การขยายวิถีการลู่เข้าสำหรับโคปูล่าการขึ้นต่อกันบริเวณหางสุดขีด

นายขจรศักดิ์ บำรุงสินมั่น

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์ ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์
คณะวิทยาศาสตร์ จุฬาลงกรณ์มหาวิทยาลัย
ปีการศึกษา 2556

GENERALIZING CONVERGENCE PATH FOR EXTREME TAIL DEPENDENCE COPULAS

Mr. Khajohnsak Bamrungsinman

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics
Department of Mathematics and Computer Science
Faculty of Science
Chulalongkorn University
Academic Year 2013

Thesis Title	GENERALIZING CONVERGENCE PATH FOR EXTREME TAIL DEPENDENCE COPULAS
By	Mr. Khajohnsak Bamrungsinman
Field of Study	Mathematics
Thesis Advisor	Assistant Professor Songkiat Sumetkijakan, Ph.D.
Thesis Co-advisor	Assistant Professor Nattakarn Chaidee, Ph.D.

Accepted by the Faculty of Science, Chulalongkorn University in Partial Fulfillment of the Requirements for the Master's Degree

..............................Dean of the Faculty of Science
(Professor Supot Hannongbua, Dr.rer.nat.)

THESIS COMMITTEE

..............................Chairman
(Professor Kritsana Neammanee, Ph.D.)

..............................Thesis Advisor
(Assistant Professor Songkiat Sumetkijakan, Ph.D.)

..............................Thesis Co-adivisor
(Assistant Professor Nattakarn Chaidee, Ph.D.)

..............................Examiner
(Kittipat Wong, Ph.D.)

..............................External Examiner
(Poomjai Nacaskul, Ph.D.)

ขจรศักดิ์ บำรุงสินมั่น : การขยายวิถีการลู่เข้าสำหรับโคปูล่าการขึ้นต่อกันบริเวณหางสุดขีด (GENERALIZING CONVERGENCE PATH FOR EXTREME TAIL DEPENDENCE COPULAS) อ.ที่ปรึกษาวิทยานิพนธ์หลัก: ผศ.ดร.ทรงเกียรติ สุเมธกิจการ, อ.ที่ปรึกษาวิทยานิพนธ์ร่วม : ผศ.ดร.ณัฐกาญจน์ ใจดี, 51 หน้า.

เราศึกษาการลู่เข้าของโคปูล่าการขึ้นต่อกันบริเวณหางสุดขีดที่เกี่ยวเนื่องกับโคปูล่าแบบอาร์คิมีดีสแท้ซึ่งหางเป็นอาณาบริเวณสี่เหลี่ยมผืนผ้าใด ๆ ด้วยมุมที่ลู่เข้าสู่จุดกำเนิด นอกจากนั้นเรายังสร้างโคปูล่าการขึ้นต่อกันบริเวณหางสุดขีดที่เกี่ยวเนื่องกับโคปูล่าแบบอาร์คิมีดีสไม่แท้และศึกษาตัวอย่างเพื่อสังเกตพฤติกรรมการลู่เข้า

ภาควิชาคณิตศาสตร์และวิทยาการคอมพิวเตอร์ ลายมือชื่อนิสิต..
สาขาวิชา คณิตศาสตร์ ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์หลัก.........................
ปีการศึกษา 2556 ลายมือชื่อ อ.ที่ปรึกษาวิทยานิพนธ์ร่วม ....

5373898023 : MAJOR MATHEMATICS
KEY WORDS : COPULA / STRICT ARCHIMEDEAN COPULA / NON-STRICT ARCHIMEDEAN COPULA / GENERATOR / REGULAR VARIATION

KHAJOHNSAK BAMRUNGSINMAN : GENERALIZING CONVERGENCE PATH FOR EXTREME TAIL DEPENDENCE COPULAS
ADVISOR : ASSIST. PROF. SONGKIAT SUMETKIJAKAN, Ph.D.,
CO-ADVISOR : ASSIST. PROF. NATTAKARN CHAIDEE, Ph.D.,
51 pp.

We study the convergence of extreme tail dependence copulas relative to a strict Archimedean copula where tails are arbitrary rectangles with corners converging to $(0, 0)$. Moreover, we also construct extreme tail dependence copulas relative to a non-strict Archimedean copula and study an example to observe its convergence behavior.

Department	: Mathematics and Computer Science	Student's Signature : ....
Field of Study	: ....Mathematics..........	Advisor's Signature : ....
Academic Year	: ....2013.................	Co-advisor's Signature : ....

ACKNOWLEDGEMENTS

First I would like to show my deep gratitude to Assistant Professor Dr. Songkiat Sumetkijakan, my thesis advisor, for many good advices. I would like to thank Assistant Professor Dr. Nattakarn Chaidee, my thesis co-advisor, for insightful suggestions on my work. I also would like to give thank to my thesis committees, Professor Dr. Kritsana Neammanee, Dr. Kittipat Wong, and Dr. Poomjai Nacaskul.

In particular, I feel thankful to my friends for my good times at Chulalongkorn University. Moreover, I am indebted to all of my teachers who have taught me for my knowledge and skills.

Special appreciation is also given to Rajamangala University of Technology Phra Nakhon scholarship for long term financial support and for granting me great opportunity to study Mathematics.

And last but not least, I would like to give thanks to my family, for their love and support in all my lifetime.

CONTENTS

LIST OF FIGURES

page

CHAPTER I
INTRODUCTION

Natural phenomena, such as tsunamis, cyclone storms, tornado storms, etc., bring damages to the assets and loss of lives. Moreover, extreme phenomenon with high casualties could cause a national disaser.

Insurance companies that do not have systematic plans in place especially on maximum payouts to their clients might face serious problems from paying larger amount of money than they could. If the problem persists, it could lead to bankruptcy.

One example of how an insurance company might avoid such a problem is to set a maximum threshold, say 50,000 baht, for which the company is directly responsible. Responsibility of all claims over the threshold, e.g. 150,000 baht, shall be splitted into two parts. The first 50,000 is covered by the company. However, the excess of 100,000 baht is covered by a larger insurance company from which the first company has bought a reinsurance. This way, the first insurance company can protect itself from unexpectedly large claims.

From the above example, we let X denote the amount of each claim and u denote the maximum threshold. The larger insurance company is surely interested in the distribution of the conditional excess over the threshold. As the threshold u increases to infinity, the distribution of the conditional excess can be approximated by the generalized Pareto distribution ($G_{\xi,\beta}$) for some $\beta > 0$ and $\xi \in \mathbb{R}$. For $\xi > 0$, $G_{\xi,\beta}$ is defined by

$$G_{\xi,\beta}(x) = 1 - \left(1 + \frac{\xi x}{\beta}\right)^{-\frac{1}{\xi}} \text{for } x \in [0, \infty). \tag{1.1}$$

For $\xi < 0$, $G_{\xi,\beta}$ is defined on $\left[0, -\frac{\beta}{\xi}\right]$ by the same formula. And $G_{0,\beta}$ is defined as the limit of (1.1) as $\xi \to 0^+$, i.e., $G_{0,\beta}(x) = 1 - \exp(-\frac{x}{\beta})$.

In the case of two separate claims, of which the amounts are denoted by X and Y, the study of the joint distribution of conditional excesses over a common threshold leads to simpler limit theorems for extreme tail dependence copulas ([6], 2002). The copula of two continuous random variables is the marginal-free joint distribution function. It captures dependence structure and discards the marginal distributions. Juri and Wüthrich proved that the extreme tail dependence copulas relative to an Archimedean copula C satisfying certain conditions always converge to a Clayton copula. Note that the extreme tail dependence copulas in [6] are defined by conditioning that both random variables are less than the same threshold u.

In this thesis, we study the convergence of extreme tail dependence copulas relative to a strict Archimedean copula where tails need not be squares, i.e., different thresholds may be used for two random variables. We show that these extreme tail dependence copulas still converge to a Clayton copula as tails converge to $(0, c)$ along any path for some $c \in [0, 1]$. We also define extreme tail dependence copulas relative to a non-strict Archimedean copula but are not successful in proving a convergence theorem in this case.

CHAPTER II
PRELIMINARIES

In this chapter, we will give basic concepts of copulas and Archimedean copulas. Moreover, we will give concepts of regular variations used in our thesis. An introduction on copulas can be found in [8].

Definition 2.1. [8, p.10] A **copula** is a function C from $[0,1]^2$ to $[0,1]$ with the following properties.

(1) For every $u \in [0,1]$, $C(u,0) = 0 = C(0,u)$ and $C(u,1) = u = C(1,u)$.

(2) C is 2**-increasing**, i.e., for every $u_1, u_2, v_1, v_2 \in [0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$V_C([u_1,u_2] \times [v_1,v_2]) = C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0.$$

Example. [8, p.11] *Some examples of copulas are*

(1) $M(u,v) = \min(u,v)$ *;*

(2) $W(u,v) = \max(u+v-1,0)$ *; and*

(3) $\prod(u,v) = uv$.

Remark. [8, p.11] *Let C be a copula. Then for every $(u,v) \in [0,1]^2$,*

$$W(u,v) \leq C(u,v) \leq M(u,v).$$

Theorem 2.2. [8, p.11] *Let C be a copula. Then for every $u_1, u_2, v_1, v_2 \in [0,1]$,*

$$|C(u_2,v_2) - C(u_1,v_1)| \leq |u_2 - u_1| + |v_2 - v_1|.$$

Hence C is uniformly continuous on $[0,1]^2$.

Definition 2.3. [8, p.21] Let F be a distribution function. Then a **quasi-inverse** of F is any function $F^{[-1]}$ with domain $[0,1]$ such that

(1) if $t \in F([0,1])$, then $F^{[-1]}(t)$ is any number $x \in \bar{\mathbb{R}}$ such that $F(x) = t$, i.e., for all $t \in F([0,1])$, $F(F^{[-1]}(t)) = t$;

(2) if $t \notin F([0,1])$, then

$$F^{[-1]}(t) = \inf\{x|F(x) \geq t\} = \sup\{x|F(x) \leq t\}.$$

If F is strictly increasing, then it has but a single quasi-inverse, which is of course the **usual inverse**, for which we use the customary notation F^{-1}.

Definition 2.4. [8, p.63] Let $\{J_i\}_{i\in\Lambda}$ denote a partition of $[0,1]$, that is, a (possibly infinite) collection of closed, non-overlapping (except at common endpoints) nondegenerate intervals $J_i = [a_i, b_i]$ whose union is $[0,1]$. Let $\{C_i\}$ be a collection of copulas with the same indexing as $\{J_i\}_{i\in\Lambda}$. Then the **ordinal sum** of $\{C_i\}$ with respect to $\{J_i\}_{i\in\Lambda}$ is the copula C given by

$$C(u,v) = \begin{cases} a_i + (b_i - a_i)C_i(\frac{u-a_i}{b_i-a_i}, \frac{v-a_i}{b_i-a_i}) & \text{if } (u,v) \in J_i^2; \\ M(u,v), & \text{otherwise.} \end{cases}$$

Definition 2.5. [8, p.110] Let $\psi : [0,1] \to [0,\infty]$ be continuous, strictly decreasing such that $\psi(1) = 0$. The **pseudo-inverse** $\psi^{[-1]} : [0,\infty] \to [0,1]$ of ψ is defined by

$$\psi^{[-1]}(s) = \begin{cases} \psi^{-1}(s) & \text{if } 0 \leq s \leq \psi(0); \\ 0 & \text{if } \psi(0) \leq s \leq \infty. \end{cases}$$

Lemma 2.6. [8, p.110] *Let $\psi : [0,1] \to [0,\infty]$ be continuous, strictly decreasing such that $\psi(1) = 0$, and let $\psi^{[-1]}$ be the pseudo-inverse of ψ defined as in Definition 2.5. Define $C_\psi : [0,1]^2 \to [0,1]$ given as*

$$C_\psi(x,y) = \psi^{[-1]}(\psi(x) + \psi(y)) \tag{2.1}$$

for $x, y \in [0,1]$. Then C_ψ satisfies the conditions as in Definition 2.1(1) for a copula.

Lemma 2.7. [8, p.111] *Let ψ, $\psi^{[-1]}$ and C_ψ satisfy the hypotheses of Lemma 2.6. Then C_ψ is 2-increasing if and only if for all $y \in [0,1]$, if $0 \leq u_1 \leq u_2 \leq 1$, then $C_\psi(u_2, v) - C_\psi(u_1, v) \leq u_2 - u_1$.*

Theorem 2.8. [8, p.111] *Let $\psi : [0,1] \to [0,\infty]$ be continuous, strictly decreasing such that $\psi(1) = 0$, and let $\psi^{[-1]}$ be the pseudo-inverse of ψ defined as in Definition 2.5. Then $C_\psi : [0,1]^2 \to [0,1]$ given by (2.1) is a copula if and only if ψ is convex.*

Definition 2.9. [8, p.112] A copula C_ψ as (2.1) is called an **Archimedean copula**. The function ψ is called a **generator** of the copula. If $\psi(0) = \infty$, then ψ and C_ψ are called **strict** and $\psi^{[-1]}$ is the **usual inverse** of ψ.

Example. [6, p.408] *The **Clayton copula** with parameter $\alpha > 0$ is the Archimedean copula generated by $\psi(t) = t^{-\alpha} - 1$ for $t \in [0,1]$ and has the corresponding form*

$$C^\alpha(x, y) = (x^{-\alpha} + y^{-\alpha} - 1)^{-1/\alpha}$$

for $x, y \in [0,1]$. Then the Clayton copula is strict.

Theorem 2.10. [8, p.113] *Let C be an associative copula, i.e., $C(C(u,v), w) = C(u, C(v,w))$ for all $u, v, w \in [0,1]$, such that $C(t,t) < t$ for all $t \in (0,1)$. Then C is Archimedean.*

Theorem 2.11. [8, p.130] *Let C be an Archimedean copula with generator ψ. Then, for almost all $u, v \in [0,1]$,*

$$\psi'(u)\frac{\partial C(u,v)}{\partial v} = \psi'(v)\frac{\partial C(u,v)}{\partial u}.$$

Theorem 2.12. [8, p.139] *Let $\{C_\theta | \theta > 0\}$ be a family of Archimedean copulas with differentiable generators ψ_θ. Then $C = \lim_{\theta \downarrow 0} C_\theta$ is an Archimedean copula if and only if there exists a function ψ such that, for all $s, t \in (0,1)$,*

$$\lim_{\theta \downarrow 0} \frac{\psi_\theta(s)}{\psi_\theta'(t)} = \frac{\psi(s)}{\psi'(t)}.$$

Theorem 2.13. [8, p.140] *Let* $\{C_\theta | \theta > 0\}$ *be a family of Archimedean copulas with differentiable generators* ψ_θ*. Then* $\lim_{\theta \downarrow 0} C_\theta(u,v) = M(u,v)$ *if and only if*

$$\lim_{\theta \downarrow 0} \frac{\psi_\theta(t)}{\psi'_\theta(t)} = 0$$

for $t \in (0,1)$*.*

Definition 2.14. [1, p.18,83] A measurable function $f : (0,\infty) \to (0,\infty)$ is called **regularly varying** at 0 with index $\rho \in \mathbb{R}$ if, for any $x > 0$,

$$\lim_{u \downarrow 0} \frac{f(ux)}{f(u)} = x^\rho.$$

In the special case where $\rho = 0$, the function f is also called **slowly varying** at 0. If f is such that

$$\lim_{u \downarrow 0} \frac{f(ux)}{f(u)} = \begin{cases} \infty & \text{if } x < 1; \\ 1 & \text{if } x = 1; \\ 0 & \text{if } x > 1, \end{cases}$$

then f is said to be **rapidly varying** at 0 with index $-\infty$. Similarly, if

$$\lim_{u \downarrow 0} \frac{f(ux)}{f(u)} = \begin{cases} 0 & \text{if } x < 1; \\ 1 & \text{if } x = 1; \\ \infty & \text{if } x > 1, \end{cases}$$

then f is said to be **rapidly varying** at 0 with index $+\infty$.

Remark. [1, p.18][6, p.408] *The set of functions which are regularly (rapidly) varying at* 0 *with index* $\rho \in [-\infty, +\infty]$ *is denoted by* $\Re_\rho$*. In particular, for* $f \in \Re_\rho$*, we have*

$$\lim_{u \downarrow 0} \frac{f(ux)}{f(uy)} = \left(\frac{x}{y}\right)^\rho$$

for all $x, y > 0$*, where, for* $\rho = \pm\infty$*, the quotient* $(\frac{x}{y})^\rho$ *has to be interpreted as the limit of* $(\frac{x}{y})^\rho$ *for* $\rho \to \pm\infty$*.*

Theorem 2.15. [1, p.17,18] *Let* $f : (0,\infty) \to (0,\infty)$. *If* $\lim_{t\downarrow 0} \frac{f(xt)}{f(t)} = g(x)$ *for all* x *in a set of positive measure, then*

1) $\lim_{t\downarrow 0} \frac{f(xt)}{f(t)} = g(x)$ *for all* $x > 0$,

2) there exists a real number α *with* $g(x) \equiv x^{-\alpha}$ *for all* $x > 0$,

3) $f(t) = t^{-\alpha}\ell(t)$ *with* ℓ *slowly varying at* 0.

Theorem 2.16. [1, p.39] *Let* U *be given on* $(0, X]$, $X \in \mathbb{R}^+$, *by*

$$U(x) = \int_0^x u(y)dy$$

for some $u \in L^1[0, X]$. *If* $\lim_{x\downarrow 0} \frac{U(x)}{cx^\rho \ell(x)} = 1$, *where* $c \in \mathbb{R}$, $\rho \geq 0$, $\lim_{x\downarrow 0} \frac{\ell(\lambda x)}{\ell(x)} = 1$ *for all* $\lambda > 0$, *and if* u *is monotone in some right neighbourhood of* 0, *then*

$$\lim_{x\downarrow 0} \frac{u(x)}{c\rho x^{\rho-1}\ell(x)} = 1.$$

Definition 2.17. [6, p.410] For a copula C and $u \in (0, 1)$ such that $C(u, u) > 0$, let $F_u : [0, 1] \to [0, 1]$ given as

$$F_u(t) := \frac{C(t \wedge u, u)}{C(u, u)}$$

for $t \in [0, 1]$. Moreover, define $C_u : [0, 1]^2 \to [0, 1]$ is given as

$$C_u(x, y) := \frac{C(F_u^{-1}(x), F_u^{-1}(y))}{C(u, u)}$$

for $x, y \in [0, 1]$ when $F_u^{-1}(x) = \inf\{t \in [0, 1] | F_u(t) \geq x\} = \sup\{t \in [0, 1] | F_u(t) \leq x\}$. The notation C_u is called the **extreme tail dependence copula relative to** C **at the level** u.

Theorem 2.18. *For* $u \in (0, 1)$, *if* F_u *is continuous, then* $F_u(F_u^{-1}(x)) = x$ *for* $x \in [0, 1]$.

Proof Let $u \in (0, 1)$ and $x \in [0, 1]$. Assume that F_u is continuous. We will show that $F_u(F_u^{-1}(x)) = x$. Set $x_0 = F_u^{-1}(x) = \inf\{t | F_u(t) \geq x\} = \sup\{t | F_u(t) \leq x\}$. For all $n \in \mathbb{N}$, there exists $b_n \in \{t | F_u(t) \geq x\}$ such that $x_0 \leq b_n < x_0 + \frac{1}{n}$, so we have $F_u(b_n) \geq x$. By Squeeze theorem, we have $\lim_{n\to\infty} b_n = x_0$ and so by the continuity of F_u,

$$F_u(x_0) = F_u(\lim_{n\to\infty} b_n) = \lim_{n\to\infty} F_u(b_n) \geq \lim_{n\to\infty} x = x.$$

Similarly, for all $n \in \mathbb{N}$, there exists $c_n \in \{t | F_u(t) \leq x\}$ such that $x_0 - \frac{1}{n} < c_n \leq x_0$, so we have $F_u(c_n) \leq x$. By Squeeze theorem, we have $\lim_{n\to\infty} c_n = x_0$ and so by the continuity of F_u,

$$F_u(x_0) = F_u(\lim_{n\to\infty} c_n) = \lim_{n\to\infty} F_u(c_n) \leq \lim_{n\to\infty} x = x.$$

Then $F_u(F_u^{-1}(x)) = F_u(x_0) = x$. □

Theorem 2.19. *Let ψ be defined as in Definition 2.5. If ψ is strict, then ψ^{-1} is strictly decreasing on $[0, \infty]$.*

Proof Assume that ψ is strict. Let $x, y \in [0, \infty]$. Assume that $x < y$. We will show that $\psi^{-1}(x) > \psi^{-1}(y)$. Suppose that $\psi^{-1}(x) \leq \psi^{-1}(y)$. If $\psi^{-1}(x) = \psi^{-1}(y)$, then $x = \psi(\psi^{-1}(x)) = \psi(\psi^{-1}(y)) = y$, a contradiction. If $\psi^{-1}(x) < \psi^{-1}(y)$, then we set $s = \psi^{-1}(x)$ and $t = \psi^{-1}(y)$. Then $s, t \in [0, 1]$ and $s < t$. Since ψ is strictly decreasing, we have $\psi(s) > \psi(t)$, i.e., $\psi(\psi^{-1}(x)) > \psi(\psi^{-1}(y))$. Since ψ is strict, we have $x > y$, a contradiction. Then $\psi^{-1}(x) > \psi^{-1}(y)$. Hence ψ^{-1} is strictly decreasing on $[0, \infty]$. □

Theorem 2.20. [6, p.410] *Let C be a strict Archimedean copula. Then C_u is also a strict Archimedean copula and its generator ψ_u is given by*

$$\psi_u(t) = \psi(F_u^{-1}(t)) - \psi(u) = \psi(t\psi^{-1}(2\psi(u))) - 2\psi(u),$$

for $t \in [0, 1]$, where ψ is the generator of C.

Theorem 2.21. [2] *Let C be a strict Archimedean copula with generator ψ, whose the derivative is denoted by ψ'. Let $0 \leq \alpha \leq \infty$. Then $\psi \in \Re_{-\alpha}$ if and only if* $\lim_{u\downarrow 0} \dfrac{u\psi'(u)}{\psi(u)} = -\alpha$.

Theorem 2.22. [6, p.413] *Let C be an Archimedean copula having a differentiable generator $\psi \in \Re_{-\alpha}$ with $0 < \alpha < \infty$. Then, for all $x, y \in [0,1]$,*

$$\lim_{u \downarrow 0} C_u(x,y) = C^{\alpha}(x,y).$$

Theorem 2.23. [6, p.415] *Let C be an Archimedean copula with a differentiable generator $\psi \in \Re_{-\infty}$. Then, for all $x, y \in [0,1]$,*

$$\lim_{u \downarrow 0} C_u(x,y) = M(x,y).$$

CHAPTER III

EXTREME TAIL DEPENDENCE COPULAS OF ORDINAL SUMS

In this chapter, we study an ordinal sum of a strict Archimedean copula, and any copula. We will prove that the extreme tail dependence copula of the above ordinal sum at the level u is strict Archimedean when restricted to an area of the strict Archimedean copula and converges to a Clayton copula.

Let C be a strict Archimedean copula with a generator ψ, D any copula, $a \in (0, 1)$, and E an ordinal sum of C and D as given by

$$E(x,y) = \begin{cases} aC(\frac{x}{a}, \frac{y}{a}) & \text{if } (x,y) \in [0,a)^2; \\ (1-a)D(\frac{x-a}{1-a}, \frac{y-a}{1-a}) & \text{if } (x,y) \in [a,1]^2; \\ M(x,y) & \text{otherwise.} \end{cases} \tag{3.1}$$

It is easy to check that E is a copula.

Definition 3.1. For $u \in (0, 1)$ such that $E(u, u) > 0$, let $\tilde{F}_u : [0, 1] \to [0, 1]$ be given as

$$\tilde{F}_u(t) = \frac{E(t \wedge u, u)}{E(u, u)}$$

for $t \in [0, 1]$. Moreover, define $E_u : [0, 1]^2 \to [0, 1]$ given as

$$E_u(x,y) = \frac{E(\tilde{F}_u^{-1}(x), \tilde{F}_u^{-1}(y))}{E(u,u)}$$

for $x, y \in [0, 1]$.

Before we will prove that E_u is a strict Archimedean copula, we know that $\tilde{F}_u^{-1}(1) \leq u$ because $\tilde{F}_u(u) = 1$. Now, we will show that $\tilde{F}_u$ is equal to $\tilde{F}_{u_0}$ when $u_0 = \tilde{F}_u^{-1}(1)$.

Lemma 3.2. *Let $u \in (0,a)$. Then $\tilde{F}_u \equiv \tilde{F}_{u_0}$ when $u_0 = \tilde{F}_u^{-1}(1)$.*

Proof Observe that

$$\tilde{F}_u(x) = \frac{E(x \wedge u, u)}{E(u,u)} = \begin{cases} 1 & \text{if } x > u; \\ \frac{E(x,u)}{E(u,u)} & \text{if } x \leq u. \end{cases}$$

Let $x \in [0,1]$. If $x > u$, then $\tilde{F}_u(x) = 1 = \tilde{F}_{u_0}(x)$ because $u \geq u_0$. If $x \leq u$, then $\tilde{F}_u(x) = \dfrac{E(x,u)}{E(u,u)}$ and $\tilde{F}_{u_0}(x) = \dfrac{E(x,u_0)}{E(u_0,u_0)}$. It suffices to show that $E(u,u) = E(u_0,u_0)$ and $E(x,u) = E(x,u_0)$.

To show that $E(u,u) = E(u_0,u_0)$, we will show that $\tilde{F}_u(u_0) = 1$.

Clearly, $\tilde{F}_u(u_0) \leq 1$. Since $u_0 = \tilde{F}_u^{-1}(1) = \inf\{t | \tilde{F}_u(t) \geq 1\}$, for each $n \in \mathbb{N}$, there exists $a_n \in \{t | \tilde{F}_u(t) \geq 1\}$ such that $u_0 \leq a_n < u_0 + \frac{1}{n}$. Then $\tilde{F}_u(a_n) \geq 1$. By Squeeze theorem, we have $\lim\limits_{n\to\infty} a_n = u_0$ and so by the continuity of $\tilde{F}_u$,

$$\tilde{F}_u(u_0) = \tilde{F}_u(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} \tilde{F}_u(a_n) \geq \lim_{n\to\infty} 1 = 1.$$

Then $\tilde{F}_u(u_0) = 1$. Thus $\dfrac{E(u_0,u)}{E(u,u)} = \tilde{F}_u(u_0) = 1 = \tilde{F}_u(u) = \dfrac{E(u,u)}{E(u,u)}$. Since C is strict Archimedean, so C is symmetric. Then, for $u_0 \leq u < a$, we have

$$E(u,u) = E(u_0,u) = aC\Big(\frac{u_0}{a}, \frac{u}{a}\Big) = aC\Big(\frac{u}{a}, \frac{u_0}{a}\Big) = E(u,u_0).$$

Then

$$\begin{aligned} V_E([u_0,u]^2) &= E(u,u) - E(u,u_0) - E(u_0,u) + E(u_0,u_0) \\ &= E(u,u_0) - E(u,u_0) - E(u,u_0) + E(u_0,u_0) \\ &= E(u_0,u_0) - E(u,u_0). \end{aligned}$$

Since $V_E([u_0,u]^2) \geq 0$, we have $E(u_0,u_0) \geq E(u,u_0)$. Since E is increasing, we have $E(u_0, u_0) \le E(u, u_0)$. Then $E(u_0, u_0) = E(u, u_0) = E(u, u)$. Thus

$$\begin{aligned} V_E([0,u]^2 \backslash [0,u_0]^2) &= V_E([0,u]^2) - V_E([0,u_0]^2) \\ &= E(u,u) - E(u_0,u_0) \\ &= 0. \end{aligned}$$

We know that $[0, x] \times [0, u] \backslash [0, x] \times [0, u_0] \subseteq [0, u]^2 \backslash [0, u_0]^2$.

Then $0 = V_E([0, x] \times [0, u] \backslash [0, x] \times [0, u_0]) = E(x, u) - E(x, u_0)$. That is, $E(x, u) = E(x, u_0)$. Thus

$$\tilde{F}_u(x) = \frac{E(x,u)}{E(u,u)} = \frac{E(x,u_0)}{E(u_0,u_0)} = \tilde{F}_{u_0}(x).$$

Hence, $\tilde{F}_u \equiv \tilde{F}_{u_0}$. □

Lemma 3.3. *Let C be a strict Archimedean copula with a generator ψ, and $\tilde{F}_u$ defined as in Definition 3.1 for each $u \in (0, 1)$. Then, for $a \in (0, 1)$ and $u < a$, $\tilde{F}_u$ is strictly increasing on $[0, u]$.*

Proof Let $u, a \in (0, 1)$ and $u < a$. Recall that, for $t \in [0, u]$,

$$\tilde{F}_u(t) = \frac{E(t,u)}{E(u,u)} = \frac{C(\frac{t}{a}, \frac{u}{a})}{C(\frac{u}{a}, \frac{u}{a})}.$$

Since C is a strict Archimedean copula with a generator ψ, we have

$$\tilde{F}_u(t) = \frac{\psi^{-1}(\psi(\frac{t}{a}) + \psi(\frac{u}{a}))}{\psi^{-1}(2\psi(\frac{u}{a}))}.$$

We will show that $\tilde{F}_u$ is strictly increasing on $[0, u]$, i.e., for $x, y \in [0, u]$, if $x < y$, then $\tilde{F}_u(x) < \tilde{F}_u(y)$.

Let $x, y \in [0, u]$. Assume that $x < y$. Since ψ is strict, by Theorem 2.19, ψ^{-1} is strictly decreasing on $[0, \infty]$. Then

$$\begin{aligned}
\frac{x}{a} &< \frac{y}{a} \\
\psi\Big(\frac{x}{a}\Big) &> \psi\Big(\frac{y}{a}\Big) \\
\psi\Big(\frac{x}{a}\Big) + \psi\Big(\frac{u}{a}\Big) &> \psi\Big(\frac{y}{a}\Big) + \psi\Big(\frac{u}{a}\Big) \\
\psi^{-1}\bigg(\psi\Big(\frac{x}{a}\Big) + \psi\Big(\frac{u}{a}\Big)\bigg) &< \psi^{-1}\bigg(\psi\Big(\frac{y}{a}\Big) + \psi\Big(\frac{u}{a}\Big)\bigg) \\
\frac{\psi^{-1}\Big(\psi(\frac{x}{a}) + \psi(\frac{u}{a})\Big)}{\psi^{-1}\Big(2\psi(\frac{u}{a})\Big)} &< \frac{\psi^{-1}\Big(\psi(\frac{y}{a}) + \psi(\frac{u}{a})\Big)}{\psi^{-1}\Big(2\psi(\frac{u}{a})\Big)} \\
\tilde{F}_u(x) &< \tilde{F}_u(y).
\end{aligned}$$

Hence $\tilde{F}_u$ is strictly increasing on $[0, u]$. □

Next, we will prove that E_u is a strict Archimedean copula.

Theorem 3.4. *Let C be a strict Archimedean copula with a generator ψ and $u, a \in (0, 1)$, and D any copula. Let E, $\tilde{F}_u$ and E_u be defined as in equation (3.1) and Definition 3.1, respectively. Assume that $u < a$. Then E_u is a strict Archimedean copula with a generator*

$$\psi_{E_u}(x) = \psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right) - \psi\Big(\frac{u}{a}\Big)$$

for $x \in [0, 1]$.

Proof First, we will show that E_u is a copula.

Note that, for $x \in [0, 1]$, $\tilde{F}_u^{-1}(x) \in [0, u] \subseteq [0, a)$ and $\tilde{F}_u^{-1}(0) = 0$. Then

$$E_u(x, 0) = \frac{E(\tilde{F}_u^{-1}(x), \tilde{F}_u^{-1}(0))}{E(u, u)} = \frac{C(\frac{\tilde{F}_u^{-1}(x)}{a}, \frac{\tilde{F}_u^{-1}(0)}{a})}{C(\frac{u}{a}, \frac{u}{a})} = 0.$$

Similarly, $E_u(0, x) = 0$.

By Lemma 3.2, without loss of generality, we assume that $\tilde{F}_u^{-1}(1) = u$. Because $\tilde{F}_u(u) = 1 \geq x$, we have $\tilde{F}_u^{-1}(x) \leq u$. Because $\tilde{F}_u$ is continuous, by

Theorem 2.18, we have

$$E_u(x,1) = \frac{E(\tilde{F}_u^{-1}(x), \tilde{F}_u^{-1}(1))}{E(u,u)} = \frac{E(\tilde{F}_u^{-1}(x), u)}{E(u,u)} = \tilde{F}_u(\tilde{F}_u^{-1}(x)) = x.$$

Similarly, $E_u(1,x) = x$.

Assume that $x_1, x_2, y_1, y_2 \in [0,1]$, $x_1 < x_2$ and $y_1 < y_2$. As $u < a$ and $\tilde{F}_u^{-1}$ is increasing, we have $0 \leq \tilde{F}_u^{-1}(x_1) \leq \tilde{F}_u^{-1}(x_2) \leq u < a < 1$ and $0 \leq \tilde{F}_u^{-1}(y_1) \leq \tilde{F}_u^{-1}(y_2) \leq u < a < 1$. Since C is a copula, we have

$$\begin{aligned}
V_{E_u}([x_1,x_2]\times[y_1,y_2]) &= E_u(x_2,y_2) - E_u(x_2,y_1) - E_u(x_1,y_2) + E_u(x_1,y_1) \\
&= \frac{E(\tilde{F}_u^{-1}(x_2), \tilde{F}_u^{-1}(y_2))}{E(u,u)} - \frac{E(\tilde{F}_u^{-1}(x_2), \tilde{F}_u^{-1}(y_1))}{E(u,u)} \\
&\quad - \frac{E(\tilde{F}_u^{-1}(x_1), \tilde{F}_u^{-1}(y_2))}{E(u,u)} + \frac{E(\tilde{F}_u^{-1}(x_1), \tilde{F}_u^{-1}(y_1))}{E(u,u)} \\
&= \frac{C\big(\frac{\tilde{F}_u^{-1}(x_2)}{a}, \frac{\tilde{F}_u^{-1}(y_2)}{a}\big)}{C(\frac{u}{a}, \frac{u}{a})} - \frac{C\big(\frac{\tilde{F}_u^{-1}(x_2)}{a}, \frac{\tilde{F}_u^{-1}(y_1)}{a}\big)}{C(\frac{u}{a}, \frac{u}{a})} \\
&\quad - \frac{C\big(\frac{\tilde{F}_u^{-1}(x_1)}{a}, \frac{\tilde{F}_u^{-1}(y_2)}{a}\big)}{C(\frac{u}{a}, \frac{u}{a})} + \frac{C\big(\frac{\tilde{F}_u^{-1}(x_1)}{a}, \frac{\tilde{F}_u^{-1}(y_1)}{a}\big)}{C(\frac{u}{a}, \frac{u}{a})} \\
&\geq 0.
\end{aligned}$$

Then E_u is a copula. Next, we will prove that E_u is strict Archimedean by the same arguments as in Theorem 2.20. First, we will show that E_u is Archimedean by using Theorem 2.10. That is, we will show that,

1) $E_u(x,x) < x$ for $x \in (0,1)$, and

2) $E_u(E_u(x,y),z) = E_u(x, E_u(y,z))$ for $x,y,z \in [0,1]$.

To show that $E_u(x,x) < x$ for any $x \in (0,1)$, let $x \in (0,1)$. By Lemma 3.3 and the continuity of $\tilde{F}_u$, we have $x = \tilde{F}_u(t)$ for some $t \in (0,u)$. Then $t = \tilde{F}_u^{-1}(x)$ and

$$x = \tilde{F}_u(t) = \frac{E(t,u)}{E(u,u)} = \frac{C(\frac{t}{a}, \frac{u}{a})}{C(\frac{u}{a}, \frac{u}{a})} = \frac{\psi^{-1}(\psi(\frac{t}{a}) + \psi(\frac{u}{a}))}{\psi^{-1}(2\psi(\frac{u}{a}))}.$$

Therefore

$$\begin{aligned} x\psi^{-1}\left(2\psi\left(\frac{u}{a}\right)\right) &= \psi^{-1}\left(\psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right)+\psi\left(\frac{u}{a}\right)\right) \\ \psi\left(x\psi^{-1}\left(2\psi\left(\frac{u}{a}\right)\right)\right) &= \psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right)+\psi\left(\frac{u}{a}\right). \end{aligned} \tag{3.2}$$

Since ψ is strict, by Theorem 2.19, ψ^{-1} is strictly decreasing on $[0,\infty]$. Because $\tilde{F}_u^{-1}(x) < u$, we have

$$\begin{aligned} \frac{\tilde{F}_u^{-1}(x)}{a} &< \frac{u}{a} \\ \psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right) &> \psi\left(\frac{u}{a}\right) \\ 2\psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right) &> \psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right)+\psi\left(\frac{u}{a}\right) \\ 2\psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right) &> \psi\left(x\psi^{-1}\left(2\psi\left(\frac{u}{a}\right)\right)\right) \qquad \Big(\text{by (3.2)}\Big) \\ \psi^{-1}\left(2\psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right)\right) &< x\psi^{-1}\left(2\psi\left(\frac{u}{a}\right)\right). \end{aligned}$$

This implies that

$$E_u(x,x) = \frac{C\left(\frac{\tilde{F}_u^{-1}(x)}{a}, \frac{\tilde{F}_u^{-1}(x)}{a}\right)}{C(\frac{u}{a},\frac{u}{a})} = \frac{\psi^{-1}\left(2\psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right)\right)}{\psi^{-1}(2\psi(\frac{u}{a}))} < x. \tag{3.3}$$

Next, we will show that $E_u(E_u(x,y),z) = E_u(x,E_u(y,z))$ for $x,y,z \in [0,1]$.

Let $x, y, z \in [0,1]$. By (3.2),

$$\begin{aligned}
\psi\left(\frac{\tilde{F}_u^{-1}(E_u(x,y))}{a}\right) &= \psi\left(E_u(x,y)\cdot\psi^{-1}\left(2\psi\left(\frac{u}{a}\right)\right)\right) - \psi\left(\frac{u}{a}\right)\\
&= \psi\left(\frac{C\left(\frac{\tilde{F}_u^{-1}(x)}{a}, \frac{\tilde{F}_u^{-1}(y)}{a}\right)}{\psi^{-1}(2\psi(\frac{u}{a}))}\cdot\psi^{-1}\left(2\psi\left(\frac{u}{a}\right)\right)\right) - \psi\left(\frac{u}{a}\right)\\
&= \psi\left(C\left(\frac{\tilde{F}_u^{-1}(x)}{a}, \frac{\tilde{F}_u^{-1}(y)}{a}\right)\right) - \psi\left(\frac{u}{a}\right)\\
&= \psi\left(\psi^{-1}\left(\psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right) + \psi\left(\frac{\tilde{F}_u^{-1}(y)}{a}\right)\right)\right) - \psi\left(\frac{u}{a}\right)\\
&= \psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right) + \psi\left(\frac{\tilde{F}_u^{-1}(y)}{a}\right) - \psi\left(\frac{u}{a}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
C\left(\frac{\tilde{F}_u^{-1}(E_u(x,y))}{a}, \frac{\tilde{F}_u^{-1}(z)}{a}\right) &= \psi^{-1}\left[\psi\left(\frac{\tilde{F}_u^{-1}(E_u(x,y))}{a}\right) + \psi\left(\frac{\tilde{F}_u^{-1}(z)}{a}\right)\right]\\
&= \psi^{-1}\Bigg[\psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right) + \psi\left(\frac{\tilde{F}_u^{-1}(y)}{a}\right)\\
&\qquad - \psi\left(\frac{u}{a}\right) + \psi\left(\frac{\tilde{F}_u^{-1}(z)}{a}\right)\Bigg]\\
&= \psi^{-1}\Bigg[\psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right) + \psi\left(\frac{\tilde{F}_u^{-1}(y)}{a}\right)\\
&\qquad + \psi\left(\frac{\tilde{F}_u^{-1}(z)}{a}\right) - \psi\left(\frac{u}{a}\right)\Bigg]\\
&= \psi^{-1}\left[\psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right) + \psi\left(\frac{\tilde{F}_u^{-1}(E_u(y,z))}{a}\right)\right]\\
&= C\left(\frac{\tilde{F}_u^{-1}(x)}{a}, \frac{\tilde{F}_u^{-1}(E_u(y,z))}{a}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
E_u(E_u(x,y),z) &= \frac{E\Big(\tilde{F}_u^{-1}(E_u(x,y)), \tilde{F}_u^{-1}(z)\Big)}{E(u,u)} \\
&= \frac{aC\Big(\frac{\tilde{F}_u^{-1}(E_u(x,y))}{a}, \frac{\tilde{F}_u^{-1}(z)}{a}\Big)}{E(u,u)} \\
&= \frac{aC\Big(\frac{\tilde{F}_u^{-1}(x)}{a}, \frac{\tilde{F}_u^{-1}(E_u(y,z))}{a}\Big)}{E(u,u)} \\
&= \frac{E\Big(\tilde{F}_u^{-1}(x), \tilde{F}_u^{-1}(E_u(y,z))\Big)}{E(u,u)} \\
&= E_u(x, E_u(y,z)). \qquad (3.4)
\end{aligned}$$

By (3.3) and (3.4), E_u is Archimedean. Now, we will find a generator ψ_{E_u} corresponding to E_u. We note that

$$\begin{aligned}
\partial_1 E_u(x,y) &= \frac{\partial}{\partial x}\left[\frac{E(\tilde{F}_u^{-1}(x), \tilde{F}_u^{-1}(y))}{E(u,u)}\right] \\
&= \frac{\partial}{\partial x}\left[\frac{C\Big(\frac{\tilde{F}_u^{-1}(x)}{a}, \frac{\tilde{F}_u^{-1}(y)}{a}\Big)}{C(\frac{u}{a}, \frac{u}{a})}\right] \\
&= \frac{1}{C(\frac{u}{a}, \frac{u}{a})} \cdot \frac{\partial}{\partial\Big(\frac{\tilde{F}_u^{-1}}{a}\Big)(x)}\left[C\left(\frac{\tilde{F}_u^{-1}(x)}{a}, \frac{\tilde{F}_u^{-1}(y)}{a}\right) \cdot \left(\frac{\tilde{F}_u^{-1}}{a}\right)'(x)\right]
\end{aligned}$$

and

$$\begin{aligned}
\partial_2 E_u(x,y) &= \frac{\partial}{\partial y}\left[\frac{E(\tilde{F}_u^{-1}(x), \tilde{F}_u^{-1}(y))}{E(u,u)}\right] \\
&= \frac{\partial}{\partial y}\left[\frac{C\Big(\frac{\tilde{F}_u^{-1}(x)}{a}, \frac{\tilde{F}_u^{-1}(y)}{a}\Big)}{C(\frac{u}{a}, \frac{u}{a})}\right] \\
&= \frac{1}{C(\frac{u}{a}, \frac{u}{a})} \cdot \frac{\partial}{\partial\Big(\frac{\tilde{F}_u^{-1}}{a}\Big)(y)}\left[C\left(\frac{\tilde{F}_u^{-1}(x)}{a}, \frac{\tilde{F}_u^{-1}(y)}{a}\right) \cdot \left(\frac{\tilde{F}_u^{-1}}{a}\right)'(y)\right].
\end{aligned}$$

By Theorem 2.11, the generator ψ_{E_u} of E_u satisfies $\dfrac{\psi'_{E_u}(x)}{\psi'_{E_u}(y)} = \dfrac{\partial_1 E_u(x,y)}{\partial_2 E_u(x,y)}$ for

$x, y \in [0,1]$. Then ψ_{E_u} necessarily satisfies

$$\begin{aligned}\frac{\psi'_{E_u}(x)}{\psi'_{E_u}(y)} &= \frac{\partial_1 C\left(\frac{\tilde{F}_u^{-1}(x)}{a}, \frac{\tilde{F}_u^{-1}(y)}{a}\right) \cdot \left(\frac{\tilde{F}_u^{-1}}{a}\right)'(x)}{\partial_2 C\left(\frac{\tilde{F}_u^{-1}(x)}{a}, \frac{\tilde{F}_u^{-1}(y)}{a}\right) \cdot \left(\frac{\tilde{F}_u^{-1}}{a}\right)'(y)} \\ &= \frac{\psi'\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right) \cdot \left(\frac{\tilde{F}_u^{-1}}{a}\right)'(x)}{\psi'\left(\frac{\tilde{F}_u^{-1}(y)}{a}\right) \cdot \left(\frac{\tilde{F}_u^{-1}}{a}\right)'(y)} \\ &= \frac{\left(\psi \circ \frac{\tilde{F}_u^{-1}}{a}\right)'(x)}{\left(\psi \circ \frac{\tilde{F}_u^{-1}}{a}\right)'(y)}\end{aligned}$$

for all $x, y \in [0,1]$. Choose $y = \dfrac{1}{2}$. Then $\psi'_{E_u}(x) = \dfrac{\psi'_{E_u}(\frac{1}{2})}{\left(\psi \circ \frac{\tilde{F}_u^{-1}}{a}\right)'(\frac{1}{2})} \cdot \left(\psi \circ \dfrac{\tilde{F}_u^{-1}}{a}\right)'(x)$.

Thus $\displaystyle\int_x^1 \psi'_{E_u}(t)dt = \frac{\psi'_{E_u}(\frac{1}{2})}{\left(\psi \circ \frac{\tilde{F}_u^{-1}}{a}\right)'(\frac{1}{2})} \int_x^1 \left(\psi \circ \frac{\tilde{F}_u^{-1}}{a}\right)'(t)dt$. By $\psi_{E_u}(1) = 0$, we have $\psi_{E_u}(x) = c_{E_u}\left[\psi\left(\dfrac{\tilde{F}_u^{-1}(x)}{a}\right) - \psi\left(\dfrac{u}{a}\right)\right]$ when c_{E_u} is a constant. Since $c_{E_u}\left[\psi\left(\dfrac{\tilde{F}_u^{-1}(x)}{a}\right) - \psi\left(\dfrac{u}{a}\right)\right]$ and $\psi\left(\dfrac{\tilde{F}_u^{-1}(x)}{a}\right) - \psi\left(\dfrac{u}{a}\right)$ generate the same copula, we choose $c_{E_u} = 1$ and $\psi_{E_u}(x) = \psi\left(\dfrac{\tilde{F}_{u,}^{-1}(x)}{a}\right) - \psi\left(\dfrac{u}{a}\right)$. Note that ψ_{E_u} is a continuous strictly decreasing function and

$$\psi_{E_u}(t) = \psi\left(\frac{\tilde{F}_u^{-1}(t)}{a}\right) - \psi\left(\frac{u}{a}\right) = \psi\left(t\psi^{-1}\left(2\psi\left(\frac{u}{a}\right)\right)\right) - 2\psi\left(\frac{u}{a}\right). \tag{3.5}$$

To show that ψ_{E_u} is a generator of E_u, we will prove that

$$E_u(x, y) = \psi_{E_u}^{-1}(\psi_{E_u}(x) + \psi_{E_u}(y))$$

for all $x, y \in [0,1]$.

Let $x, y \in [0,1]$. Since ψ_{E_u} is continuous, $x = \psi_{E_u}(t)$ for some $t \in [0,1]$. Then

$$\begin{aligned}
x &= \psi\left(t\psi^{-1}\left(2\psi\left(\frac{u}{a}\right)\right)\right) - 2\psi\left(\frac{u}{a}\right) \\
\psi\left(t\psi^{-1}\left(2\psi\left(\frac{u}{a}\right)\right)\right) &= x + 2\psi\left(\frac{u}{a}\right) \\
t\psi^{-1}\left(2\psi\left(\frac{u}{a}\right)\right) &= \psi^{-1}\left(x + 2\psi\left(\frac{u}{a}\right)\right) \\
t &= \frac{\psi^{-1}\big(x + 2\psi(\frac{u}{a})\big)}{\psi^{-1}\big(2\psi(\frac{u}{a})\big)}
\end{aligned}$$

That is, $\psi_{E_u}^{-1}(x) = \dfrac{\psi^{-1}\big(x + 2\psi(\frac{u}{a})\big)}{\psi^{-1}\big(2\psi(\frac{u}{a})\big)}$. Thus, for $u < a$,

$$\begin{aligned}
E_u(x,y) &= \frac{E(\tilde{F}_u^{-1}(x), \tilde{F}_u^{-1}(y))}{E(u,u)} \\
&= \frac{C\left(\frac{\tilde{F}_u^{-1}(x)}{a}, \frac{\tilde{F}_u^{-1}(y)}{a}\right)}{C(\frac{u}{a}, \frac{u}{a})} \\
&= \frac{\psi^{-1}\left(\psi\left(\frac{\tilde{F}_u^{-1}(x)}{a}\right) + \psi\left(\frac{\tilde{F}_u^{-1}(y)}{a}\right)\right)}{\psi^{-1}(2\psi(\frac{u}{a}))} \\
&= \frac{\psi^{-1}\Big(\psi\big(x\psi^{-1}(2\psi(\frac{u}{a}))\big) + \psi\big(y\psi^{-1}(2\psi(\frac{u}{a}))\big) - 2\psi(\frac{u}{a})\Big)}{\psi^{-1}(2\psi(\frac{u}{a}))} \qquad \Big(\text{by (3.2)}\Big) \\
&= \frac{\psi^{-1}\big(\psi_{E_u}(x) + \psi_{E_u}(y) + 2\psi(\frac{u}{a})\big)}{\psi^{-1}(2\psi(\frac{u}{a}))} \qquad \Big(\text{by (3.5)}\Big) \\
&= \psi_{E_u}^{-1}(\psi_{E_u}(x) + \psi_{E_u}(y)).
\end{aligned}$$

This proved that ψ_{E_u} is a generator of E_u. Since ψ is strict, we have

$$\psi_{E_u}(0) = \psi\left(\frac{\tilde{F}_u^{-1}(0)}{a}\right) - \psi\left(\frac{u}{a}\right) = \psi\left(\frac{0}{a}\right) - \psi\left(\frac{u}{a}\right) = \psi(0) - \psi\left(\frac{u}{a}\right) = \infty.$$

Then ψ_{E_u} is strict. Hence, for $0 < u < a < 1$, E_u is strict Archimedean with the generator ψ_{E_u}. $\square$

Moreover, we can verify that E_u converges to a Clayton copula as $u \downarrow 0$.

Theorem 3.5. *Let C be a strict Archimedean copula having a differentiable generator $\psi \in \Re_{-\alpha}$ with $0 < \alpha < \infty$, and D any copula. Let E and E_u be defined as in equation (3.1) and Definition 3.1, respectively. Then, for all $x, y \in [0,1]$,*

$$\lim_{u \downarrow 0} E_u(x,y) = C^{\alpha}(x,y).$$

Proof We follow exactly the same arguments as Theorem 2.22.

Since $\psi \in \Re_{-\alpha}$, ψ is measurable, $\psi > 0$ and, for any $t > 0$, we have $\lim_{v \downarrow 0} \dfrac{\psi(vt)}{\psi(v)} = t^{-\alpha}$. By Theorem 2.15, $\psi(t) = t^{-\alpha}\ell(t)$ for some $\ell \in \Re_0$. Then $t^{-\alpha} \neq 0$ and $\lim_{t \downarrow 0} \dfrac{\psi(t)}{t^{-\alpha}\ell(t)} = 1$. Since ψ is convex, ψ' is increasing and monotone in some right neighbourhood of 0. By Theorem 2.16, we have $\lim_{t \downarrow 0} \dfrac{\psi'(t)}{-\alpha t^{-\alpha-1}\ell(t)} = 1$. We know that E_u is strict Archimedean with the generator

$$\psi_{E_u}(t) = \psi\left(\frac{\tilde{F}_u^{-1}(t)}{a}\right) - \psi\left(\frac{u}{a}\right) = \psi\left(t\psi^{-1}\left(2\psi\left(\frac{u}{a}\right)\right)\right) - 2\psi\left(\frac{u}{a}\right),$$

so this implies that, for $s, t \in (0,1)$,

$$\lim_{u \downarrow 0} \frac{\psi_{E_u}(s)}{\psi'_{E_u}(t)} = \lim_{u \downarrow 0} \frac{\psi(s\psi^{-1}(2\psi(\frac{u}{a}))) - 2\psi(\frac{u}{a})}{\psi^{-1}(2\psi(\frac{u}{a}))\psi'(t\psi^{-1}(2\psi(\frac{u}{a})))}.$$

Let $v = \psi^{-1}\left(2\psi\left(\dfrac{u}{a}\right)\right)$. Because $\psi^{-1}\left(2\psi\left(\dfrac{u}{a}\right)\right) = C\left(\dfrac{u}{a}, \dfrac{u}{a}\right)$ and C is continuous, $u \downarrow 0$ implies $v = C\left(\frac{u}{a}, \frac{u}{a}\right) \downarrow 0$. Then

$$\begin{aligned}
\lim_{u\downarrow 0} \frac{\psi_{E_u}(s)}{\psi'_{E_u}(t)} &= \lim_{v\downarrow 0} \frac{\psi(sv) - \psi(v)}{v\psi'(tv)} \\
&= \lim_{v\downarrow 0} \frac{(sv)^{-\alpha}\ell(sv) - v^{-\alpha}\ell(v)}{-\alpha v(tv)^{-\alpha-1}\ell(tv)} \\
&= \lim_{v\downarrow 0} \frac{s^{-\alpha}\ell(sv) - \ell(v)}{-\alpha t^{-\alpha-1}\ell(tv)} \\
&= \frac{\lim_{v\downarrow 0} \frac{s^{-\alpha}\ell(sv)}{\ell(v)} - 1}{\lim_{v\downarrow 0} \frac{-\alpha t^{-\alpha-1}\ell(tv)}{\ell(v)}} \\
&= \frac{s^{-\alpha} - 1}{-\alpha t^{-\alpha-1}} \qquad \text{(since } \ell \in \Re_0\text{)} \\
&= \frac{\phi_\alpha(s)}{\phi'_\alpha(t)}.
\end{aligned}$$

Since $\phi_\alpha(t) = t^{-\alpha} - 1$ is the generator of C^α, by Theorem 2.12, E_u converges to C^α as $u \downarrow 0$. □

Theorem 3.6. *Let C be a strict Archimedean copula with a differentiable generator $\psi \in \Re_{-\infty}$, and D any copula. Let E and E_u be defined as in equation (3.1) and Definition 3.1, respectively. Then, for all $x, y \in [0,1]$,*

$$\lim_{u\downarrow 0} E_u(x,y) = M(x,y).$$

Proof We follow exactly the same arguments as Theorem 2.23.

By Theorem 2.13, it suffices to show that, for $t \in (0,1)$, $\lim_{u\downarrow 0} \frac{\psi_{E_u}(t)}{\psi'_{E_u}(t)} = 0$. Applying Theorem 3.5, we have for $\psi \in \Re_{-\infty}$, $\lim_{v\downarrow 0} \frac{\psi(tv)}{\psi(v)} = +\infty$ and, by Theorem 2.21, $\lim_{v\downarrow 0} \frac{v\psi'(v)}{\psi(v)} = -\infty$ when $v = \psi^{-1}\left(2\psi\left(\frac{u}{a}\right)\right)$. Then

$$\begin{aligned}
\lim_{u\downarrow 0} \frac{\psi_{E_u}(t)}{\psi'_{E_u}(t)} &= \lim_{v\downarrow 0} \frac{\psi(tv) - \psi(v)}{v\psi'(tv)} \\
&= \lim_{v\downarrow 0} t\left(\frac{\psi(tv)}{tv\psi'(tv)} - \frac{\psi(tv)}{tv\psi'(tv)}\frac{\psi(v)}{\psi(tv)}\right) \\
&= 0.
\end{aligned}$$

By Theorem 2.13, E_u converges to M as $u \downarrow 0$. $\square$

CHAPTER IV

EXTREME TAIL DEPENDENCE COPULAS ALONG ARBITRARY CONVERGENCE PATHS

Let C be an Archimedean copula having a generator ψ. Juri and Wüthrich [6] studied tail copulas of C conditioning on $[0,u] \times [0,u]$ as u goes down to 0. In this chapter, we assume that C is strict and investigate tail copulas of C conditioning on $[0,u] \times [0,v]$ as (u,v) approaches $(0,0)$ along any path. We obtain the same results as those in [6].

Let two random variables X, Y be uniformly distributed on the unit interval $[0,1]$ and have joint distribution function C.

Definition 4.1. For $u \in (0,1)$, we define $F_{u,} : [0,1] \to [0,1]$ as

$$F_{u,}(t) \equiv F^C_{u,}(t) \equiv P[X \le t | X \le u, Y \le 1] = P[X \le t | X \le u] = \frac{\min(t,u)}{u}$$

for $t \in [0,1]$, and define $G_{u,} : [0,1] \to [0,1]$ by

$$G_{u,}(t) \equiv G^C_{u,}(t) \equiv P[Y \le t | X \le u, Y \le 1] = P[Y \le t | X \le u] = \frac{C(u,t)}{u}$$

for $t \in [0,1]$. Then define $C_{u,} : [0,1]^2 \to [0,1]$ as

$$C_{u,}(x,y) = \frac{C(F^{-1}_{u,}(x), G^{-1}_{u,}(y))}{u}$$

for $x, y \in [0,1]$ when $F^{-1}_{u,}(x) = \inf\{t \in [0,1] | F_{u,}(t) \ge x\} = \sup\{t \in [0,1] | F_{u,}(t) \le x\}$.

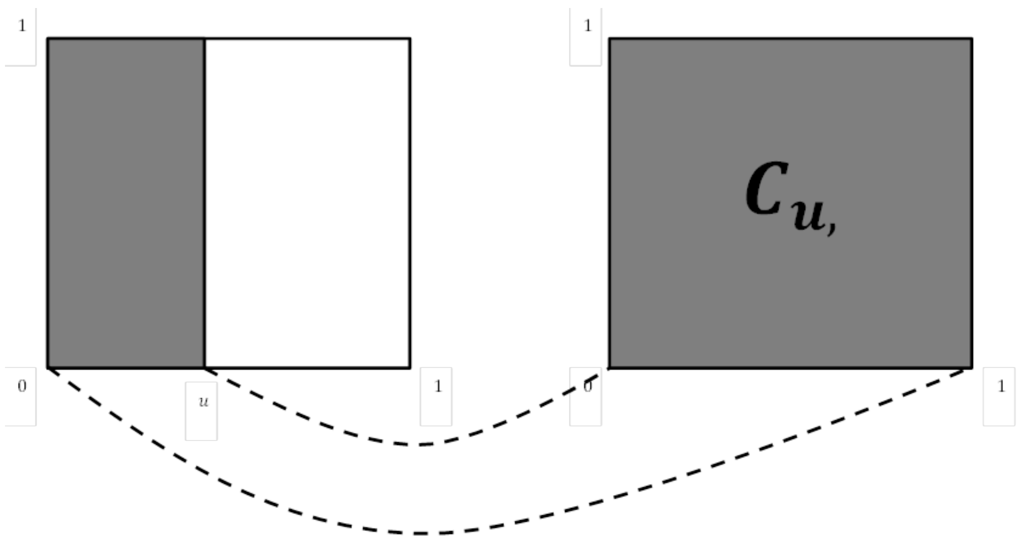

Fig.4.1 A copula $C_{u,}$

Proposition 4.2. *For every $u \in (0,1)$,*

(1) $F_{u,}^{-1}(0) = 0$, $G_{u,}^{-1}(0) = 0$,

(2) $F_{u,}^{-1}(1) = u$, $G_{u,}^{-1}(1) = 1$.

Proof We shall only prove the statements involving $F_{u,}$ as the equations concerning $G_{u,}$ can be shown in a similar fashion.

(1) Since $F_{u,}(0) = 0$, we have $F_{u,}^{-1}(0) = \inf\{t | F_{u,}(t) \geq 0\} \leq 0$. Clearly, $F_{u,}^{-1}(0) \geq 0$. Then $F_{u,}^{-1}(0) = 0$.

(2) Since $F_{u,}(u) = 1$, we have $F_{u,}^{-1}(1) = \inf\{t | F_{u,}(t) \geq 1\} \leq u$ and $F_{u,}^{-1}(1) = \sup\{t | F_{u,}(t) \leq 1\} \geq u$. Then $F_{u,}^{-1}(1) = u$. □

Lemma 4.3. *Let C be a strict Archimedean copula with a generator ψ, $F_{u,}$ and $G_{u,}$ defined as in Definition 4.1 for each $u \in (0,1)$. Then*

(1) $F_{u,}$ is strictly increasing on $[0,u]$, and

(2) $G_{u,}$ is strictly increasing on $[0,1]$.

Proof Let $u \in (0,1)$. Recall that, for $t \in [0,u]$, $F_{u,}(t) = \dfrac{t}{u}$.

(1) We will show that $F_{u,}$ is strictly increasing on $[0,u]$, i.e., for $x, y \in [0,u]$, if $x < y$, then $F_{u,}(x) < F_{u,}(y)$. Let $x, y \in [0,u]$. Assume that $x < y$. Then

$$F_{u,}(x) = \frac{x}{u} < \frac{y}{u} = F_{u,}(y).$$

Hence $F_{u,}$ is strictly increasing on $[0, u]$.

(2) Since C is strict Archimedean with a generator ψ, we have

$$G_{u,}(t) = \frac{C(u,t)}{u} = \frac{\psi^{-1}(\psi(u) + \psi(t))}{u}$$

for $t \in [0, 1]$. We will show that $G_{u,}$ is strictly increasing on $[0, 1]$, i.e., for $x, y \in [0, 1]$, if $x < y$, then $G_{u,}(x) < G_{u,}(y)$. Let $x, y \in [0, 1]$. Assume that $x < y$. Since ψ is strict, by Theorem 2.19, ψ^{-1} is strictly decreasing on $[0, \infty]$. Then

$$\begin{aligned}
\psi(x) &> \psi(y) \\
\psi(u) + \psi(x) &> \psi(u) + \psi(y) \\
\frac{\psi^{-1}(\psi(u) + \psi(x))}{u} &< \frac{\psi^{-1}(\psi(u) + \psi(y))}{u} \\
G_{u,}(x) &< G_{u,}(y).
\end{aligned}$$

Hence $G_{u,}$ is strictly increasing on $[0, 1]$. □

Now, we will prove that $C_{u,}$ is a strict Archimedean copula.

Theorem 4.4. *Let C be a strict Archimedean copula with a generator ψ, $F_{u,}$, $G_{u,}$ and $C_{u,}$ defined as in Definition 4.1 for each $u \in (0, 1)$. Then $C_{u,}$ is a strict Archimedean copula with the generator*

$$\psi_{u,}(x) = \psi(F_{u,}^{-1}(x)) - \psi(u) = \psi(G_{u,}^{-1}(x)) = \psi(xu) - \psi(u)$$

for $x \in [0, 1]$.

Proof We will show that $C_{u,}$ is a copula. Firstly, by Proposition 4.2(1), we have

$$C_{u,}(x, 0) = \frac{C(F_{u,}^{-1}(x), G_{u,}^{-1}(0))}{u} = \frac{C(F_{u,}^{-1}(x), 0)}{u} = 0$$

and

$$C_{u,}(0, y) = \frac{C(F_{u,}^{-1}(0), G_{u,}^{-1}(y))}{u} = \frac{C(0, G_{u,}^{-1}(y))}{u} = 0$$

for $x, y \in [0,1]$. Secondly, because of Proposition 4.2(2), $F_{u,}^{-1}(x) \leq u$ and the continuity of $F_{u,}$, by Theorem 2.18, we have

$$C_{u,}(x,1) = \frac{C(F_{u,}^{-1}(x), G_{u,}^{-1}(1))}{u} = \frac{C(F_{u,}^{-1}(x), 1)}{u} = F_{u,}(F_{u,}^{-1}(x)) = x$$

for $x \in [0,1]$. Similarly, because of Proposition 4.2(2) and the continuity of $G_{u,}$, by Theorem 2.18, we have

$$C_{u,}(1,y) = \frac{C(F_{u,}^{-1}(1), G_{u,}^{-1}(y))}{u} = \frac{C(u, G_{u,}^{-1}(y))}{u} = G_{u,}(G_{u,}^{-1}(y)) = y$$

for $y \in [0,1]$. Finally, let $(x_1, y_1), (x_2, y_2) \in [0,1]^2$ be such that $x_1 \leq x_2$ and $y_1 \leq y_2$. As $F_{u,}^{-1}$ and $G_{u,}^{-1}$ are increasing, we have $0 \leq F_{u,}^{-1}(x_1) \leq F_{u,}^{-1}(x_2) \leq u < 1$ and $0 \leq G_{u,}^{-1}(y_1) \leq G_{u,}^{-1}(y_2) \leq 1$. Since C is a copula, we have

$$\begin{aligned}
V_{C_{u,}}([x_1,x_2] \times [y_1,y_2]) &= C_{u,}(x_2,y_2) - C_{u,}(x_2,y_1) - C_{u,}(x_1,y_2) + C_{u,}(x_1,y_1) \\
&= \frac{C(F_{u,}^{-1}(x_2), G_{u,}^{-1}(y_2))}{u} - \frac{C(F_{u,}^{-1}(x_2), G_{u,}^{-1}(y_1))}{u} \\
&\quad - \frac{C(F_{u,}^{-1}(x_1), G_{u,}^{-1}(y_2))}{u} + \frac{C(F_{u,}^{-1}(x_1), G_{u,}^{-1}(y_1))}{u} \\
&= \frac{V_C([F_{u,}^{-1}(x_1), F_{u,}^{-1}(x_2)] \times [G_{u,}^{-1}(y_1), G_{u,}^{-1}(y_2)])}{u} \\
&\geq 0.
\end{aligned}$$

Hence $C_{u,}$ is a copula.

Now, we will show that $C_{u,}$ is Archimedean by using Theorem 2.10. That is, we will show that,

1) $C_{u,}(x,x) < x$ for $x \in (0,1)$, and

2) $C_{u,}(C_{u,}(x,y),z) = C_{u,}(x, C_{u,}(y,z))$ for $x,y,z \in [0,1]$.

First, we will show that $C_{u,}(x,x) < x$ for $x \in (0,1)$.

Let $x \in (0,1)$. We know that C is strict Archimedean with a generator ψ, i.e., for $s,t \in [0,1]$, $C(s,t) = \psi^{-1}(\psi(s) + \psi(t))$. Recall that $F_{u,}(t) = \dfrac{\min(t,u)}{u} = \dfrac{t}{u}$ for $t < u$. By Lemma 4.3(1), and the continuity of $F_{u,}$, we have $x = F_{u,}(t)$ for some

$t \in (0, u)$. Then $t = F_{u,}^{-1}(x)$. Since $F_{u,}^{-1}(x) \leq u$, we have

$$\begin{aligned} F_{u,}^{-1}(x) &= xu && (4.1)\\ \psi(F_{u,}^{-1}(x)) &= \psi(xu). && (4.2)\end{aligned}$$

Similarly, recall that $G_{u,}(t) = \dfrac{C(u,t)}{u} = \dfrac{\psi^{-1}(\psi(u) + \psi(t))}{u}$. By Lemma 4.3(2), and the continuity of $G_{u,}$, we have $x = G_{u,}(t)$ for some $t \in (0, 1)$. Then $t = G_{u,}^{-1}(x)$. Since ψ is strict, we have

$$\begin{aligned} x &= \frac{\psi^{-1}(\psi(u) + \psi(G_{u,}^{-1}(x)))}{u}\\ xu &= \psi^{-1}(\psi(u) + \psi(G_{u,}^{-1}(x)))\\ \psi(xu) &= \psi(u) + \psi(G_{u,}^{-1}(x))\\ \psi(G_{u,}^{-1}(x)) &= \psi(xu) - \psi(u). && (4.3)\end{aligned}$$

Since ψ is strict, by Theorem 2.19, ψ^{-1} is strictly decreasing on $[0, \infty]$. Since $xu < u$, we have

$$\begin{aligned} \psi(xu) &> \psi(u)\\ 2\psi(xu) &> \psi(xu) + \psi(u)\\ 2\psi(xu) - \psi(u) &> \psi(xu)\\ \psi(F_{u,}^{-1}(x)) + \psi(G_{u,}^{-1}(x)) &> \psi(xu) \qquad \Big(\text{by (4.2) and (4.3)}\Big)\\ \psi^{-1}(\psi(F_{u,}^{-1}(x)) + \psi(G_{u,}^{-1}(x))) &< xu\\ C(F_{u,}^{-1}(x), G_{u,}^{-1}(x)) &< xu\\ \frac{C(F_{u,}^{-1}(x), G_{u,}^{-1}(x))}{u} &< x\\ C_{u,}(x, x) &< x. && (4.4)\end{aligned}$$

Now, let $x, y, z \in [0,1]$. Note that

$$\begin{aligned}
\psi(F_{u,}^{-1}(C_{u,}(x,y))) &= \psi(uC_{u,}(x,y)) \\
&= \psi(C(F_{u,}^{-1}(x), G_{u,}^{-1}(y))) \\
&= \psi(\psi^{-1}(\psi(F_{u,}^{-1}(x)) + \psi(G_{u,}^{-1}(y)))) \\
&= \psi(F_{u,}^{-1}(x)) + \psi(G_{u,}^{-1}(y)). \qquad (4.5)
\end{aligned}$$

Similarly, we have

$$\psi(G_{u,}^{-1}(C_{u,}(x,y))) = \psi(F_{u,}^{-1}(x)) + \psi(G_{u,}^{-1}(y)) - \psi(u). \qquad (4.6)$$

By (4.5) and (4.6),

$$\begin{aligned}
C\big(F_{u,}^{-1}(C_{u,}(x,y)), G_{u,}^{-1}(z)\big) &= \psi^{-1}[\psi\big(F_{u,}^{-1}(C_{u,}(x,y))\big) + \psi(G_{u,}^{-1}(z))] \\
&= \psi^{-1}[\psi(F_{u,}^{-1}(x)) + \psi(G_{u,}^{-1}(y)) + \psi(G_{u,}^{-1}(z))] \\
&= \psi^{-1}[\psi(F_{u,}^{-1}(x)) + \psi(yu) - \psi(u) + \psi(G_{u,}^{-1}(z))] \\
&= \psi^{-1}[\psi(F_{u,}^{-1}(x)) + \psi(F_{u,}^{-1}(y)) + \psi(G_{u,}^{-1}(z)) - \psi(u)] \\
&= \psi^{-1}\Big[\psi(F_{u,}^{-1}(x)) + \psi\big(G_{u,}^{-1}(C_{u,}(y,z))\big)\Big] \\
&= C\big(F_{u,}^{-1}(x), G_{u,}^{-1}(C_{u,}(y,z))\big).
\end{aligned}$$

Hence

$$C_{u,}(C_{u,}(x,y), z) = C_{u,}(x, C_{u,}(y,z)). \qquad (4.7)$$

By (4.4) and (4.7), $C_{u,}$ is Archimedean. Now, we will find a generator $\psi_{u,}$ corresponding to $C_{u,}$. By Theorem 2.11, the generator ψ of C must satisfy

$$\frac{\psi'(x)}{\psi'(y)} = \frac{\partial_1 C(x,y)}{\partial_2 C(x,y)} \qquad (4.8)$$

for all $x, y \in [0,1]$ and likewise $\psi_{u,}$ is the solution of $\dfrac{\psi'_{u,}(x)}{\psi'_{u,}(y)} = \dfrac{\partial_1 C_{u,}(x,y)}{\partial_2 C_{u,}(x,y)}$, $x, y \in$

$[0,1]$. Then

$$\begin{aligned}\partial_1 C_{u,}(x,y) &= \frac{\partial}{\partial x}\left[\frac{C(F_{u,}^{-1}(x), G_{u,}^{-1}(y))}{u}\right]\\ &= \frac{1}{u}\cdot \partial_1 C(F_{u,}^{-1}(x), G_{u,}^{-1}(y))\cdot (F_{u,}^{-1})'(x).\end{aligned}$$

Similarly, $\partial_2 C_{u,}(x,y) = \dfrac{1}{u}\cdot \partial_2 C(F_{u,}^{-1}(x), G_{u,}^{-1}(y))\cdot (G_{u,}^{-1})'(y)$.

Then $\psi_{u,}$ necessarily satisfies

$$\begin{aligned}\frac{\psi'_{u,}(x)}{\psi'_{u,}(y)} &= \frac{\partial_1 C(F_{u,}^{-1}(x), G_{u,}^{-1}(y))\cdot (F_{u,}^{-1})'(x)}{\partial_2 C(F_{u,}^{-1}(x), G_{u,}^{-1}(y))\cdot (G_{u,}^{-1})'(y)}\\ &= \frac{\psi'(F_{u,}^{-1}(x))\cdot (F_{u,}^{-1})'(x)}{\psi'(G_{u,}^{-1}(y))\cdot (G_{u,}^{-1})'(y)}\\ &= \frac{(\psi\circ F_{u,}^{-1})'(x)}{(\psi\circ G_{u,}^{-1})'(y)}\end{aligned}$$

for all $x, y \in [0,1]$, where we have used (4.8) in the second equality. Putting $y = \dfrac{1}{2}$ gives $\psi'_{u,}(x) = \dfrac{\psi'_{u,}(\frac{1}{2})}{(\psi\circ G_{u,}^{-1})'(\frac{1}{2})}\cdot(\psi\circ F_{u,}^{-1})'(x)$. Integrating both sides yields $\displaystyle\int_x^1 \psi'_{u,}(t)dt = \frac{\psi'_{u,}(\frac{1}{2})}{(\psi\circ G_{u,}^{-1})'(\frac{1}{2})}\int_x^1 (\psi\circ F_{u,}^{-1})'(t)dt$. By $\psi_{u,}(1) = 0$ and $F_{u,}^{-1}(1) = u$, we have $\psi_{u,}(x) = c_u[\psi(F_{u,}^{-1}(x)) - \psi(u)]$ where c_u is a constant. Since $c_u[\psi(F_{u,}^{-1}(x)) - \psi(u)]$ and $\psi(F_{u,}^{-1}(x)) - \psi(u)$ are generators of the same copula, we put $c_u = 1$ and $\psi_{u,}(x) = \psi(F_{u,}^{-1}(x)) - \psi(u)$. Since $\psi(F_{u,}^{-1}(x)) = \psi(xu)$ and $\psi(G_{u,}^{-1}(x)) = \psi(xu) - \psi(u)$, we also have $\psi_{u,}(x) = \psi(F_{u,}^{-1}(x)) - \psi(u) = \psi(G_{u,}^{-1}(x))$. Since ψ is strict, we have $\psi_{u,}(0) = \psi(G_{u,}^{-1}(0)) = \psi(0) = \infty$. Then $\psi_{u,}$ is strict. Recall that

$$\psi_{u,}(t) = \psi(F_{u,}^{-1}(t)) - \psi(u) = \psi(tu) - \psi(u). \tag{4.9}$$

Let $x, y \in [0,1]$. We will show that $C_{u,}(x,y) = \psi_{u,}^{-1}(\psi_{u,}(x) + \psi_{u,}(y))$. Since $\psi_{u,}$ is continuous, we have $x = \psi_{u,}(t)$ for some $t \in [0, 1]$. Then

$$\begin{aligned} x &= \psi(tu) - \psi(u) \\ \psi(tu) &= x + \psi(u) \\ tu &= \psi^{-1}(x + \psi(u)) \\ t &= \frac{\psi^{-1}(x + \psi(u))}{u} \end{aligned}$$

That is, $\psi_{u,}^{-1}(x) = \dfrac{\psi^{-1}(x + \psi(u))}{u}$. Thus, by (4.2), (4.3) and (4.9),

$$\begin{aligned} C_{u,}(x, y) &= \frac{C(F_{u,}^{-1}(x), G_{u,}^{-1}(y))}{u} \\ &= \frac{\psi^{-1}(\psi(F_{u,}^{-1}(x)) + \psi(G_{u,}^{-1}(y)))}{u} \\ &= \frac{\psi^{-1}(\psi(xu) + \psi(yu) - \psi(u))}{u} \\ &= \frac{\psi^{-1}(\psi_{u,}(x) + \psi(u) + \psi_{u,}(y))}{u} \\ &= \psi_{u,}^{-1}(\psi_{u,}(x) + \psi_{u,}(y)). \end{aligned}$$

Hence $C_{u,}$ is strict Archimedean with the generator $\psi_{u,}$. □

The next theorem investigates the convergence of $C_{u,}$ to a Clayton copula C^{α} as $u \downarrow 0$.

Theorem 4.5. *Let C be a strict Archimedean copula having a differentiable generator $\psi \in \Re_{-\alpha}$ for some $\alpha \in (0, \infty)$, and $C_{u,}$ defined as in Definition 4.1. Then, for all $x, y \in [0, 1]$,*

$$\lim_{u \downarrow 0} C_{u,}(x, y) = C^{\alpha}(x, y).$$

Proof Since $\psi \in \Re_{-\alpha}$, ψ is measurable, $\psi > 0$ and, for any $t > 0$, we have $\lim_{u \downarrow 0} \dfrac{\psi(ut)}{\psi(u)} = t^{-\alpha}$. By Theorem 2.15, $\psi(t) = t^{-\alpha}\ell(t)$ for some $\ell \in \Re_0$. Then $t^{-\alpha} \neq 0$ and $\lim_{t \downarrow 0} \dfrac{\psi(t)}{t^{-\alpha}\ell(t)} = 1$. Since ψ is convex, ψ' is increasing and monotone in some right neighbourhood of 0. By Theorem 2.16, we have $\lim_{t \downarrow 0} \dfrac{\psi'(t)}{-\alpha t^{-\alpha-1}\ell(t)} = 1$. By

Theorem 4.4, $C_{u,}$ is strict Archimedean with the generator

$$\psi_{u,}(t) = \psi(tu) - \psi(u)$$

for $t \in [0,1]$. Then, for $s, t \in (0,1)$, we have

$$\begin{aligned}
\lim_{u\downarrow 0} \frac{\psi_{u,}(s)}{\psi'_{u,}(t)} &= \lim_{u\downarrow 0} \frac{\psi(su) - \psi(u)}{u\psi'(tu)} \\
&= \lim_{u\downarrow 0} \frac{(su)^{-\alpha}\ell(su) - u^{-\alpha}\ell(u)}{-\alpha u(tu)^{-\alpha-1}\ell(tu)} \\
&= \lim_{u\downarrow 0} \frac{s^{-\alpha}\ell(su) - \ell(u)}{-\alpha t^{-\alpha-1}\ell(tu)} \\
&= \frac{\lim_{u\downarrow 0} \frac{s^{-\alpha}\ell(su)}{\ell(u)} - 1}{\lim_{u\downarrow 0} \frac{-\alpha t^{-\alpha-1}\ell(tu)}{\ell(u)}} \\
&= \frac{s^{-\alpha} - 1}{-\alpha t^{-\alpha-1}} \qquad \text{(since } \ell \in \Re_0\text{)} \\
&= \frac{\phi_\alpha(s)}{\phi'_\alpha(t)}
\end{aligned}$$

where $\phi_\alpha(t) = t^{-\alpha} - 1$ is the generator corresponding to C^α. By Theorem 2.12, $C_{u,}$ converges to C^α as $u \downarrow 0$. $\square$

Moreover, if C is strict Archimedean with a generator $\psi \in \Re_{-\infty}$, then $C_{u,}$ converges to M as $u \downarrow 0$.

Theorem 4.6. *Let C be a strict Archimedean copula with a differentiable generator $\psi \in \Re_{-\infty}$, and $C_{u,}$ defined as in Definition 4.1. Then, for all $x, y \in [0,1]$,*

$$\lim_{u\downarrow 0} C_{u,}(x,y) = M(x,y).$$

Proof By Theorem 2.13, it suffices to show that, for $t \in (0,1)$, $\lim_{u\downarrow 0} \frac{\psi_{u,}(t)}{\psi'_{u,}(t)} = 0$. Applying Theorem 4.5, we have for $\psi \in \Re_{-\infty}$, $\lim_{u\downarrow 0} \frac{\psi(tu)}{\psi(u)} = +\infty$ and, by Theorem

2.21, $\lim_{u\downarrow 0} \frac{u\psi'(u)}{\psi(u)} = -\infty$. Then

$$\begin{aligned}
\lim_{u\downarrow 0} \frac{\psi_{u,}(t)}{\psi'_{u,}(t)} &= \lim_{u\downarrow 0} \frac{\psi(tu) - \psi(u)}{u\psi'(tu)} \\
&= \lim_{u\downarrow 0} t \left(\frac{\psi(tu)}{tu\psi'(tu)} - \frac{\psi(tu)}{tu\psi'(tu)} \frac{\psi(u)}{\psi(tu)} \right) \\
&= 0.
\end{aligned}$$

By Theorem 2.13, $C_{u,}$ converges to M as $u \downarrow 0$. □

We shall also define tail copulas of C conditioning on $[0, 1] \times [0, v]$ denoted by $C_{,v}$.

Definition 4.7. For $v \in (0, 1)$, we define $F_{,v} : [0, 1] \to [0, 1]$ as

$$F_{,v}(t) \equiv F^C_{,v}(t) \equiv P[X \le t | X \le 1, Y \le v] = P[X \le t | Y \le v] = \frac{C(t, v)}{v}$$

for $t \in [0, 1]$, and define $G_{,v} : [0, 1] \to [0, 1]$ by

$$G_{,v}(t) \equiv G^C_{,v}(t) \equiv P[Y \le t | X \le 1, Y \le v] = P[Y \le t | Y \le v] = \frac{\min(t, v)}{v}$$

for $t \in [0, 1]$. Then define $C_{,v} : [0, 1]^2 \to [0, 1]$ as

$$C_{,v}(x, y) = \frac{C(F^{-1}_{,v}(x), G^{-1}_{,v}(y))}{v}$$

for $x, y \in [0, 1]$.

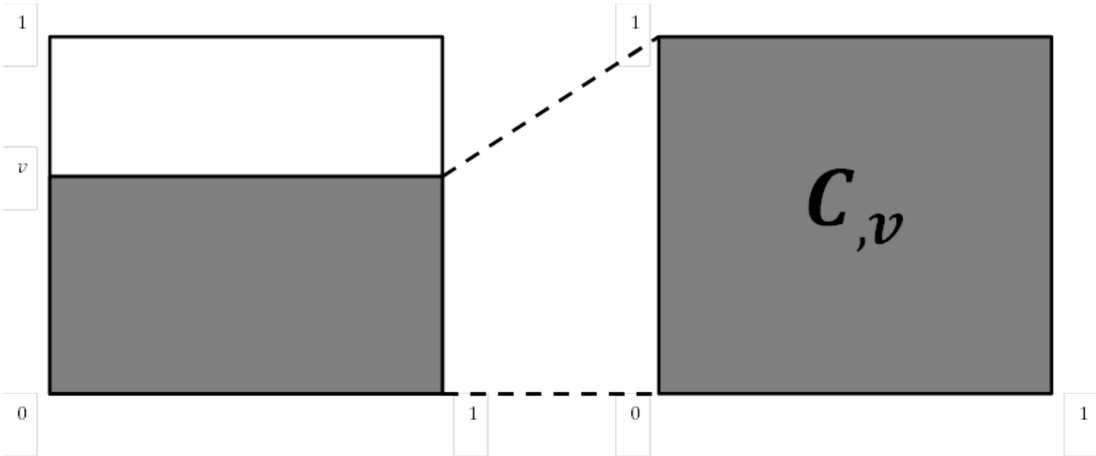

Fig.4.2 A copula $C_{,v}$

For $t \in [0,1]$, we observe that

$$F^{C}_{,v}(t) = \frac{C(t,v)}{v} = \frac{C^T(v,t)}{v} = G^{C^T}_{v,}(t),$$

and

$$G^{C}_{,v}(t) = \frac{C(1, t\wedge v)}{v} = \frac{\min(t,v)}{v} = \frac{C^T(t\wedge v, 1)}{v} = F^{C^T}_{v,}(t).$$

Moreover,

$$\begin{aligned}
C_{,v}(x,y) &= \frac{C(F^{-1}_{,v}(x), G^{-1}_{,v}(y))}{v} \\
&= \frac{C((F^{C}_{,v})^{-1}(x), (G^{C}_{,v})^{-1}(y))}{v} \\
&= \frac{C((G^{C^T}_{v,})^{-1}(x), (F^{C^T}_{v,})^{-1}(y))}{v} \\
&= \frac{C^T((F^{C^T}_{v,})^{-1}(y), (G^{C^T}_{v,})^{-1}(x))}{v} \\
&= (C^T)_{v,}(y,x),
\end{aligned}$$

where $(C^T)_{v,}$ is a strict Archimedean copula with a generator $(\psi^T)_{v,}$, by applying Theorem 4.4. Now, we will use the above properties to prove these theorems.

Theorem 4.8. *Let C be a strict Archimedean copula with a generator ψ, $F_{,v}$, $G_{,v}$ and $C_{,v}$ defined as in Definition 4.7 for each $v \in (0,1)$. Then $C_{,v}$ is also a strict Archimedean* *copula with a generator*

$$\psi_{,v}(x) = \psi((F_{,v}^{-1}(x)) = \psi(G_{,v}^{-1}(x)) - \psi(v) = \psi(xv) - \psi(v)$$

for $x \in [0, 1]$.

Proof Since $C_{,v} = (C^T)_{v,}$, we can apply Theorem 4.4 to show that $C_{,v} = (C^T)_{v,}$ is a strict Archimedean copula with a generator $\psi_{,v} = (\psi^T)_{v,}$. That is, for $x \in [0, 1]$, $\psi_{,v}(x) = (\psi^T)_{v,}(x) = \psi((G_{v,}^{C^T})^{-1}(x)) = \psi((F_{v,}^{C^T})^{-1}(x)) - \psi(v)$. Because $G_{v,}^{C^T} = F_{,v}$ and $F_{v,}^{C^T} = G_{,v}$, we have $\psi_{,v}(x) = \psi((F_{,v}^{-1}(x)) = \psi(G_{,v}^{-1}(x)) - \psi(v)$. □

Similarly, we can verify that $C_{,v}$ converges to the Clayton copula.

Theorem 4.9. *Let C be a strict Archimedean copula having a differentiable generator $\psi \in \Re_{-\alpha}$ for some $\alpha \in (0, \infty)$, and $C_{,v}$ defined as in Definition 4.7. Then, for all $x, y \in [0, 1]$,*

$$\lim_{v \downarrow 0} C_{,v}(x, y) = C^{\alpha}(x, y).$$

Proof We can prove similarly as Theorem 4.5 by using $C_{,v} = (C^T)_{v,}$. □

Moreover, if C is strict Archimedean with a generator $\psi \in \Re_{-\infty}$, then $C_{,v}$ converges to M as $v \downarrow 0$.

Theorem 4.10. *Let C be a strict Archimedean copula with a differentiable generator $\psi \in \Re_{-\infty}$, and $C_{,v}$ defined as in Definition 4.7. Then, for all $x, y \in [0, 1]$,*

$$\lim_{v \downarrow 0} C_{,v}(x, y) = M(x, y).$$

Proof We can prove similarly as Theorem 4.6 by using $C_{,v} = (C^T)_{v,}$. □

And then, we will define a new copula $C_{u,v}$.

Definition 4.11. For $u, v \in (0, 1)$, we define $F_{u,v} : [0, 1] \to [0, 1]$ as

$$F_{u,v}(t) = \frac{C_{u,}(t, v)}{v} = \frac{C(F_{u,}^{-1}(t), G_{u,}^{-1}(v))}{uv}$$

for $t \in [0,1]$, and define $G_{u,v} : [0,1] \to [0,1]$ by

$$G_{u,v}(t) = \frac{C_{u,}(1, t \wedge v)}{C_{u,}(1,v)} = \frac{\min(t,v)}{v}$$

for $t \in [0,1]$. Then define $C_{u,v} : [0,1]^2 \to [0,1]$ as

$$C_{u,v}(x,y) = \frac{C_{u,}(F_{u,v}^{-1}(x), G_{u,v}^{-1}(y))}{v} = \frac{C(F_{u,}^{-1}(F_{u,v}^{-1}(x)), G_{u,}^{-1}(G_{u,v}^{-1}(y)))}{uv}$$

for $x, y \in [0,1]$.

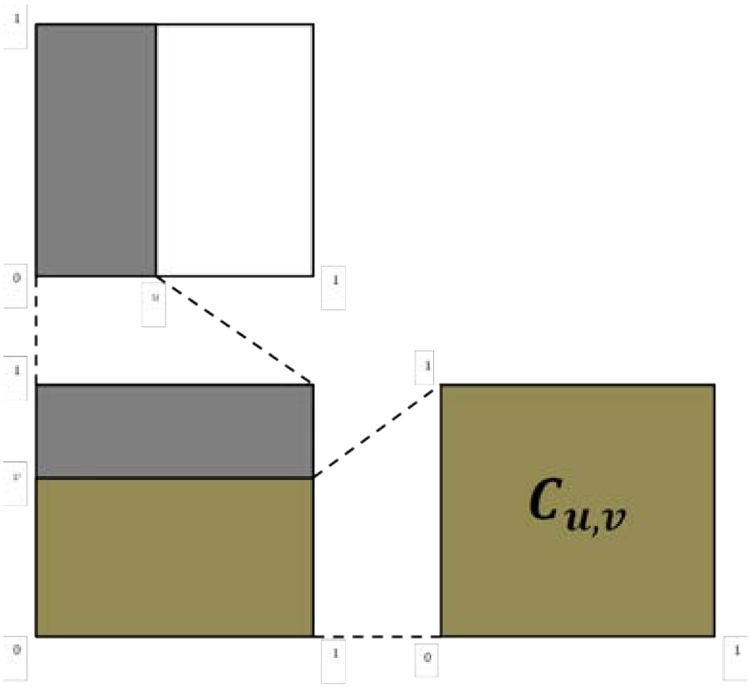

Fig.4.3 A copula $C_{u,v}$

Note that $C_{u,v} = (C_{u,})_{,v}$.

Theorem 4.12. *Let C be a strict Archimedean copula with a generator ψ, $F_{u,v}$, $G_{u,v}$ and $C_{u,v}$ defined as in Definition 4.11 for each $u, v \in (0,1)$. Then $C_{u,v}$ is also a strict Archimedean copula with a generator*

$$\psi_{u,v}(x) = \psi(xuv) - \psi(uv)$$

for $x \in [0,1]$.

Proof Since $C_{u,}$ and $C_{,v}$ are strict Archimedean copulas, by Theorems 4.4 and 4.8, $C_{u,v}$ is a strictly Archimedean copula. Next, we will find a generator $\psi_{u,v}$ for $C_{u,v}$. By definitions and equations (4.2) and (4.3),

$$\begin{aligned}
F_{u,v}(t) &= \frac{C(F_{u,}^{-1}(t), G_{u,}^{-1}(v))}{uv} \\
&= \frac{\psi^{-1}(\psi(F_{u,}^{-1}(t)) + \psi(G_{u,}^{-1}(v)))}{uv} \\
&= \frac{\psi^{-1}(\psi(tu) + \psi(vu) - \psi(u))}{uv}.
\end{aligned}$$

Since $F_{u,v}$ is continuous, we have $x = F_{u,v}(t)$ for some $t \in [0,1]$. Then $t = F_{u,v}^{-1}(x)$. Since ψ is strict, we have

$$\begin{aligned}
x &= \frac{\psi^{-1}(\psi(uF_{u,v}^{-1}(x)) + \psi(vu) - \psi(u))}{uv} \\
xuv &= \psi^{-1}(\psi(uF_{u,v}^{-1}(x)) + \psi(vu) - \psi(u)) \\
\psi(xuv) &= \psi(uF_{u,v}^{-1}(x)) + \psi(vu) - \psi(u) \\
\psi(uF_{u,v}^{-1}(x)) &= \psi(xuv) + \psi(u) - \psi(vu)
\end{aligned}$$

Since $\psi(F_{u,}^{-1}(x)) = \psi(xu)$ (equation (4.2)), we have

$$\psi(F_{u,}^{-1}(F_{u,v}^{-1}(x))) = \psi(uF_{u,v}^{-1}(x)) = \psi(xuv) + \psi(u) - \psi(vu).$$

By definition, $G_{u,v}(t) = \dfrac{t}{v}$ for $t < v$. Since $G_{u,v}$ is continuous, we have $x = G_{u,v}(t)$ for some $t \in [0,v]$. Then $t = G_{u,v}^{-1}(x)$. Since ψ is strict, we have

$$\begin{aligned}
x &= \frac{G_{u,v}^{-1}(x)}{v} \\
G_{u,v}^{-1}(x) &= xv \\
G_{u,}^{-1}(G_{u,v}^{-1}(x)) &= G_{u,}^{-1}(xv) \\
\psi(G_{u,}^{-1}(G_{u,v}^{-1}(x))) &= \psi(G_{u,}^{-1}(xv)).
\end{aligned}$$

Since $\psi(G_{u,}^{-1}(x)) = \psi(xu) - \psi(u)$ (equation (4.3)), we have $\psi(G_{u,}^{-1}(xv)) = \psi(xuv) - \psi(u)$. Then $\psi(G_{u,}^{-1}(G_{u,v}^{-1}(x))) = \psi(xuv) - \psi(u)$. Since $\psi_{u,v} = (\psi_{u,})_{,v}$, we can apply Theorems 4.8 and 4.4 to obtain

$$\psi_{u,v}(x) = \psi_{u,}(xv) - \psi_{u,}(v) = \psi(xuv) - \psi(u) - \psi(uv) + \psi(u) = \psi(xuv) - \psi(uv).$$

Hence $C_{u,v}$ is strict Archimedean with the generator $\psi_{u,v}$. $\square$

Next, we will compare the copula $C_{u,v}$ and Juri and Wuthrich's copula C_u.

Theorem 4.13. *Let C be a strict Archimedean copula with a generator ψ, C_u and $C_{u,v}$ defined as in Definition 2.17 and Definition 4.11, respectively. Then, for every $u \in (0,1)$, $C_u \equiv C_{u,v}$ where $v = \dfrac{C(u,u)}{u}$.*

Proof It suffices to show that $\psi_u = \psi_{u,v}$ where $v = \dfrac{C(u,u)}{u}$.
Substituting $v = \dfrac{C(u,u)}{u} = \dfrac{\psi^{-1}(2\psi(u))}{u}$,

$$\begin{aligned}
\psi_{u,v}(x) &= \psi(xuv) - \psi(uv) \\
&= \psi(x\psi^{-1}(2\psi(u))) - \psi(\psi^{-1}(2\psi(u)) \\
&= \psi(x\psi^{-1}(2\psi(u))) - 2\psi(u) \\
&= \psi_u(x)
\end{aligned}$$

which is the generator of C_u. Hence $C_u \equiv C_{u,v}$. $\square$

It is shown by Juri et al. that $C_{u,\frac{C(u,u)}{u}} = C_u$ converges to a Clayton copula. C_u is by definition the extreme tail dependence of C given that $X \le u$ and $Y \le u$. So it is natural to investigate the extreme tail dependence of C given that $X \le u$ and $Y \le f(u)$. We shall show that the conditional copula is $C_{u,v}$ where $v = \dfrac{C(u, f(u))}{u}$ and that $C_{u,\frac{C(u,f(u))}{u}}$ converges to a Clayton copula as $u \downarrow 0$.

Let random variables X, Y be uniformly distributed on the unit interval $[0,1]$ and have joint distribution function C. Let f map $(0,\delta)$ into $(0,1)$ for some $\delta \in (0,1]$ and suppose that $C(u, f(u)) > 0$ for all $u > 0$. For $u \in (0,\delta)$, define $F_{u:f} : [0,1] \to [0,1]$ by

$$F_{u:f}(t) = P[X \le t | X \le u, Y \le f(u)] = \frac{C(t \wedge u, f(u))}{C(u, f(u))}$$

for $t \in [0,1]$, and define $G_{u:f} : [0,1] \to [0,1]$ by

$$G_{u:f}(t) = P[Y \le t | X \le u, Y \le f(u)] = \frac{C(u, t \wedge f(u))}{C(u, f(u))}$$

for $t \in [0,1]$. Then define $C_{u:f} : [0,1]^2 \to [0,1]$ as

$$C_{u:f}(x,y) = \frac{C(F^{-1}_{u:f}(x), G^{-1}_{u:f}(y))}{C(u, f(u))} \tag{4.10}$$

for $x, y \in [0,1]$.

Applying the proof of Theorem 2.20, we have $C_{u:f}$ is a copula. Since C is strict Archimedean with the generator ψ, by definition, we have

$$F_{u:f}(t) = \frac{C(t, f(u))}{C(u, f(u))} = \frac{\psi^{-1}(\psi(t) + \psi(f(u)))}{\psi^{-1}(\psi(u) + \psi(f(u)))}$$

for $t < u$. Set $x = F_{u:f}(t)$ so that $t = F^{-1}_{u:f}(x)$. Since ψ is strict, we have

$$\begin{aligned}
x &= \frac{\psi^{-1}(\psi(F^{-1}_{u:f}(x)) + \psi(f(u)))}{\psi^{-1}(\psi(u) + \psi(f(u)))} \\
x\psi^{-1}(\psi(u) + \psi(f(u))) &= \psi^{-1}(\psi(F^{-1}_{u:f}(x)) + \psi(f(u))) \\
\psi(x\psi^{-1}(\psi(u) + \psi(f(u)))) &= \psi(F^{-1}_{u:f}(x)) + \psi(f(u)) \\
\psi(F^{-1}_{u:f}(x)) &= \psi(x\psi^{-1}(\psi(u) + \psi(f(u)))) - \psi(f(u)).
\end{aligned}$$

Similarly, by definition, we have

$$G_{u:f}(t) = \frac{C(u, t)}{C(u, f(u))} = \frac{\psi^{-1}(\psi(u) + \psi(t))}{\psi^{-1}(\psi(u) + \psi(f(u)))}$$

for $t < f(u)$. Set $x = G_{u:f}(t)$ so that $t = G^{-1}_{u:f}(x)$. Since ψ is strict, we have

$$\begin{aligned}
x &= \frac{\psi^{-1}(\psi(u) + \psi(f(G^{-1}_{u:f}(x))))}{\psi^{-1}(\psi(u) + \psi(f(u)))} \\
x\psi^{-1}(\psi(u) + \psi(f(u))) &= \psi^{-1}(\psi(u) + \psi(f(G^{-1}_{u:f}(x)))) \\
\psi(x\psi^{-1}(\psi(u) + \psi(f(u)))) &= \psi(u) + \psi(f(G^{-1}_{u:f}(x))) \\
\psi(G^{-1}_{u:f}(x)) &= \psi(x\psi^{-1}(\psi(u) + \psi(f(u)))) - \psi(u).
\end{aligned}$$

Theorem 4.14. *Let C be a strict Archimedean copula with a generator ψ, $C_{u,v}$ and $C_{u:f}$ defined as in Definition 4.11 and equation (4.10), respectively. Then, for $\delta \in (0,1]$, $u \in (0,\delta)$ and f maps $(0,\delta)$ into $(0,1)$, $C_{u:f} \equiv C_{u,v}$ is strict Archimedean where* $v = \dfrac{C(u,f(u))}{u}$.

Proof We will show that $C_{u:f}$ is strict Archimedean by using the process of $C_{u,v}$.

Let $v = \dfrac{C(u,f(u))}{u} = \dfrac{\psi^{-1}(\psi(u)+\psi(f(u)))}{u}$. Then

$$\begin{aligned}
C_{u,v}(x,y) &= \frac{C(F_{u,}^{-1}(F_{u,v}^{-1}(x)), G_{u,}^{-1}(G_{u,v}^{-1}(y)))}{uv} \\
&= \frac{\psi^{-1}(\psi(xuv)+\psi(yuv)-\psi(vu))}{C(u,f(u))} \\
&= \frac{\psi^{-1}(\psi(xC(u,f(u)))+\psi(yC(u,f(u)))-\psi(\psi^{-1}(\psi(u)+\psi(f(u)))))}{C(u,f(u))} \\
&= \frac{\psi^{-1}(\psi(xC(u,f(u)))+\psi(yC(u,f(u)))-\psi(u)-\psi(f(u)))}{C(u,f(u))} \\
&= \frac{\psi^{-1}(\psi(xC(u,f(u)))-\psi(f(u))+\psi(yC(u,f(u)))-\psi(u))}{C(u,f(u))} \\
&= \frac{\psi^{-1}(\psi(F_{u:f}^{-1}(x))+\psi(G_{u:f}^{-1}(y)))}{C(u,f(u))} \\
&= \frac{C(F_{u:f}^{-1}(x), G_{u:f}^{-1}(y))}{C(u,f(u))} \\
&= C_{u:f}(x,y).
\end{aligned}$$

Hence $C_{u,v} \equiv C_{u:f}$. Since $C_{u,v}$ is strict Archimedean, so is $C_{u:f}$. $\square$

Then we can compute the generator $\psi_{u:f}$ from finding $\psi_{u,v}$ when $v = \dfrac{C(u,f(u))}{u}$.

$$\begin{aligned}
\psi_{u,v}(x) &= \psi(xuv) - \psi(uv) \\
&= \psi(xC(u,f(u))) - \psi(C(u,f(u))) \\
&= \psi(x\psi^{-1}(\psi(u)+\psi(f(u)))) - \psi(u) - \psi(f(u)).
\end{aligned}$$

So $\psi_{u:f}(x) = \psi(x\psi^{-1}(\psi(u)+\psi(f(u)))) - \psi(u) - \psi(f(u))$.

Now, we will show that $C_{u:f}$ converges to a Clayton copula as $u \downarrow 0$.

Theorem 4.15. *Let C be a strict Archimedean copula having a differentiable generator $\psi \in \Re_{-\alpha}$ for some $\alpha \in (0, \infty)$, $C_{u:f}$ defined as in equation (4.10) and, for $\delta \in (0, 1]$, let $f : (0, \delta) \to (0, 1)$ have right limit at 0. Then, for all $x, y \in [0, 1]$,*

$$\lim_{u \downarrow 0} C_{u:f}(x, y) = C^{\alpha}(x, y).$$

Proof Since $\psi \in \Re_{-\alpha}$, ψ is measurable, $\psi > 0$ and, for any $t > 0$, we have $\lim_{v \downarrow 0} \dfrac{\psi(vt)}{\psi(v)} = t^{-\alpha}$. By Theorem 2.15, $\psi(t) = t^{-\alpha}\ell(t)$ for some $\ell \in \Re_0$. Then $t^{-\alpha} \neq 0$ and $\lim_{t \downarrow 0} \dfrac{\psi(t)}{t^{-\alpha}\ell(t)} = 1$. Since ψ is convex, ψ' is increasing and monotone in some right neighbourhood of 0. By Theorem 2.16, we have $\lim_{t \downarrow 0} \dfrac{\psi'(t)}{-\alpha t^{-\alpha-1}\ell(t)} = 1$. We know that $C_{u:f}$ is strict Archimedean with the generator

$$\psi_{u:f}(t) = \psi(t\psi^{-1}(\psi(u) + \psi(f(u)))) - \psi(u) - \psi(f(u))$$

for $t \in [0, 1]$. Then, for $s, t \in (0, 1)$,

$$\lim_{u \downarrow 0} \frac{\psi_{u:f}(s)}{\psi'_{u:f}(t)} = \lim_{u \downarrow 0} \frac{\psi(s\psi^{-1}(\psi(u) + \psi(f(u)))) - \psi(u) - \psi(f(u))}{\psi^{-1}(\psi(u) + \psi(f(u)))\psi'(t\psi^{-1}(\psi(u) + \psi(f(u))))}.$$

Let $v = \psi^{-1}(\psi(u) + \psi(f(u)))$. Because $\psi^{-1}(\psi(u) + \psi(f(u))) = C(u, f(u))$, C is continuous on $[0, 1]^2$ and f has a right limit at 0,

$$\lim_{u \downarrow 0} v = \lim_{u \downarrow 0} \psi^{-1}(\psi(u) + \psi(f(u))) = \lim_{u \downarrow 0} C(u, f(u)) = C(0, \lim_{u \downarrow 0} f(u)) = 0.$$

Hence, $v = C(u, f(u)) \downarrow 0$. Then

$$\begin{aligned}
\lim_{u\downarrow 0} \frac{\psi_{u:f}(s)}{\psi'_{u:f}(t)} &= \lim_{v\downarrow 0} \frac{\psi(sv) - \psi(v)}{v\psi'(tv)} \\
&= \lim_{v\downarrow 0} \frac{(sv)^{-\alpha}\ell(sv) - v^{-\alpha}\ell(v)}{-\alpha v(tv)^{-\alpha-1}\ell(tv)} \\
&= \lim_{v\downarrow 0} \frac{s^{-\alpha}\ell(sv) - \ell(v)}{-\alpha t^{-\alpha-1}\ell(tv)} \\
&= \frac{\lim_{v\downarrow 0} \frac{s^{-\alpha}\ell(sv)}{\ell(v)} - 1}{\lim_{v\downarrow 0} \frac{-\alpha t^{-\alpha-1}\ell(tv)}{\ell(v)}} \\
&= \frac{s^{-\alpha} - 1}{-\alpha t^{-\alpha-1}} \qquad \text{(since } \ell \in \Re_0\text{)} \\
&= \frac{\phi_\alpha(s)}{\phi'_\alpha(t)}
\end{aligned}$$

where $\phi_\alpha(t) = t^{-\alpha} - 1$ is the generator corresponding to C^α. By Theorem 2.12, $C_{u:f}$ converges to C^α as $u \downarrow 0$. □

Furthermore, if C is strict Archimedean with a generator $\psi \in \Re_{-\infty}$, then $C_{u:f}$ converges to M as $u \downarrow 0$.

Theorem 4.16. *Let C be a strict Archimedean copula with a differentiable generator $\psi \in \Re_{-\infty}$, $C_{u:f}$ defined as in equation (4.10) and, for $\delta \in (0, 1]$, a function f mapping $(0, \delta)$ into $(0, 1)$ have right limit at 0. Then, for all $x, y \in [0, 1]$,*

$$\lim_{u\downarrow 0} C_{u:f}(x, y) = M(x, y).$$

Proof By Theorem 2.13, it suffices to show that, for $t \in (0, 1)$, $\lim_{u\downarrow 0} \frac{\psi_{u:f}(t)}{\psi'_{u:f}(t)} = 0$. Applying Theorem 4.15, we have for $\psi \in \Re_{-\infty}$, $\lim_{v\downarrow 0} \frac{\psi(tv)}{\psi(v)} = +\infty$ and, by Theorem 2.21, $\lim_{v\downarrow 0} \frac{v\psi'(v)}{\psi(v)} = -\infty$ when $v = \psi^{-1}(\psi(u) + \psi(f(u)))$. Then

$$\begin{aligned}
\lim_{u\downarrow 0} \frac{\psi_{u:f}(t)}{\psi'_{u:f}(t)} &= \lim_{v\downarrow 0} \frac{\psi(tv) - \psi(v)}{v\psi'(tv)} \\
&= \lim_{v\downarrow 0} t\left(\frac{\psi(tv)}{tv\psi'(tv)} - \frac{\psi(tv)}{tv\psi'(tv)}\frac{\psi(v)}{\psi(tv)}\right) \\
&= 0.
\end{aligned}$$

□

As an example, we will construct a copula $C_{u:f}$ where $f(u) = 2u$ by the process of the copula $C_{u,v}$. We will consider the characteristic of $C_{u:f}$ when $f(u) = 2u$. For $u \in (0, \frac{1}{2})$, define $F_{u:f} : [0,1] \to [0,1]$ given as

$$F_{u:f}(t) = P[X \leq t | X \leq u, Y \leq 2u] = \frac{C(t \wedge u, 2u)}{C(u, 2u)}.$$

Moreover, we define $G_{u:f} : [0,1] \to [0,1]$ given as

$$G_{u:f}(t) = P[Y \leq t | X \leq u, Y \leq 2u] = \frac{C(u, t \wedge 2u)}{C(u, 2u)}.$$

Next, we define $C_{u:f} : [0,1]^2 \to [0,1]$ when $f(u) = 2u$ given as

$$C_{u:f}(x, y) = \frac{C(F_{u:f}^{-1}(x), G_{u:f}^{-1}(y))}{C(u, 2u)}.$$

Note that $C_{u:f}$ when $f(u) = 2u$ is a copula. Since C is strict Archimedean with the generator ψ, we have $\psi(F_{u:f}^{-1}(x)) = \psi(x\psi^{-1}(\psi(u) + \psi(2u))) - \psi(2u)$ and $\psi(G_{u:f}^{-1}(y)) = \psi(y\psi^{-1}(\psi(u) + \psi(2u))) - \psi(u)$.

It then follows from Theorem 4.14 that $C_{u:f} = C_{u,v}$, where $v = \dfrac{C(u, 2u)}{u}$, is strict Archimedean provided that C is strict Archimedean.

Applying Theorem 4.15 and Theorem 4.16, we have, for $f(u) = 2u$, $C_{u:f}$ converges to a Clayton copula as $u \downarrow 0$.

CHAPTER V

EXTREME TAIL DEPENDENCE COPULAS OF NON-STRICT ARCHIMEDEAN COPULAS

In this chapter, we shall consider non-strict Archimedean copulas and study its tail dependence copulas.

Definition 5.1. Let random variables X, Y be uniformly distributed on the unit interval $[0,1]$ and have joint distribution function C. Assume that $V_C([u_1,u]^2) > 0$ for $0 < u_1 < u < 1$ and define $F_u : [0,1] \to [0,1]$ by

$$F_u(t) = P[u_1 \leq X \leq t | X, Y \in [u_1, u]] = \frac{V_C([u_1, u \wedge t] \times [u_1, u])}{V_C([u_1, u]^2)}$$

for $t \in [0,1]$. Moreover, we define $C_u : [0,1]^2 \to [0,1]$ by

$$\begin{aligned} C_u(x,y) &= P[u_1 \leq X \leq F_u^{-1}(x), u_1 \leq Y \leq F_u^{-1}(y) | X, Y \in [u_1, u]] \\ &= \frac{V_C([u_1, F_u^{-1}(x)] \times [u_1, F_u^{-1}(y)])}{V_C([u_1, u]^2)} \end{aligned}$$

for $x, y \in [0,1]$.

Before we prove that C_u is a copula, we know that $F_u^{-1}(1) \leq u$ because $F_u(u) = 1$. Now, we will show that F_u is equal to F_{u_0} when $u_0 = F_u^{-1}(1)$.

Lemma 5.2. *Let C be a symmetric copula for which there is $u_1 \in (0,1)$ such that $C(u_1,u_1) = 0$ and $V_C([u_1,u]^2) > 0$ for $0 < u_1 < u < 1$. Then $F_u \equiv F_{u_0}$ when $u_0 = F_u^{-1}(1)$.*

Proof Observe that

$$F_u(x) = \frac{V_C([u_1, u \wedge x] \times [u_1, u])}{V_C([u_1, u]^2)} = \begin{cases} 1 & \text{if } x > u; \\ \frac{V_C([u_1,x] \times [u_1,u])}{V_C([u_1,u]^2)} & \text{if } u_1 \leq x \leq u; \\ 0 & \text{if } 0 \leq x < u_1. \end{cases}$$

Let $x \in [0,1]$. If $x > u$, then $F_u(x) = 1 = F_{u_0}(x)$ because $u \geq u_0$. If $x < u_1$, then $F_u(x) = 0 = F_{u_0}(x)$. Assume that $u_1 \leq x \leq u$. Then

$$F_u(x) = \frac{V_C([u_1, x] \times [u_1, u])}{V_C([u_1, u]^2)}$$

and

$$F_{u_0}(x) = \frac{V_C([u_1, x] \times [u_1, u_0])}{V_C([u_1, u_0]^2)}.$$

It suffices to show that $V_C([u_1, x] \times [u_1, u]) = V_C([u_1, x] \times [u_1, u_0])$ and $V_C([u_1, u]^2) = V_C([u_1, u_0]^2)$.

To show that $V_C([u_1, u]^2) = V_C([u_1, u_0]^2)$, we will show that $F_u(u_0) = 1$.

Clearly, $F_u(u_0) \leq 1$. Since $u_0 = F_u^{-1}(1) = \inf\{t | F_u(t) \geq 1\}$, for each $n \in \mathbb{N}$, there exists $a_n \in \{t | F_u(t) \geq 1\}$ such that $u_0 \leq a_n < u_0 + \frac{1}{n}$. Then $F_u(a_n) \geq 1$. By Squeeze theorem, we have $\lim_{n\to\infty} a_n = u_0$ and so by the continuity of F_u,

$$F_u(u_0) = F_u(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} F_u(a_n) \geq \lim_{n\to\infty} 1 = 1.$$

Then $F_u(u_0) = 1$. Since

$$\frac{V_C([u_1, u_0] \times [u_1, u])}{V_C([u_1, u]^2)} = F_u(u_0) = 1 = F_u(u) = \frac{V_C([u_1, u]^2)}{V_C([u_1, u]^2)},$$

we have $V_C([u_1, u_0] \times [u_1, u]) = V_C([u_1, u]^2)$, which by definition of V_C implies that $C(u_0, u) - C(u_0, u_1) = C(u, u) - C(u, u_1)$. Hence $V_C([u_0, u] \times [u_1, u]) = C(u, u) - C(u, u_1) - C(u_0, u) + C(u_0, u_1) = 0$. Since C is symmetric, we also have

$$V_C([u_1, u] \times [u_0, u]) = C(u, u) - C(u_1, u) - C(u, u_0) + C(u_1, u_0) = 0.$$

Since $[u_1, u_0] \times [u_0, u] \subseteq [u_1, u] \times [u_0, u]$, we have $V_C([u_1, u_0] \times [u_0, u]) = 0$. Then $V_C([u_1, u]^2 \backslash [u_1, u_0]^2) = V_C([u_0, u] \times [u_1, u]) + V_C([u_1, u_0] \times [u_0, u]) = 0$. Note that

$$V_C([u_1, u]^2) = V_C([u_1, u_0]^2) + V_C([u_1, u]^2 \backslash [u_1, u_0]^2).$$

Since $V_C([u_1,u]^2\backslash[u_1,u_0]^2) = 0$, we have $V_C([u_1,u]^2) = V_C([u_1,u_0]^2)$.

We know that $V_C([u_1,u]^2\backslash[u_1,u_0]^2) = 0$ and $\Big([u_1,x]\times[u_1,u]\Big)\backslash\Big([u_1,x]\times[u_1,u_0]\Big) \subseteq [u_1,u]^2\backslash[u_1,u_0]^2$. Then

$$0 = V_C\Bigg(\Big([u_1,x]\times[u_1,u]\Big)\backslash\Big([u_1,x]\times[u_1,u_0]\Big)\Bigg) = V_C([u_1,x]\times[u_1,u]) - V_C([u_1,x]\times[u_1,u_0]).$$

Thus $V_C([u_1,x]\times[u_1,u]) = V_C([u_1,x]\times[u_1,u_0])$. Finally,

$$F_u(x) = \frac{V_C([u_1,x]\times[u_1,u])}{V_C([u_1,u]^2)} = \frac{V_C([u_1,x]\times[u_1,u_0])}{V_C([u_1,u_0]^2)} = F_{u_0}(x).$$

Then $F_u \equiv F_{u_0}$. $\square$

Next, we will prove that C_u is a copula.

Theorem 5.3. *Let C be a symmetric copula for which there exists $u_1 \in (0,1)$ such that $C(u_1,u_1) = 0$ and $V_C([u_1,u]^2) > 0$ for $0 < u_1 < u < 1$, and C_u defined as in Definition 5.1. Then, for $u \in (0,1)$, C_u is a copula.*

Proof Note that $F_u^{-1}(0) = \inf\{t|F_u(t) \geq 0\}$ and $F_u(0) = \dfrac{V_C([u_1,0]\times[u_1,u])}{V_C([u_1,u]^2)} = 0$. Then $F_u^{-1}(0) = 0$. Thus, for $0 \leq a \leq 1$, we have

$$\begin{aligned} C_u(a,0) &= P[u_1 \leq X \leq F_u^{-1}(a), u_1 \leq Y \leq F_u^{-1}(0)|X,Y \in [u_1,u]] \\ &= P[u_1 \leq X \leq F_u^{-1}(a), u_1 \leq Y \leq 0|X,Y \in [u_1,u]] = 0 \\ &= 0. \end{aligned}$$

Similarly, $C_u(0,a) = 0$.

Since C is a symmetric copula, by Lemma 5.2, without loss of generality, we assume that $F_u^{-1}(1) = u$. Since F_u is continuous, by Theorem 2.18, we have

$$C_u(x,1) = \frac{V_C([u_1,F_u^{-1}(x)]\times[u_1,F_u^{-1}(1)])}{V_C([u_1,u]^2)} = \frac{V_C([u_1,F_u^{-1}(x)]\times[u_1,u])}{V_C([u_1,u]^2)} = F_u(F_u^{-1}(x)) = x.$$

Now, let $(x_1,y_1),(x_2,y_2) \in [0,1]^2$ be such that $x_1 \leq x_2$ and $y_1 \leq y_2$. Since F_u^{-1} is increasing, $0 \leq F_u^{-1}(x_1) \leq F_u^{-1}(x_2) \leq u < 1$ and $0 \leq F_u^{-1}(y_1) \leq F_u^{-1}(y_2) \leq u <$

1. Since C is a copula, we have

$$\begin{aligned}
V_{C_u}([x_1,x_2]\times[y_1,y_2]) &= C_u(x_2,y_2)-C_u(x_2,y_1)-C_u(x_1,y_2)+C_u(x_1,y_1)\\
&= \frac{V_C\big([u_1,F_u^{-1}(x_2)]\times[u_1,F_u^{-1}(y_2)]\big)}{V_C([u_1,u]^2)}\\
&\quad -\frac{V_C\big([u_1,F_u^{-1}(x_2)]\times[u_1,F_u^{-1}(y_1)]\big)}{V_C([u_1,u]^2)}\\
&\quad -\frac{V_C\big([u_1,F_u^{-1}(x_1)]\times[u_1,F_u^{-1}(y_2)]\big)}{V_C([u_1,u]^2)}\\
&\quad +\frac{V_C\big([u_1,F_u^{-1}(x_1)]\times[u_1,F_u^{-1}(y_1)]\big)}{V_C([u_1,u]^2)}\\
&= \frac{C(F_u^{-1}(x_2),F_u^{-1}(y_2))}{V_C([u_1,u]^2)}-\frac{C(F_u^{-1}(x_2),F_u^{-1}(y_1))}{V_C([u_1,u]^2)}\\
&\quad -\frac{C(F_u^{-1}(x_1),F_u^{-1}(y_2))}{V_C([u_1,u]^2)}+\frac{C(F_u^{-1}(x_1),F_u^{-1}(y_1))}{V_C([u_1,u]^2)}\\
&\geq 0.
\end{aligned}$$

Hence C_u is a copula. □

We are unable to show that C_u is Archimedean. However, we have the following example for non-strict Archimedean copula C.

Example. *If* $C_\theta(u,v)=\max(1-[(1-u)^\theta+(1-v)^\theta]^{\frac{1}{\theta}},0)$ *where* $\theta>1$*, then*

$$\lim_{u\downarrow u_1}(C_\theta)_u(x,y)=xy$$

where $u_1=1-(\frac{1}{2})^{\frac{1}{\theta}}$.

Proof It is easy to show that $C_\theta(u,u)=0$ if and only if $u\leq 1-\left(\frac{1}{2}\right)^{\frac{1}{\theta}}$. Set $u_1=1-(\frac{1}{2})^{\frac{1}{\theta}}$. We have to check that $V_{C_\theta}([u_1,u]^2)>0$, for all $u>u_1$. Note that $V_{C_\theta}([u_1,u]^2)=2^{1-\frac{1}{\theta}}[1+2(1-u)^\theta]^{\frac{1}{\theta}}-2^{\frac{1}{\theta}}(1-u)-1$. Set $f(u)=2^{1-\frac{1}{\theta}}[1+2(1-u)^\theta]^{\frac{1}{\theta}}-2^{\frac{1}{\theta}}(1-u)-1$. Then $f'(u)=2^{\frac{1}{\theta}}-2^{2-\frac{1}{\theta}}(1-u)^{\theta-1}[1+2(1-u)^\theta]^{\frac{1}{\theta}-1}$. If $f'(u)=0$, then $2^{\frac{1}{\theta}}-2^{2-\frac{1}{\theta}}(1-u)^{\theta-1}[1+2(1-u)^\theta]^{\frac{1}{\theta}-1}=0$. By solving the equation, we have $u=1-\left(\frac{1}{2}\right)^{\frac{1}{\theta}}$.

Consider $u>1-\left(\frac{1}{2}\right)^{\frac{1}{\theta}}$. We will show that f is strictly increasing by showing that $f'(u)>0$. First, note that $1-u<\left(\frac{1}{2}\right)^{\frac{1}{\theta}}$. Then $(1-u)^\theta<\frac{1}{2}$ and $(1-u)^{\theta-1}<$

$\left(\frac{1}{2}\right)^{1-\frac{1}{\theta}}$. Thus

$$\begin{aligned} f'(u) &= 2^{\frac{1}{\theta}} - 2^{2-\frac{1}{\theta}}(1-u)^{\theta-1}[1+2(1-u)^{\theta}]^{\frac{1}{\theta}-1} \\ &> 2^{\frac{1}{\theta}} - 2^{2-\frac{1}{\theta}}\left(\frac{1}{2}\right)^{1-\frac{1}{\theta}}\left[1+2\left(\frac{1}{2}\right)\right]^{\frac{1}{\theta}-1} \\ &= 2^{\frac{1}{\theta}} - 2^{2-\frac{1}{\theta}} \cdot 2^{\frac{1}{\theta}-1} \cdot 2^{\frac{1}{\theta}-1} \\ &= 2^{\frac{1}{\theta}} - 2^{\frac{1}{\theta}} \\ &= 0. \end{aligned}$$

Then f is strictly increasing. Since $f\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\theta}}\right) = 0$ and f is strictly increasing, $f(u) > 0$ for all $u > 1-\left(\frac{1}{2}\right)^{\frac{1}{\theta}}$, i.e., $V_{C_\theta}\left(\left[1-\left(\frac{1}{2}\right)^{\frac{1}{\theta}}, u\right]^2\right) > 0$, or $V_{C_\theta}([u_1,u]^2) > 0$. Set $u_1 = 1-\left(\frac{1}{2}\right)^{\frac{1}{\theta}}$ and $u \in (0,1)$ such that $u > u_1$. Therefore,

$$\begin{aligned} F_u(t) &= \frac{V_{C_\theta}([u_1, u \wedge t] \times [u_1, u])}{V_{C_\theta}([u_1,u]^2)} \\ &= \begin{cases} 1 & \text{if } t \geq u; \\ \frac{[(1-t)^{\theta}+\frac{1}{2}]^{\frac{1}{\theta}} + [(1-u)^{\theta}+\frac{1}{2}]^{\frac{1}{\theta}} - [(1-t)^{\theta}+(1-u)^{\theta}]^{\frac{1}{\theta}} - 1}{2^{1-\frac{1}{\theta}}[1+2(1-u)^{\theta}]^{\frac{1}{\theta}} - 2^{\frac{1}{\theta}}(1-u) - 1} & \text{if } u_1 < t < u; \\ 0 & \text{if } 0 \leq t \leq u_1 < u. \end{cases} \end{aligned}$$

Next, we will find $(C_\theta)_u$. Note that, for $x, y \in [0,1]$,

$$\begin{aligned} (C_\theta)_u(x,y) &= \frac{V_{C_\theta}([u_1, F_u^{-1}(x)] \times [u_1, F_u^{-1}(y)])}{V_{C_\theta}([u_1,u]^2)} \\ &= \frac{V_{C_\theta}([u_1, s] \times [u_1, t])}{V_{C_\theta}([u_1,u]^2)} \\ &= \frac{[(1-s)^{\theta}+\frac{1}{2}]^{\frac{1}{\theta}} + [(1-t)^{\theta}+\frac{1}{2}]^{\frac{1}{\theta}} - [(1-s)^{\theta}+(1-t)^{\theta}]^{\frac{1}{\theta}} - 1}{2^{1-\frac{1}{\theta}}[1+2(1-u)^{\theta}]^{\frac{1}{\theta}} - 2^{\frac{1}{\theta}}(1-u) - 1}, \end{aligned}$$

where $s = F_u^{-1}(x)$ and $t = F_u^{-1}(y)$.

Case 1 : $x = y$. Then $s = t$. We have

$$(C_\theta)_u(x,x) = \frac{2[(1-s)^{\theta}+\frac{1}{2}]^{\frac{1}{\theta}} - [2(1-s)^{\theta}]^{\frac{1}{\theta}} - 1}{2^{1-\frac{1}{\theta}}[1+2(1-u)^{\theta}]^{\frac{1}{\theta}} - 2^{\frac{1}{\theta}}(1-u) - 1}.$$

By L'Hospital's rule and squeeze theorem, we have

$$\lim_{u \downarrow u_1} (C_\theta)_u(x,x) = \lim_{u \downarrow u_1} \frac{A_s(\frac{d^2 s}{du^2}) - B_s(\frac{ds}{du})^2}{D_u}$$

where

$$\begin{aligned} A_s &= 2^{\frac{1}{\theta}} - 2^{2-\frac{1}{\theta}}(1-s)^{\theta-1}[1+2(1-s)^\theta]^{\frac{1}{\theta}-1}, \\ B_s &= 2^{2-\frac{1}{\theta}}(1-\theta)(1-s)^{\theta-2}[1+2(1-s)^\theta]^{\frac{1}{\theta}-2} \quad \text{and} \\ D_u &= 2^{2-\frac{1}{\theta}}(\theta-1)(1-u)^{\theta-2}[1+2(1-u)^\theta]^{\frac{1}{\theta}-2}. \end{aligned}$$

Note that

$$\lim_{u \downarrow u_1} \left[A_s \left(\frac{d^2 s}{du^2} \right) - B_s \left(\frac{ds}{du} \right)^2 \right] = 2^{\frac{2}{\theta}-1}(\theta-1) \left[\lim_{u \downarrow u_1} \left(\frac{ds}{du} \right)^2 \right]$$

and $\lim_{u \downarrow u_1} D_u = 2^{\frac{2}{\theta}-1}(\theta-1)$. Then $\lim_{u \downarrow u_1} (C_\theta)_u(x,x) = \lim_{u \downarrow u_1} \left(\frac{ds}{du} \right)^2$.

Since $s = F_u^{-1}(x)$, we have

$$\begin{aligned} x &= F_u(s) \\ &= \frac{[(1-s)^\theta + \frac{1}{2}]^{\frac{1}{\theta}} + [(1-u)^\theta + \frac{1}{2}]^{\frac{1}{\theta}} - [(1-s)^\theta + (1-u)^\theta]^{\frac{1}{\theta}} - 1}{2^{1-\frac{1}{\theta}}[1+2(1-u)^\theta]^{\frac{1}{\theta}} - 2^{\frac{1}{\theta}}(1-u) - 1}. \end{aligned}$$

Then

$$\left[(1-s)^\theta + \frac{1}{2}\right]^{\frac{1}{\theta}} + \left[(1-u)^\theta + \frac{1}{2}\right]^{\frac{1}{\theta}} - [(1-s)^\theta + (1-u)^\theta]^{\frac{1}{\theta}} - 1 = (2^{1-\frac{1}{\theta}}[1+2(1-u)^\theta]^{\frac{1}{\theta}} - 2^{\frac{1}{\theta}}(1-u) - 1)x.$$

By implicit differentiation, we have $\frac{ds}{du} = \frac{xA_u + (1-u)^{\theta-1}B_{u,s}}{(1-s)^{\theta-1}D_{u,s}}$ where

$$\begin{aligned} A_u &= 2^{\frac{1}{\theta}} - 2^{2-\frac{1}{\theta}}(1-u)^{\theta-1}[1+2(1-u)^\theta]^{\frac{1}{\theta}-1}, \\ B_{u,s} &= \left[(1-u)^\theta + \frac{1}{2}\right]^{\frac{1}{\theta}-1} - [(1-s)^\theta + (1-u)^\theta]^{\frac{1}{\theta}-1} \quad \text{and} \\ D_{u,s} &= \left[(1-s)^\theta + \frac{1}{2}\right]^{\frac{1}{\theta}-1} - [(1-s)^\theta + (1-u)^\theta]^{\frac{1}{\theta}-1}. \end{aligned}$$

Next, we will find $\lim_{u \downarrow u_1} \frac{ds}{du}$ by L'Hospital's rule and squeeze theorem. Then we have $\lim_{u \downarrow u_1} \frac{ds}{du} = x$, so $\lim_{u \downarrow u_1} \left(\frac{ds}{du}\right)^2 = x^2$. Hence, $\lim_{u \downarrow u_1} (C_\theta)_u(x,x) = x^2$.

Case 2 : $x \neq y$. Then $s \neq t$. We have

$$\lim_{u \downarrow u_1} (C_\theta)_u(x,y) = \lim_{u \downarrow u_1} \frac{[(1-s)^\theta + \frac{1}{2}]^{\frac{1}{\theta}} + [(1-t)^\theta + \frac{1}{2}]^{\frac{1}{\theta}} - [(1-s)^\theta + (1-t)^\theta]^{\frac{1}{\theta}} - 1}{2^{1-\frac{1}{\theta}}[1 + 2(1-u)^\theta]^{\frac{1}{\theta}} - 2^{\frac{1}{\theta}}(1-u) - 1}.$$

By L'Hospital's rule, squeeze theorem and the same method as Case 1, we have

$$\lim_{u \downarrow u_1} (C_\theta)_u(x,y) = \lim_{u \downarrow u_1} \left[\left(\frac{ds}{du}\right)\left(\frac{dt}{du}\right)\right] = xy.$$

$\square$

REFERENCES

[1] N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular variation*, Cambridge University Press, 1987.

[2] A. Charpentier, J. Segers, Lower tail dependence for Archimedean copulas: characterizations and pitfalls, *Insurance: Mathematics & Economics*, **40** (2007), 525–532.

[3] S. Coles, *An Introduction to Statistical Modeling of Extreme Values*, Springer, London, 2001.

[4] F. Durante, P. Jaworski, R. Mesiar, Invariant dependence structures and Archimedean copulas, *Statistics and Probability Letters*, **81** (2011), 1995–2003.

[5] P. Embrechts, C. Klüppelberg, T. Mikosch, *Modelling extremal events for insurance and finance*, Springer, Berlin, 1997.

[6] A. Juri, M.V. Wüthrich, Copula convergence theorems for tail events, *Insurance: Mathematics & Economics*, **30** (2002), 405–420.

[7] A. Juri, M.V. Wüthrich, Tail dependence from a distributional point of view, *Extremes*, **6** (2004), 213–246.

[8] R.B. Nelsen, *An introduction to copulas*, Springer, New York, 2006.

[9] H. Royden, P. Fitzpatrick, *Real analysis*, Pearson Education, Inc., 2010.

[10] G. Salvadori, C. De Michele, N.T. Kottegoda, R. Rosso, *Extreme in nature: An approach using copulas*, Springer, Netherlands, 2007.

[11] W.R. Wade, *An introduction to analysis*, Pearson Education, Inc., 2010.

VITA

Name	Mr. Khajohnsak Bamrungsinman
Date of Birth	24 July 1986
Place of Birth	Bangkok, Thailand
Education	B.Sc. (Mathematics), Chulalongkorn University, 2008
Scholarship	Rajamangala University of Technology Phra Nakhon scholarship