k-1$ for each $R \subseteq S \smallsetminus \{s_0\}$ where $Y_R \neq \varnothing$. Define $g : V(G) \to \{1, 2, \ldots, k\}$ by

$$g(v) = \begin{cases} 1, & \text{if } v \in S \\ 2, & \text{if } v \in N_G(s_0) \\ 3, & \text{if } v \in A \\ \min(R)+2, & \text{if } v \in Y_R \text{ where } R = S \smallsetminus (\{s_0\} \cup N_S(v)) \text{ and} \\ & \quad \min(R) \leq k-2 \\ k, & \text{if } v \in Y_R \text{ where } R = S \smallsetminus (\{s_0\} \cup N_S(v)) \text{ and} \\ & \quad \min(R) = k-1. \end{cases}$$

To claim that g is a proper k-clique-coloring of G, suppose that G has a monocolored maximal clique Q of size at least two, say colored by m. Since S is an independent set, $m \neq 1$. If $m \leq k-1$, then we have that s_{m-2} is adjacent to all vertices of Q, a contradiction.

Assume $m = k$. Then $Q \subseteq \bigcup\{Y_R \mid R \subseteq S \smallsetminus \{s_0\} \text{ and } k-2 \leq \min(R) \leq k-1\}$. Since $Y_R = \varnothing$ for all $R \subseteq S \smallsetminus \{s_0\}$ where $|R| \geq k-1$, we consider only Y_R where $|R| \leq k-2$. Thus if $k-2 \leq \min(R) \leq k-1$, then $R = \{s_1, s_2, \ldots, s_{k-3}, s_t\}$ where $k-2 \leq t \leq \alpha(G)-1$. Since G is diamond-free and $\alpha(G)-1 \geq k$, Y_R is an independent set, and then $|Q \cap Y_R| \leq 1$ for each $R \subseteq S \smallsetminus \{s_0\}$. If $|Q| \geq 3$, then there exists a diamond induced by a vertex in $S \smallsetminus \{s_0\}$ and three vertices in Q, a contradiction. So $|Q| = 2$. Let $Q \subseteq Y_{R_1} \cup Y_{R_2}$ for some $R_1, R_2 \subseteq S \smallsetminus \{s_0\}$ where

$R_1 \neq R_2$ and $k-2 \leq \min(R_1), \min(R_2) \leq k-1$. Then $|R_1 \cup R_2| \leq k-1$. Since $\alpha(G) - 1 \geq k$, there exists a vertex in $S \setminus \{s_0\}$ that is adjacent to all vertices of Q, a contradiction. Hence $\chi_c(G) \leq k$. $\square$

Gravier et al. [9] showed that all $(P_2 + 2P_1)$-free graphs are 3-clique-colorable, and this bound is sharp by the cycle C_5. In the next theorem, we give a subclass of $(P_2 + 2P_1)$-free graphs with clique-chromatic number two.

Theorem 3.6. *Let G be a $(P_2 + 2P_1$,diamond$)$-free graph where $G \neq C_5$ and $E(G) \neq \varnothing$. Then $\chi_c(G) = 2$.*

Proof. Let S be a maximum independent set of G. If $|S| \leq 2$, then $\chi_c(G) \leq 2$ by Theorem 3.3. Assume $|S| \geq 3$. Let $x \in S$ and $M(x) = V(G) \setminus (S \cup N_G(x))$. Since G is $(P_2 + 2P_1)$-free, each vertex in $M(x)$ is adjacent to all vertices of $S \setminus \{x\}$. Assign color 1 to all vertices of S and color 2 to all vertices of $V(G) \setminus S$.

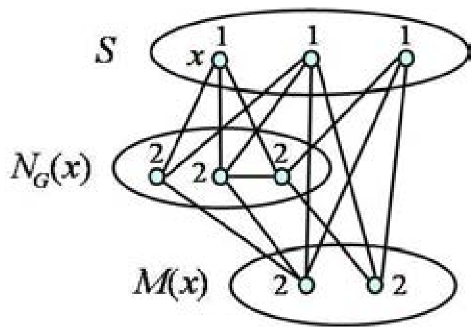

Figure 3.2: A $(P_2 + 2P_1$, diamond$)$-free graph G

Suppose that G has a monocolored maximal clique Q of size at least two. Since S is an independent set, all vertices of Q are colored by 2. If $Q \subseteq N_G(x)$, then x is adjacent to all vertices of Q, a contradiction. Thus Q is not a subset of $N_G(x)$. Since G is diamond-free, $M(x)$ is an independent set. Thus $|Q \cap M(x)| = 1$, say $a \in Q \cap M(x)$. If $|Q| \geq 3$, then the vertices x and a and two vertices in

$Q \cap N_G(x)$ induce a diamond, a contradiction. Thus $|Q| = 2$. Let $Q = \{a, b\}$ where $b \in N_G(x)$. Since G is $(P_2 + 2P_1)$-free and $|S| \geq 3$, each vertex in $N_G(x)$ is adjacent to a vertex in $S \setminus \{x\}$. Then there exists a vertex in $S \setminus \{x\}$ that is adjacent to both a and b, a contradiction. Thus there is no monocolored maximal clique of G of size at least two, so the coloring is a proper 2-clique-coloring of G. Since $E(G) \neq \varnothing$, $\chi_c(G) = 2$. □

3.2 $(P_3 + kP_1)$-free Graphs

It follows from Theorem 3.2 that every $(P_3 + kP_1)$-free graph is $(2k+1)$-clique-colorable. We give a better bound in the next theorem.

Theorem 3.7. *For $k \geq 3$, a $(P_3 + kP_1)$-free graph is $(k+2)$-clique-colorable.*

Proof. Let G be a (P_3+kP_1)-free graph. Let $S = \{s_1, s_2, \ldots, s_{\alpha(G)}\}$ be a maximum independent set of G. If $\alpha(G) \leq k+1$, then $\chi_c(G) \leq k+1$ by Theorem 3.3. Assume $\alpha(G) \geq k + 2$. Let $A = \{v \in V(G) \setminus S \mid N_S(v) = S\}$. For $1 \leq i \leq \alpha(G)$, let $X_i = \{v \in V(G) \setminus S \mid N_S(v) = \{s_i\}\}$. Suppose that there is an edge, say $x_i x_j$, between X_i and X_j where $i \neq j$. Then there exist k vertices in $S \setminus \{s_i, s_j\}$ together with s_i, x_i, x_j form an induced subgraph $P_3 + kP_1$ of G, a contradiction. Thus there is no edge between any two X_i's.

For $\varnothing \neq R \subseteq S$ where $|R| \neq \alpha(G) - 1$, define $Y_R = \{v \in V(G) \setminus S \mid N_S(v) = S \setminus R\}$ and $\min(R) = \min\{i \in \mathbb{N} \mid s_i \notin R\}$. Note that $V(G)$ is the disjoint union of S, A, X_i where $1 \leq i \leq \alpha(G)$, and Y_R where $\varnothing \neq R \subseteq S$ and $|R| \neq \alpha(G) - 1$.

Let f be the coloring of G defined by

$$f(v) = \begin{cases} 1, & \text{if} \quad v \in S \\ 2, & \text{if} \quad v \in A \cup \left(\bigcup_{i=1}^{\alpha(G)} X_i\right) \\ \min(R) + 2, & \text{if} \quad v \in Y_R \text{ where } R = S \setminus N_S(v). \end{cases}$$

To claim that f is a $(k+2)$-coloring of G, let $R \subseteq S$ where $Y_R \neq \varnothing$, and let $y \in Y_R$. If $R = S$, then $N_S(y) = \varnothing$; so $S \cup \{y\}$ is an independent set of G. It contradicts the maximality of S. If $k \leq |R| \leq \alpha(G) - 2$, then the subgraph of G induced by $S \cup \{y\}$ contains an induced subgraph $P_3 + kP_1$, a contradiction. Thus $|R| \leq k - 1$, and it follows that $\min(R) \leq k$. Hence f is a $(k+2)$-coloring of G.

Now, suppose that G has a monocolored maximal clique Q of size at least two, say colored by m. Since S is an independent set, $m \neq 1$. If $m = 2$, then $Q \subseteq A \cup X_i$ for some i. We have that s_i is adjacent to all vertices of Q, a contradiction. Now, assume $m \geq 3$. Since $s_{\min(R)}$ is adjacent to all vertices of Y_R, s_{m-2} is adjacent to all vertices of Q, a contradiction. Thus f is a proper $(k+2)$-clique-coloring of G, and hence $\chi_c(G) \leq k + 2$. □

Similarly to $(P_2 + kP_1)$-free graphs, the result for $(P_3 + kP_1)$-free graphs in Theorem 3.7 has not been proved to be sharp. Theorem 3.8 gives its subclass of graphs using at most $k + 1$ colors.

Theorem 3.8. *For $k \geq 3$, a $(P_3 + kP_1$,diamond$)$-free graph is $(k+1)$-clique-colorable.*

Proof. Let G be a $(P_3 + kP_1$,diamond$)$-free graph. Let $S = \{s_1, s_2, \ldots, s_{\alpha(G)}\}$ be a maximum independent set of G. If $\alpha(G) \leq k + 1$, then $\chi_c(G) \leq k + 1$ by Theorem 3.3. Assume $\alpha(G) \geq k + 2$. Let $A = \{v \in V(G) \setminus S \mid N_S(v) = S\}$. For $1 \leq i \leq \alpha(G)$, let $X_i = \{v \in V(G) \setminus S \mid N_S(v) = \{s_i\}\}$. For $\varnothing \neq R \subseteq S$ where $|R| \neq \alpha(G) - 1$, define $Y_R = \{v \in V(G) \setminus S \mid N_S(v) = S \setminus R\}$ and $\min(R) = \min\{i \in \mathbb{N} \mid s_i \notin R\}$. Note that $V(G)$ is the disjoint union of S, A, X_i where $1 \leq i \leq \alpha(G)$, and Y_R where $\varnothing \neq R \subseteq S$ and $|R| \neq \alpha(G) - 1$. By the same argument as in the proof of Theorem 3.7, there is no edge between any two X_i's and $\min(R) \leq k$ for each $R \subseteq S$ where $Y_R \neq \varnothing$.

Define $g : V(G) \to \{1, 2, \ldots, k+1\}$ by

$$g(v) = \begin{cases} 1, & \text{if} \quad v \in S \\ 2, & \text{if} \quad v \in A \cup (\bigcup_{i=1}^{\alpha(G)} X_i) \\ \min(R) + 2, & \text{if} \quad v \in Y_R \text{ where } R = S \setminus N_S(v) \text{ and } \min(R) \leq k-1 \\ k+1, & \text{if} \quad v \in Y_R \text{ where } R = S \setminus N_S(v) \text{ and } \min(R) = k. \end{cases}$$

To claim that g is a proper $(k+1)$-clique-coloring of G, suppose that G has a monocolored maximal clique Q of size at least two, say colored by m. Since S is an independent set, $m \neq 1$. If $m = 2$, then $Q \subseteq A \cup X_i$ for some i; so s_i is adjacent to all vertices of Q, a contradiction. If $3 \leq m \leq k$, then we have that s_{m-2} is adjacent to all vertices of Q, a contradiction.

Assume $m = k+1$. Then $Q \subseteq \bigcup\{Y_R \mid R \subseteq S \text{ and } k-1 \leq \min(R) \leq k\}$. Since G is diamond-free and $\alpha(G) \geq k+2$, Y_R is an independent set. Thus $|Q \cap Y_R| \leq 1$ for each $R \subseteq S$. If $|Q| \geq 3$, then there exist a vertex in S together with any three vertices in Q which induce a diamond, a contradiction. So $|Q| = 2$. Since $\alpha(G) \geq k+2$, there exists a vertex in S that is adjacent to all vertices of Q, a contradiction. Hence $\chi_c(G) \leq k+1$. $\square$

Since the 4-chromatic Mycielski's graph G_4 is ($P_3 + 3P_1$, diamond)-free, the upper bound in Theorem 3.8 for the case $k = 3$ is sharp.

3.3 $(P_k + P_m)$-free Graphs

It follows from Theorem 3.2 that every $(P_k + P_m)$-free graph is $(k+m-1)$-clique-colorable. This bound is sharp when $\max\{k, m\} \leq 3$ and not both of k and m are 3 (see more in [9]). In this section, we improve this bound when $\max\{k, m\} \geq 4$.

In [9], Gravier et. al gave an upper bound of the clique-chromatic numbers of P_k-free graphs where $k \geq 4$, as follows.

Theorem 3.9. [9] *For $k \geq 4$, a P_k-free graph is $(k-2)$-clique-colorable.*

Remark 3.10.

(i) If G is a P_2-free graph, then G is an edgeless graph, so $\chi_c(G) = 1$.

(ii) If G is a P_3-free graph, then each component of G is a clique, so $\chi_c(G) \leq 2$.

Theorem 3.11. *For $k, m \in \mathbb{N}$ where $\max\{k, m\} \geq 4$, a $(P_k + P_m)$-free graph is $(k+m-2)$-clique-colorable.*

Proof. Let G be a (P_k+P_m)-free graph. Without lost of generality, assume $k \geq m$. If G contains no induced path P_m, then G is P_m-free. Thus G is also P_k-free. Since $k \geq 4$, $\chi_c(G) \leq k-2 \leq k+m-2$ by Theorem 3.9.

Assume that G contains an induced path P_m. Let $V(P_m) = \{v_1, v_2, \ldots, v_m\}$ and $M = V(G) \setminus (V(P_m) \cup (\bigcup_{i=1}^{m} N_G(v_i)))$.

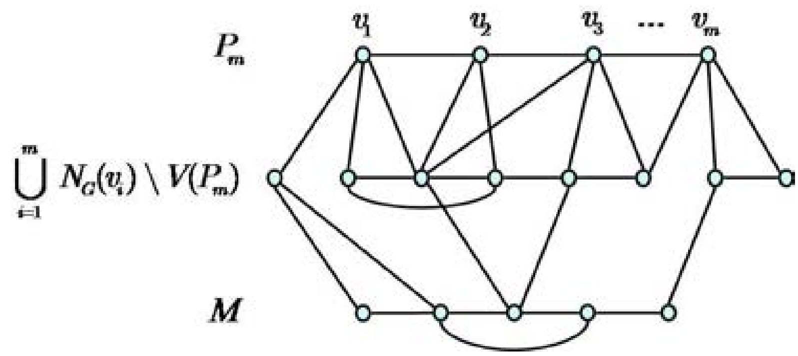

Figure 3.3: A $(P_k + P_m)$-free graph G

Note that each vertex v_i is not adjacent to a vertex in M. Since G is (P_k+P_m)-free, an induced subgraph $G[M]$ is P_k-free. Since $k \geq 4$, $\chi_c(G[M]) \leq k-2$ by Theorem 3.9. Thus there exists a proper $(k-2)$-clique-coloring of $G[M]$, using colors $1, 2, \ldots, k-2$. Then we label vertices of P_m alternately by 1,2,1,2,....

Next, we label all vertices of $N_{G-V(P_m)}(v_1)$ by color $k-1$, label all vertices of $N_{G-V(P_m)}(v_2) \smallsetminus N_G(v_1)$ by color k and so on. Finally, we label all vertices of $N_{G-V(P_m)}(v_m) \smallsetminus \bigcup_{i=1}^{m-1} N_G(v_i)$ by color $k+m-2$.

Suppose that G has a monocolored maximal clique Q of size at least two. It is clear that every maximal clique of size at least two in M is not monocolored. Thus $Q \subseteq \bigcup_{i=1}^{m} N_G(v_i) \smallsetminus V(P_m)$. Then there exists a vertex v_i in $V(P_m)$ which is adjacent to all vertices of Q, a contradiction. Thus the coloring is a proper $(k+m-2)$-clique-coloring of G, and hence $\chi_c(G) \leq k+m-2$. □

Since the cycle C_5 is (P_4+P_1)-free and the 4-chromatic Mycielski's graph G_4 is (P_4+P_2)-free, the upper bound in Theorem 3.11 is sharp when ($k=4$ and $m=1$), and ($k=4$ and $m=2$).

3.4 Claw-free Graphs

Since a claw is not a vertex-disjoint union of paths, by Theorem 3.2, the family of claw-free graphs has no bounded clique-chromatic number. In 2004, Bacsó et al. [1] proved that all (claw, odd hole)-free graphs are 2-clique-colorable. In this section, we focus on some other subclasses of the family of claw-free graphs, namely, (claw, paw)-free graphs and (claw, diamond)-free graphs.

3.4.1 (Claw, paw)-free Graphs

In 1988, paw-free graphs have been characterized by Olariu [16], as follows.

Theorem 3.12. [16] *If G is a paw-free graph, then each component of G is either triangle-free or complete multipartite.*

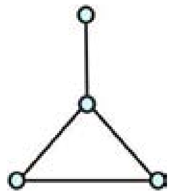

Figure 3.4: A paw

Lemma 3.13. *Let G be a complete multipartite graph with at least one edge. Then $\chi_c(G) = 2$.*

Proof. Since each maximal clique of G intersects every partite set of G, labeling all vertices of one partite set of G by color 1 and the remaining vertices by color 2 provides a proper 2-clique-coloring of G. So $\chi_c(G) = 2$. □

Lemma 3.14. *Let G be a (claw, triangle)-free graph. Then each component of G is a path or a cycle.*

Proof. Let H be a component of G. If H contains no cycle, then H is a tree. Since H is claw-free, H is a path. Now, assume that H contains an induced cycle C. Suppose $H \neq C$. Then there exists a vertex v outside C which is adjacent to some vertex u in C. Since neighborhoods of u in C are not adjacent and H is claw-free, one of them, say w, must be adjacent to v. Then $\{u, v, w\}$ forms a triangle in H, a contradiction. Hence H is a cycle. □

In the next theorem, we give the characterization of the clique-chromatic numbers of (claw, paw)-free graphs.

Theorem 3.15. *Let G be a (claw, paw)-free graph. Then*

$$\chi_c(G) = \begin{cases} 1, & \text{if } G \text{ is an edgeless graph} \\ 3, & \text{if } G \text{ has an odd hole component} \\ 2, & \text{otherwise.} \end{cases}$$

Proof. If G is an edgeless graph, then $\chi_c(G) = 1$. Assume that G has at least one edge. Without lost of generality, assume that G is connected. Since G is paw-free, by Theorem 3.12, G is either triangle-free or complete multipartite. If G is complete multipartite, then $\chi_c(G) = 2$ by Lemma 3.13. Now, assume that G is triangle-free. Then G is (claw, triangle)-free. By Lemma 3.14, G is a path or a cycle. If G is an odd cycle, then $\chi_c(G) = 3$. If G is not an odd cycle, then G is a path or an even cycle. Hence $\chi_c(G) = 2$. □

3.4.2 (Claw, diamond)-free Graphs

It is unknown whether the family of all (claw, diamond)-free graphs has a bounded clique-chromatic number. We introduce two subclasses of (claw, diamond)-free graphs having bounded clique-chromatic numbers, namely, (claw, diamond)-free graphs without even holes, and (claw, diamond)-free graphs without maximal cliques of size three.

Lemma 3.16. *Let x be a vertex in a diamond-free graph G. Then $N_G(x)$ is a disjoint union of cliques of G.*

Proof. Let H be a component of $G[N_G(x)]$. Suppose that $V(H)$ is not a clique of G. Then there are non-adjacent vertices a and b in H. Since H is connected, there is a path P between a and b. It follows that P contains an induced path P_3

of G. Then such induced path P_3 and the vertex x form an induced diamond of G, a contradiction. Hence $V(H)$ is clique of G. $\square$

Lemma 3.17. *Let G be a connected (claw, diamond, even hole)-free graph. If G has a vertex contained in only one maximal clique of G, then G is 2-clique-colorable.*

Proof. Let x be a vertex contained in only one maximal clique of G. Define $A_0 = \{x\}$, $A_1 = N_G(x)$, and $A_i = N_G(A_{i-1}) \smallsetminus (A_{i-1} \cup A_{i-2})$ for all $i \geq 2$. Then $V(G) = \bigcup_i A_i$. Note that A_1 is a clique of G. Define a coloring of G by labeling the vertices of A_i by color 1 if i is even, and by color 2 if i is odd.

Suppose that this coloring yields a monocolored maximal clique Q of size at least two. Then $Q \subseteq A_i$ for some $i \geq 2$. Let $u_i, v_i \in Q$. Then there is a vertex u_{i-1} in A_{i-1} which is adjacent to u_i. Suppose that u_{i-1} is adjacent to v_i. Since Q is a maximal clique of G, there is a vertex w in Q which is not adjacent to u_{i-1}. Then $\{u_{i-1}, u_i, v_i, w\}$ induces a diamond, a contradiction. So u_{i-1} is not adjacent to v_i. Similarly, there is a vertex v_{i-1} in A_{i-1} which is adjacent to v_i but not to u_i.

Since G is C_4-free, u_{i-1} cannot be adjacent to v_{i-1}. So $i \geq 3$. Let $u_{i-2}, v_{i-2} \in A_{i-2}$ such that u_{i-2} is adjacent to u_{i-1} and v_{i-2} is adjacent to v_{i-1}. If $u_{i-2} = v_{i-2}$, then there is a vertex u_{i-3} in A_{i-3} which is adjacent to u_{i-2}, and it follows that $\{u_{i-3}, u_{i-2}, u_{i-1}, v_{i-1}\}$ induces a claw, a contradiction. Thus $u_{i-2} \neq v_{i-2}$. Since G is claw-free, u_{i-2} is not adjacent to v_{i-1} and v_{i-2} is not adjacent to u_{i-1}. Since G is C_6-free, u_{i-2} is not adjacent to v_{i-2}. Continue this way until we have $u_1, v_1 \in A_1$. Since A_1 is a clique, we eventually have an even hole, a contradiction. Hence this coloring is a proper 2-clique-coloring of G. $\square$

The next theorem shows that the family of (claw, diamond, even hole)-free graphs has a bounded clique-chromatic number.

Theorem 3.18. *Every* (*claw, diamond, even hole*)*-free graph is* 3*-clique-colorable.*

Proof. Let G be a (claw, diamond, even hole)-free graph. Without lost of generality, assume that G is connected. Let $x \in V(G)$. By Lemma 3.16, $N_G(x)$ is a disjoint union of r cliques of G for some integer r. Since G is claw-free, $r \leq 2$. If $r = 1$, then the theorem is proved by Lemma 3.17. Now, let $N_G(x) = A_1 \cup B_1$ where A_1 and B_1 are cliques of G. Define $A_i = N_G(A_{i-1}) \smallsetminus (A_{i-1} \cup A_{i-2})$ and $B_i = N_G(B_{i-1}) \smallsetminus (B_{i-1} \cup B_{i-2})$ for all $i \geq 2$. Then $V(G) = \{x\} \cup (\bigcup_i A_i) \cup (\bigcup_j B_j)$.

Case 1. $(\bigcup_i A_i) \cap (\bigcup_j B_j) = \varnothing$. By Lemma 3.17, both of $G[(\bigcup_i A_i) \cup \{x\}]$ and $G[(\bigcup_j B_j) \cup \{x\}]$ have a proper 2-clique-coloring. Combining these two colorings by identifying the color of x yields a proper 2-clique-coloring of G, so G is 2-clique-colorable.

Case 2. $(\bigcup_i A_i) \cap (\bigcup_j B_j) \neq \varnothing$. Let G' be the subgraph of G obtained by deleting all vertices of B_1. Then G' is a connected (claw, diamond, even hole)-free graph with x satisfying the condition in Lemma 3.17. Thus G' has a proper 2-clique-coloring. We can extend this coloring to a proper 3-clique-coloring of G by labeling color 3 to all vertices of B_1, and hence G is 3-clique-colorable. □

Note that all odd cycles C_{2n+1} ($n \geq 2$) are (claw, diamond, even hole)-free and $\chi_c(C_{2n+1}) = 3$. Thus the upper bound in Theorem 3.18 is sharp.

Next, we focus on (claw, diamond)-free graphs without maximal cliques of size three. Let T be a triangle in a graph G. We say that T is *odd* if $|N_G(v) \cap V(T)|$ is odd for some $v \in V(G)$. In 1965, van Rooij and Wilf [18] gave a characterization of line graphs which is shown in the following theorem.

Theorem 3.19. [18] *A graph G is a line graph if and only if G is claw-free and no induced diamond of G has two odd triangles.*

The next corollary gives the characterization of the clique-chromatic numbers of (claw, diamond)-free graphs without maximal cliques of size three.

Corollary 3.20. *Let G be a (claw, diamond)-free graph. If G has no maximal clique of size three, then*

$$\chi_c(G) = \begin{cases} 1, & \textit{if } G \textit{ is an edgeless graph} \\ 3, & \textit{if } G \textit{ has an odd hole component} \\ 2, & \textit{otherwise.} \end{cases}$$

Proof. By Theorem 3.19, all (claw, diamond)-free graphs are line graphs, so G is a line graph. Then there is a simple graph H such that $G = L(H)$. If H has a triangle T, then T corresponds to a maximal clique of size three in $L(H) = G$, a contradiction. Thus H is triangle-free. Then the corollary follows directly from Theorem 2.14, and the fact that G has an odd hole component if and only if H has an odd hole component. $\square$

CHAPTER IV

CONCLUSION AND OPEN PROBLEMS

4.1 Conclusion

We have investigated values and bounds of the clique-chromatic numbers of graphs. Results are listed as follows:

Clique-colorings of line graphs:

The exact values of the clique-chromatic numbers of the line graphs of complete graphs are obtained in the following main theorem.

Theorem 2.8. *For $n \geq 3$, $\chi_c(L(K_n)) = m + 1$ where $R(m) \leq n < R(m + 1)$ for some positive integer m.*

Consequently, we also have these results:

1. Let $n \in \mathbb{N}$ where $R(m) \leq n < R(m + 1)$ for some positive integer m. Let $G = kK_{n-t} \vee pK_t$ where $k, p \in \mathbb{N}$ and $1 \leq t \leq n - 1$. Then $\chi_c(L(G)) = m + 1$. (Example 2.9)

2. Let $n \in \mathbb{N}$ where $R(m) \leq n < R(m + 1)$ for some positive integer m. Let G be a graph with $n + 1$ vertices and $E(G) = E(K_{n+1}) - E(K_{1,r})$ where $1 \leq r \leq n - 1$. Then $L(G)$ is $(m + 2)$-clique-colorable. In particular, if $r = 1$ or $n - m - 1 \leq r \leq n - 1$, then $\chi_c(L(G)) = m + 1$. (Theorem 2.10)

The characterization of the clique-chromatic numbers of the line graphs of triangle-free graphs is obtained in Theorem 2.14.

Theorem 2.14. *Let G be a triangle-free graph with at least one edge. Then*

$$\chi_c(L(G)) = \begin{cases} 1, & \textit{if all components of } G \textit{ are trivial} \\ 3, & \textit{if } G \textit{ has an odd hole component} \\ 2, & \textit{otherwise.} \end{cases}$$

Clique-colorings of F-free graphs:

The following is the list of bounds of the clique-chromatic number of the family of F-free graphs that are improved. (See Theorems 3.4, 3.5, 3.7, 3.8 and 3.11)

1. For $k \geq 3$, a $(P_2 + kP_1)$-free graph is $(k+1)$-clique-colorable.
2. For $k \geq 3$, a $(P_2 + kP_1$, diamond)-free graph is k-clique-colorable.
3. For $k \geq 3$, a $(P_3 + kP_1)$-free graph is $(k+2)$-clique-colorable.
4. For $k \geq 3$, a $(P_3 + kP_1$, diamond)-free graph is $(k+1)$-clique-colorable.
5. For $k, m \in \mathbb{N}$ where $max\{k, m\} \geq 4$, a $(P_k + P_m)$-free graph is $(k+m-2)$-clique-colorable.

Moreover, the clique-chromatic numbers of some subclasses of claw-free graphs are investigated. The results are as follows: (See Theorems 3.15, 3.18 and 3.20)

1. Let G be a (claw, paw)-free graph with at least one edge. Then

$$\chi_c(G) = \begin{cases} 1, & \text{if } G \text{ is an edgeless graph} \\ 3, & \text{if } G \text{ has an odd hole component} \\ 2, & \text{otherwise.} \end{cases}$$

2. Every (claw, diamond, even hole)-free graph is 3-clique-colorable.

3. Let G be a (claw, diamond)-free graph. If G has no maximal clique of size three, then

$$\chi_c(G) = \begin{cases} 1, & \text{if } G \text{ is an edgeless graph} \\ 3, & \text{if } G \text{ has an odd hole component} \\ 2, & \text{otherwise.} \end{cases}$$

4.2 Open Problems for Future Work

This dissertation brings some open problems for future work as follows:

1. Let $n \in \mathbb{N}$ where $R(m) \leq n < R(m+1)$ for some positive integer m. Let G be a graph with $n+1$ vertices and $E(G) = E(K_{n+1}) - E(K_{1,r})$ where $1 \leq r \leq n-1$. Theorem 2.10 shows that $L(G)$ is $(m+2)$-clique-colorable, and $\chi_c(L(G)) = m+1$ if $r = 1$ or $n-m-1 \leq r \leq n-1$. When $2 \leq r \leq n-m-2$, the problem is still unsolved.

2. Gravier et al. showed that all $(P_2 + 2P_1)$-free graphs are 3-clique-colorable, and this bound is sharp by the cycle C_5. Theorem 3.6 gives a subclass of (P_2+2P_1)-free graphs with clique-chromatic number two. An interesting unsolved problem is to find the characterization of the clique-chromatic numbers of $(P_2 + 2P_1)$-free graphs.

3. By Theorem 3.2, every $(P_r + kP_1)$-free graph is $(2k+r-2)$-clique-colorable where $r, k \in \mathbb{N}$. This dissertation improves the bound for $r = 2$ and 3. Improving this bound where $r \geq 4$ could be future work.

4. Theorem 3.18 shows that every (claw, diamond, even hole)-free graph is 3-clique-colorable. It is interesting to find the characterization of the clique-chromatic numbers of (claw, diamond, even hole)-free graphs.

5. In Section 3.4.2, we show that both of the family of (claw, diamond)-free graphs without even holes, and the family of (claw, diamond)-free graphs without maximal cliques of size three have a bounded clique-chromatic number. It is unknown whether the family of all (claw, diamond)-free graphs has a bounded clique-chromatic number. We conjecture that this family has a bounded clique-chromatic number.

REFERENCES

[1] Bacsó, G., Gravier, S., Gyárfás, A., Preissmann, M. and Sebő, A.: Coloring the maximal cliques of graphs, *SIAM Journal on Discrete Mathematics.* **17**, 361–376(2004).

[2] Campos, C. N., Dantasa, S. and de Mello, C. P.: Colouring clique-hypergraphs of circulant graphs, *Electronic Notes in Discrete Mathematics.* **30**, 189–194(2008).

[3] Cerioli, M. R. and Korenchendler, A. L.: Clique-coloring circular-arc graphs, *Electronic Notes in Discrete Mathematics.* **35**, 287–292(2009).

[4] Cerioli, M. R. and Petito, P.: Clique-coloring UE and UEH graphs, *Electronic Notes in Discrete Mathematics.* **30**, 201–206(2008).

[5] Chung, F. R. K.: On the Ramsey numbers $N(3, 3, ..., 3; 2)$, *Discrete Mathematics.* **5**, 317–321(1973).

[6] Defossez, D.: Clique-coloring some classes of odd-hole-free, *Journal of Graph Theory.* **53**, 233–249(2006).

[7] Duffus, D., Kierstead, H. A. and Trotter, W. T.: Fibres and ordered set coloring, *Journal of Combinatorial Theory Series A.* **58**, 158–164(1991).

[8] Duffus, D., Sands, B., Sauer, N. and Woodrow, R. E.: Two-coloring all two-element maximal antichains, *Journal of Combinatorial Theory Series A.* **57**, 109–116(1991).

[9] Gravier, S., Hoáng, C. T. and Maffray, F.: Coloring the hypergraph of maximal cliques of a graph with no long path, *Discrete Mathematics.* **272**, 285–290(2003).

[10] Gravier, S. and Škrekovski, R.: Coloring the clique hypergraph of graphs without forbidden structure, *Les cahiers du laboratoire Leibniz.* **83**(2003) (http://www-leibniz.imag.fr/LesCahiers).

[11] Greenwood, R. E. and Gleason A. M.: Combinatorial relations and chromatic graphs, *Canadian Journal of Mathematics.* **7**, 1–7(1955).

[12] Kramer R. L.: The classical Ramsey number $R(3, 3, 3, 3; 2)$ is no greater than 62, *Preprint*, Iowa State University, (1994).

[13] Kratochvíl, J. and Tuza, Zs.: On the complexity of bicoloring clique hypergraphs of graphs, *Journal of Algorithms.* **45**, 40–54(2002).

[14] Mohar, B. and Škrekovski, R.: The Grötzsch theorem for the hypergraph of maximal cliques, *Electronic Journal of Combinatorics.* **6**, #R26, 13pp. (1999).

[15] Mycielski, J.: Sur le coloriage des graphes, *Colloquium Mathematicum.* **3**, 161–162(1955).

[16] Olariu, S.: Paw-free graphs, *Information Processing Letters.* **28**, 53–54 (1988).

[17] Ramsey, F. P.: On a problem of formal logic, *Proceedings of the London Mathematical Society.* **30**, 264–286(1930).

[18] van Rooij, A. and Wilf, H. S.: The interchange graphs of a finte graph, *Acta Mathematica Academiae Scientiarum Hungaricae.* **16**, 263–269(1965).

[19] West, D. B.: *Introduction to Graph Theory*, Prentice Hall, New Jersey, (2001).

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