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BRANE COSMOLOGY

Mr. Itzadah Thongkool

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By	Mr. Itzadah Thongkool
Thesis Advisor	Dr. Rujikorn Dhanawittayapol

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....Dean of Faculty of Science
(Professor Piamsak Menasveta, Ph.D.)

THESIS COMMITTEE

....Chairman
(Ahpisit Ungkitchanukit, Ph.D.)

....Thesis Advisor
(Rujikorn Dhanawittayapol, Ph.D.)

....Member
(Assistant Professor Pornchai Pacharintanakul, Ph.D.)

....Member
(Patcha Chatraphorn, Ph.D.)

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วิทยานิพนธ์ฉบับนี้เป็นการทวนสอบแนวคิดพื้นฐานและผลของจักรวาลวิทยาแบบเบรนที่มีพื้นฐานมาจากแบบจำลองมิติที่5 ของ Randall และ Sundrum เราได้เขียนสมการไอน์สไตน์บนเบรนเวิลด์โดยใช้เทคนิคมาตรฐานในทฤษฎีสัมพัทธภาพทั่วไป ปรากฏว่าสมการไอน์สไตน์ในกรณีนี้มีเทอมพิเศษเพิ่มจากกรณีของสมการในสี่มิติ เราสามารถหาผลเฉลยเชิงจักรวาลวิทยาที่สอดคล้องกับสมการข้างต้นซึ่งได้รวมเงื่อนไขของการเกิดอินเฟลชันบนเบรนเวิลด์ ตอนท้ายของวิทยานิพนธ์เป็นการอภิปรายเพอร์เทอร์เบชันเชิงจักรวาลวิทยาในแบบของเมทริกเพอร์เทอร์เบชั่น และหาค่าแอมพลิจูดเชิงสเกลาร์ของเพอร์เทอร์เบชันบนเบรนเวิลด์ได้

ภาควิชา ฟิสิกส์ ลายมือชื่อนิสิต

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We review the basic ideas and results of brane cosmology based on the 5-dimensional model by Randall and Sundrum. The Einstein equations on the braneworld are derived by using standard techniques in general relativity. It is found that the Einstein equations in this case contain extra terms which are absent in the conventional 4-dimensional case. The corresponding cosmological solutions which include the inflationary solution are then obtained and the conditions for inflation on the braneworld are derived. Finally, the cosmological perturbations are discussed in the context of metric perturbations and the amplitude of scalar perturbations in the braneworld is obtained.

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Chapter 1

Introduction

The proposal of general relativity by Albert Einstein in 1915 made it possible to apply the gravitational theory to the universe as a whole. At the time, it was known that the universe is dominated by matter then the universe should be expanding or contracting due to the predictions of general relativity. However, from the limitation of astronomical apparatus, all accessible observations were within our galaxy and therefore there was no evidence for expanding or contracting universe. Einstein solved this shortcoming by introducing the cosmological constant term, which provides the repulsive force for balancing the attractive force from known matter in order to obtain a static universe, into his equation. In 1922, Alexander Friedmann [1] found the cosmological solution which implies an expansion (contraction) of the universe as a function of time. His solution contradicts what people had believed so that his work received a little attention until Edwin Hubble discovered the expansion of the universe by observing the redshift of galaxies in 1929. Almost twenty years later, George Gamow and his collaborators proposed the hot Big Bang model which assumes the universe must begin in a very hot and dense state. From their hypothesis, the abundance of light elements were predicted precisely from the theory of nucleosynthesis. They also predicted that the universe after decoupling time should be filled with black-body radiations which lead to the cosmic microwave background (CMB) at the present time. Thus the discovery of CMB by Penzias and Wilson in 1965 is the strong evidence for the standard Big Bang model. However, there are many serious shortcomings in the Big Bang model such as the horizon problem and the monopole problem; these motivated the idea of inflation originally proposed by Alan Guth [2] for solving these problems. Since then, many models of inflationary universe have been constructed and tested.

In 1999, Lisa Randall and Raman Sundrum wrote two remarkable papers [3, 4] proposing that our universe we live in could be a 4-dimensional subspace, whose spatial part is called brane, of a 5-dimensional spacetime. What people found very interesting in their models is that the gauge hierarchy problem in particle physics could be solved beautifully. Actually, the idea that we live in a spacetime with dimension higher than four is not new. The oldest one being the Kaluza-Klein theory [5, 6] treats spacetime as a product space of our 4-dimensional universe and the extra compact space whose size must be of the order of the Planck length. The difference between the Randall-Sundrum models and the Kaluza-Klein theory is that, in Randall-Sundrum models, the spacetime is not a product space and the size of the 5th dimension needs not be small. Moreover, gravity and matter are confined to the brane unlike the older idea of Kaluza-Klein where the gravitational field and matter fields can propagate in the extra dimensions. Since their discovery, a very large amount of research work has been devoted to the investigation of its phenomenological implications [7]. In particular, the cosmological consequences of their models are very important and resulted in the subject called braneworld cosmology, or in short, brane cosmology.

The purpose of this thesis is to explore the basic ideas and results of brane cosmology. The organization of this thesis is as follows. In Chapter 2, we will explore the Einstein equations on the braneworld. The cosmological solutions to the Einstein equations will then be obtained in Chapter 3. In Chapter 4, the cosmological perturbations which is a very important topic in cosmology will be discussed. Finally, the conclusions will be made in Chapter 5.

Chapter 2

The Einstein Equations on 3-Brane World

Many models with extra dimensions such as the Randall-Sundrum-type models are constructed in the scheme of general relativity. To investigate these models in the phenomenological aspect, we need to know whether the gravity induced on our observable 4-dimensional universe (called 3-brane world) satisfies the observational and experimental bounds.

In order to test the extra-dimensional models, we need to derive the effective Einstein equation of the models on the 3-brane world. Fortunately, the sought-after procedures are already known in general relativity since we can project the Einstein equations from 5-dimensional spacetime to our 3-brane world in the same way as to project the 4-dimensional Einstein equations to the 3-dimensional hypersurface of the instant time in the initial value formulation [8, 9].

An alternative way to derive the Einstein equation on the 3-brane world is to take advantage of the boundary-term calculations of the Lagrangian formulation in general relativity [10, 11, 12, 13]. As will be shown, the important singular behavior of the energy-momentum tensor in the braneworld scenario, known as the Israel's junction condition [14], can be obtained from the non-vanishing boundary term.

In this chapter we investigate the derivations of the Einstein field equations for a 3-brane world embedded in a 5-dimensional spacetime in detail by using different procedures. We begin with the projection procedure by introducing the mathematical tools used in the initial value formulation such as the Gaussian normal coordinates, the extrinsic curvature, etc. We then construct the Gauss-Codacci equations, which plays the crucial role in projecting the Einstein field equations on 3-brane. After that we reconsider the problem from the aspect of the Lagrangian formulation by analyzing the conventional variational method in detail. We will show that the boundary term does not vanish as normally assumed but is proportional to the scalar extrinsic curvature which leads to the junction condition.

2.1 The Mathematical Techniques from the Initial Value Formulation of General Relativity

The initial value problems in general relativity have been studied extensively. In particular we have the Arnowitt-Deser-Misner (ADM) [15] formulation that directly investigates these problems. Interestingly enough, the techniques used in the initial value formulation are exploited in some models in cosmology, for example, the singularity theory [10, 11] and the cosmological perturbation [16, 17, 18, 19].

In the initial value formulation of general relativity, we divide spacetime into "space" and "time" along which the space evolves. Once the initial condition is given on an "initial" slice of "space," the geometry of space at later time can be obtained by solving the Einstein equations. In the mathematical language, we consider our spacetime manifold $\mathcal{M}$ to be $R \times \Sigma$ where R represents the "time" coordinate ($t \in R$) and Σ represents "space," i.e., the hypersurface of instant of time. The hypersurface Σ is thus a 3-dimensional submanifold embedded in the 4-dimensional spacetime manifold. In general, the slicing defined as the diffeomorphism

$$\phi : \mathcal{M} \to R \times \Sigma \tag{2.1}$$

is arbitrary provided that the hypersurface Σ is spacelike. Each diffeomorphism therefore corresponds to a particular gauge choice [19, 20]. We will be concerned with this gauge issue later in the topic of the cosmological perturbations.

In the braneworld cosmology, a similar thing happens: Our universe is considered as a four (or more precisely, 3+1) dimensional submanifold embedded in a five (indeed, 4+1) dimensional manifold. The similar mathematical techniques thus apply. We now see that the problems of initial value and the projections of the Einstein equations on the 3-brane are almost the same, i.e., they are different only in the number of dimensions.

2.1.1 The Gaussian Normal Coordinates

In the initial value formulation, it is useful for calculation to use the Gaussian normal coordinates, or synchronous coordinates, in which the spacelike hypersurface is always orthogonal to the congruence of timelike geodesics [10, 11].

Let $\widetilde{V}_p$ and V_p be the tangent spaces to the hypersurface Σ and the manifold $\mathcal{M}$, respectively. At each point $p \in \Sigma$, there will be a vector $n^a \in V_p$, unique up to scaling, which is orthogonal to all vectors in $\widetilde{V}_p$. Thus, from this vector, we can define the induced metric h_{ab} on Σ_t, the instant hypersurface of time t, as follows:

$$h_{ab} = g_{ab} + n_a n_b. \tag{2.2}$$

The normal vector n^a is timelike and can be normalized to have unit length,

$$n^a n_a = -1. \tag{2.3}$$

Let the coordinate basis $\{X^a\}$ span the tangent space V_p to $\mathcal{M}$ at the point p. Then we can think of n^a as X^a_n which is orthogonal to the remaining $X^a_1 \ldots X^a_{n-1}$ that span the tangent space to the hypersurface Σ_t. It is easy to prove that n^a at each point p on Σ_t is always orthogonal to all vectors in the tangent space $\widetilde{V}_p$ for all the time t (that is, n^a is always orthogonal to $X^a_1 \ldots X^a_{n-1}$ as Σ_t evolves in time t) provided that n^a is the tangent vector to the timelike geodesic passing through Σ. This can be done by showing that

$$n^b \nabla_b (n_a X^a) = 0. \tag{2.4}$$

To prove the relation Eq. (2.4), we assume that $n_a X^a = 0$ at the initial time $t = 0$. Then the relation Eq. (2.4) can be proved by using the fact that the commutation of any two basis vector fields in coordinate basis vanishes which implies $n^b\nabla_b X^a = X^b\nabla_b n^a$, and the transport equation which implies the preservation of the size of n^a as Σ_t evolves along the geodesic.

2.1.2 The Projection of Tensors to a Submanifold

We now describe the method for projecting arbitrary tensor on the manifold to the one living only on the hypersurface. Given any vector $v \in V_p$ we can decompose it into a component tangent to Σ, $v_{\parallel}$, and a normal component to Σ, $v_{\perp}$:

$$v = v_{\perp} + v_{\parallel}, \tag{2.5}$$

where $v^a_{\parallel} n_a = 0$. We thus see that to project any vector v to the tangent space $\widetilde{V}_p$, we need to find a projection operator that, when acting on v, destroys its normal component $v_{\perp}$. It turns out that the opertor that does the job is just the induced metric $h^a{}_b \equiv g^{ac}h_{cb}$. Indeed if $v^a = h^a{}_b v^b$, then

$$\begin{aligned} v^a n_a &= h^a{}_b v^b n_a \\ &= (g^a{}_b + n^a n_b) v^b n_a \\ &= v^b n_b + (n^a n_a) v^b n_b \\ &= 0 \end{aligned} \tag{2.6}$$

which tells us that its normal component automatically vanishes. In general, for arbitrary tensor $T^{a_1\cdots a_k}{}_{b_1\cdots b_l}$ at $p \in \Sigma$, it is a tensor over the tangent space to Σ at p if

$$T^{a_1\cdots a_k}{}_{b_1\cdots b_l} = h^{a_1}{}_{c_1}\cdots h^{a_k}{}_{c_k} h^{d_1}{}_{b_1}\cdots h^{d_l}{}_{b_l} T^{c_1\cdots c_k}{}_{d_1\cdots d_l}. \tag{2.7}$$

For the covariant derivative, things are not so simple as one might have thought. Suppose one tries to define $h_a{}^b\nabla_b$ as a projected covariant derivative on the hypersurface. Then it is easy to see that, when acting on the above tensor (over the tangent space to Σ), the normal components of the resulting object still survive. It was found that the desired covariant derivative operator, denoted by D_a, can be defined as

$$D_c T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} = h^{a_1}{}_{d_1} \cdots h^{a_k}{}_{d_k} h^{e_1}{}_{b_1} \cdots h^{e_l}{}_{b_l} h^f{}_c \nabla_f T^{d_1 \cdots d_k}{}_{e_1 \cdots e_l}. \qquad (2.8)$$

We can see immediately that the obove object has no normal component. Furthermore, with this covariant derivative, the metric compatibility condition on the hypersurface is automatically satisfied:

$$D_a h_{bc} = h_a{}^d h_b{}^e h_c{}^f \nabla_d (g_{ef} + n_e n_f) = 0 \qquad (2.9)$$

since $\nabla_d g_{ef} = 0$ and $h_{ab} n^b$=0. Thus, D_a is the unique derivative operator associated with h_{ab}.

2.1.3 The Extrinsic Curvature

An understanding of the notion of the extrinsic curvature is crucial for the derivation of the Einstein equations on the 3-brane world. We will find later that the Gauss-Codacci equations and the boundary term of the Lagrangian of the models are in terms of the extrinsic curvature.

Not surprisingly, our conception of the curvature in everyday life is alike the notion of the extrinsic curvature rather than the familiar notion of curvature in general relativity which is indeed the intrinsic curvature. In everyday life, when we think about curvature of the surface, we normally think of the way it bends. That this "curvature" appears to our eyes is because the curved surface (mathematically speaking, a 2-dimensional manifold) is embedded in a 3-dimensional flat space we live in. In differential geometry, such a curvature is known as the extrinsic curvature which is the result of the embedding of the manifold in another manifold of higher dimension. On the contrary, the intrinsic curvature is the intrinsic property of the manifold itself and does not result from any kind of embedding.

In differential geometry the notion of intrinsic curvature comes from the failure of successive differentiation on tensor fields to cummute,

$$[\nabla_a, \nabla_b]\omega_c = \nabla_a \nabla_b \omega_c - \nabla_b \nabla_a \omega_c = R_{abc}{}^d \omega_d, \tag{2.10}$$

where $R_{abc}{}^d$ is the Riemann curvature tensor. From the geometrical description of the covariant derivative the commutator, $[\nabla_a, \nabla_b]$, can be interpreted as the parallel transportation around a small closed curve. By similar idea we can define the extrinsic curvature from the failure of the normal vector to the surface, n^a, at q to coincide with n^a at p after we parallel transport it along the surface. Note that we generally think of this normal vector field as the tangent vectors to the congruence of curves passing through the surface Thus the extrinsic curvature K_{ab} has the form

$$K_{ab} = h_a{}^c \nabla_c n_b \tag{2.11}$$

and its trace is $K \equiv K^a{}_a = h^{ab} K_{ab}$ with h_{ab} being the induced metric on the surface. It can be verified that K_{ab} is symmetric in its two indices, i.e., $K_{ab} = K_{ba}$.

The concept of the extrinsic curvature does not only help us to understand the procedure of the embedding of a surface in a higher dimensional manifold, but also gives the deeper insight about the evolution of the surface. The evolution of the surface along the congruence of geodesics passing through it is described by the so-called Raychaudhuri equation [21, 22]. Let the normal vector n^a (the tangent vector to the geodesic) be timelike and normalized to unit length, $n^a n_a = -1$, as usual. Define B_{ab} as

$$B_{ab} = \nabla_b n_a. \tag{2.12}$$

Let

$$\theta \equiv B^{ab} h_{ab}, \tag{2.13}$$

$$\sigma_{ab} \equiv B_{(ab)} - \frac{1}{3}\theta h_{ab}, \tag{2.14}$$

$$\omega_{ab} \equiv B_{[ab]}, \tag{2.15}$$

so that

$$B_{ab} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab}. \tag{2.16}$$

The Raychaudhuri equation is

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - R_{cd}n^c n^d \tag{2.17}$$

with τ being the parameter of the geodesic. If the congruence is orthogonal to the hypersurface, then $B_{ba} = K_{ab}$ so that $K = \theta$ and $\omega_{ab} = 0$. Thus the Raychaudhuri equation in this case,

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{ab}\sigma^{ab} - R_{cd}n^c n^d, \tag{2.18}$$

is nothing but the equation describing the evolution of the trace of the extrinsic curvature.

Since K_{ab} is symmetric,

$$\begin{aligned} K_{ab} &= \frac{1}{2}(\nabla_a n_b + \nabla_b n_a) && (2.19)\\ &= \frac{1}{2}\pounds_n g_{ab} && (2.20)\\ &= \frac{1}{2}\pounds_n h_{ab} && (2.21) \end{aligned}$$

where $\pounds_n$ denotes the Lie derivative along the geodesic. Thus we see that K_{ab} measures the rate of change of the spatial metric as one moves along the congruence. In other words, it measures the "bending" of the hypersurface Σ in the spacetime manifold $\mathcal{M}$. In Gaussian normal coordinate system,

$$K_{ab} = \frac{1}{2}\frac{\partial h_{ab}}{\partial \tau} \tag{2.22}$$

and its trace $K = \theta$ is the expansion of the geodesic congruence orthogonal to Σ.

2.1.4 The Gauss-Codacci Relations

It can be shown that some components of the Riemann curvature tensor depend only on the extrinsic curvature and the intrinsic curvature of the hypersurface,

${}^{(3)}R_{abc}{}^{d}$. The formulas that describe this are known as the Gauss-Codacci relations [10, 11]. The Gauss equation can be derived by using $h^{a}{}_{b}$ to project the components of

$$R_{abc}{}^{d}\omega_d = \nabla_a\nabla_b\omega_c - \nabla_b\nabla_a\omega_c \tag{2.23}$$

with ω_a being a dual vector field on the hypersurface Σ. The result is

$$\begin{aligned} {}^{(3)}R_{abc}{}^{d}\omega_d &= D_aD_b\omega_c - D_bD_a\omega_c \\ &= D_a(h_b{}^{d}h_c{}^{e}\nabla_d\omega_e) - D_b(h_a{}^{d}h_c{}^{e}\nabla_d\omega_e) \\ &= h_e{}^{a}h_b{}^{f}h_c{}^{g}h_d{}^{h}R^{e}{}_{fgh} + K^{a}{}_{c}K_{bd} - K^{a}{}_{d}K_{bc} \end{aligned} \tag{2.24}$$

which is known as the Gauss equation. A similar calculation gives the result

$$D_aK^{a}{}_{b} - D_bK = R_{ef}n^{e}h_b{}^{f} \tag{2.25}$$

known as the Codacci equation.

2.2 The Projection of the Einstien Field Equations on the 3-brane

We now use the Gauss-Codacci relations, Eqs. (2.24) and (2.25), to obtain the effective Einstein equations on the 3-brane. In this case, the Gauss-Codacci relations describe the 5-dimensional Riemann tensor $\widehat{R}^{a}_{bcd}$ in the terms of the extrinsic curvature K_{ab} and the intrinsic curvature $R^{a}{}_{bcd}$ our 4-dimensional universe, with the metric g_{ab}, treated as a (3+1)-dimensional hypersurface (brane) embedded in a (4+1)-dimensional spacetime (bulk) [8, 9].

We begin with Gauss equation for 3-brane,

$$R^{a}{}_{bcd} = g_e{}^{a}g_b{}^{f}g_c{}^{g}g_d{}^{h}\widehat{R}^{e}_{fgh} + K^{a}{}_{c}K_{bd} - K^{a}{}_{d}K_{bc}, \tag{2.26}$$

where the extrinsic curvature K_{ab} is defined as

$$K_{ab} = g_a{}^{c}\nabla_c n_b \tag{2.27}$$

with n^a being a normal vector to the brane. The Einstein tensor on the brane is thus

$$\begin{aligned} G_{ab} &= \left[\widehat{R}_{ef} + \frac{1}{2}g_{ef}\widehat{R}\right] g_a{}^e g_b{}^f + \widehat{R}_{ef}n^e n^f g_{ab} + KK_{ab} - K_a{}^e K_{be} \\ &\quad -\frac{1}{2}g_{ab}(K^2 - K^{ef}K_{ef}) - \widetilde{E}_{ab} \end{aligned} \tag{2.28}$$

where

$$\widetilde{E}_{ab} = \widehat{R}^e_{fgh} n_e n^g g_a{}^f g_b{}^h. \tag{2.29}$$

Since the formalism of general relativity is independent of the number of dimensions for dimension $d > 2$, then the 5-dimensional Einstein equation takes the familiar form,

$$\widehat{G}_{ab} = \widehat{R}_{ab} - \frac{1}{2}\hat{g}_{ab}\widehat{R} = \kappa_5^2 \widehat{T}_{ab}. \tag{2.30}$$

The decomposition of the Riemann curvature tensor into the Weyl curvature, the Ricci tensor and the scalar curvature is standard:

$$\widehat{R}_{abcd} = \frac{2}{3}\left(\hat{g}_{a[c}\widehat{R}_{d]b} - \hat{g}_{b[c}\widehat{R}_{d]a}\right) - \frac{1}{6}g_{a[c}g_{d]b}\widehat{R} + \widehat{C}_{abcd}. \tag{2.31}$$

Using Eqs. (2.30) and (2.31), the Einstein tensor on the brane now reads

$$\begin{aligned} G_{ab} &= \frac{2\kappa_5^2}{3}\left(\widehat{T}_{ef} g_a{}^e g_b{}^f + \left(\widehat{T}_{ef}n^e n^f - \frac{1}{4}\widehat{T}^e{}_e\right) g_{ab}\right) + KK_{ab} - K_a{}^e K_{be} \\ &\quad -\frac{1}{2}\left(K^2 - K^{ef}K_{ef}\right) - E_{ab} \end{aligned} \tag{2.32}$$

where the traceless tensor E_{ab} is

$$E_{ab} = \widehat{C}^e_{fgh} n_e n^g g_a{}^f g_b{}^h. \tag{2.33}$$

From the Codacci equation,

$$\nabla_a K_b{}^a - \nabla_b K = \widehat{R}_{ef} n^e g_b{}^f, \tag{2.34}$$

we get

$$\nabla_a K_b{}^a - \nabla_b K = \kappa_5^2 T_{ef} n^e g_b{}^f \tag{2.35}$$

where we have used $\hat{g}_{ef}n^e g_b{}^f = 0$. Decomposing the 5-dimensional energy-momentum tensor as

$$\widehat{T}_{ab} = -\Lambda_5 \hat{g}_{ab} + S_{ab}\delta(y) \tag{2.36}$$

where Λ_5 is a cosmological constant in the bulk, y is the coordinate of the 5th dimension, and

$$S_{ab} = -\lambda g_{ab} + T_{ab} \tag{2.37}$$

is the energy-momentum tensor on the brane located at $y = 0$. Above, λ is the vacuum energy (or cosmological constant) on the brane and T_{ab} satisfying $T_{ab}n^b = 0$ is the energy-momentum tensor of matter fields confined to the brane. Note that λ is the tension of brane in 5 dimensions. In general, S_{ab} should be evaluated by the variational principle of the 4-dimensional Lagrangian for matter fields because the normal matter except for gravity is assumed to be living only in the brane [3, 4].

The Israel's junction conditions [14] across the brane are

$$[g_{ab}] = 0, \tag{2.38}$$

$$[K_{ab}] = -\kappa_5^2 \left(S_{ab} - \frac{1}{3} g_{ab} S \right), \tag{2.39}$$

where $[X] := \lim_{X\to+0} - \lim_{X\to-0} = X^+ - X^-$ and S is the trace of S_{ab}. These conditions will be derived in the next section. Imposing the Z_2-symmetry across the brane, we get

$$K_{ab}^+ = -K_{ab}^- = -\frac{1}{2}\kappa_5^2 \left(S_{ab} - \frac{1}{3} g_{ab} S \right). \tag{2.40}$$

Substituting the explicit form of S_{ab} into Eq. (2.39), we find

$$[K_{ab}] = -\kappa_5^2 \left\{ \frac{1}{3}(\lambda - T) g_{ab} + T_{ab} \right\}. \tag{2.41}$$

Putting all the things together, we finally obtain the effective Einstein equation on the 3-brane:

$$G_{ab} = -\Lambda_4 g_{ab} + 8\pi G_N T_{ab} + \kappa_5{}^4 \Pi_{ab} - E_{ab} \tag{2.42}$$

where

$$\Lambda_{4} = \frac{1}{2}{\kappa_{5}}^{2}(\Lambda_{5} + \frac{1}{6}{\kappa_{5}}^{2}\lambda^{2}), \tag{2.43}$$

$$G_{N} = \frac{{\kappa_{5}}^{4}\lambda}{48\pi}, \tag{2.44}$$

$$\Pi_{ab} = -\frac{1}{4}T_{ae}{T_{b}}^{e} + \frac{1}{12}TT_{ab} + \frac{1}{8}g_{ab}T_{ef}T^{ef} - \frac{1}{24}g_{ab}T^{2}. \tag{2.45}$$

The salient feature of the above result is that the right-hand-side of Eq. (2.42) contains the term Π_{ab} which is bilinear in the energy-momentum tensor. As we will see, this term will play an important role in cosmology.

2.3 The Lagrangian Formulation of General Relativity

As was conventionally done, to obtain the correct form of the 4-dimensional Einstein equations from the Lagrangian formalism, the total divergence resulting from the variation of the Einstein-Hilbert action was normally set to zero. Thus we have to put some conditions to the boundary term depending on the physics that we are interested in [10]. On the other hand, it is worth noting that we might miss some crucial information that help us understand the dynamics of the models when we treat the boundary term vanished. The Israel's junction condition in the braneworld scenario is a good example [10, 11, 12, 13].

The "spacetime" manifolds normally used in physics are oriented manifolds, then the volume form associated to $\hat{g}$ is well defined. We can take the integrations over$\mathcal{M}$ and Σ with the natural volume elements $\sqrt{-\hat{g}}\,d^5x$ and $\sqrt{-g}\,d^4x$, respectively, in a local coordinate basis. Noting that the volume elements $\sqrt{-\hat{g}}\,d^5x$ and $\sqrt{-g}\,d^4x$ are always attached to the integrals $\int_{\mathcal{M}}$ and $\int_{\Sigma}$ respectively, so we can leave their notations from equations without lost of clarity.

The 5-dimensional Einstein-Hilbert action can be generalized from the 4-dimensional case as

$$S_{\hat{G}} = \int_{\mathcal{M}} \widehat{R}. \tag{2.46}$$

The variation of $S_{\widehat{G}}$ with respect to the metric $\hat{g}^{ab}$ is

$$\delta S_{\widehat{G}} = \int_{\mathcal{M}} \left[\sqrt{-\hat{g}}\hat{g}^{ab}\delta\widehat{R}_{ab} + \sqrt{-\hat{g}}\widehat{R}_{ab}\delta g^{ab} + \widehat{R}\delta\sqrt{-\hat{g}} \right]. \tag{2.47}$$

Then we use

$$\hat{g}^{ab}\delta\widehat{R}_{ab} = D^a v_a, \tag{2.48}$$

where

$$v_a = D^b(\delta\hat{g}_{ab}) - \hat{g}^{cd}D_a(\delta\hat{g}_{cd}), \tag{2.49}$$

and

$$\delta(\sqrt{-\hat{g}}) = -\frac{1}{2}\sqrt{-\hat{g}}\,\hat{g}_{ab}\delta\hat{g}^{ab}, \tag{2.50}$$

to obtain

$$\delta S_{\widehat{G}} = \int_{\mathcal{M}} D^a v_a + \int_{\mathcal{M}} \left(\widehat{R}_{ab} + \frac{1}{2}\widehat{R}\hat{g}_{ab} \right) \hat{g}^{ab}. \tag{2.51}$$

As the equations of motion come from varying the action with respect to the metric $\hat{g}^{ab}$, setting $\delta S_{\widehat{G}} = 0$ and assuming the boundary term equal to zero, we get

$$\widehat{R}_{ab} + \frac{1}{2}\widehat{R}\hat{g}_{ab} = 0, \tag{2.52}$$

which is the usual vacuum gravitational field equation.

In reality, the boundary term may not be neglected. The Stoke's theorem tells us that

$$\int_{\mathcal{M}} D^a v_a = \int_{\Sigma} v_a n^a \tag{2.53}$$

where Σ is the boundary of $\mathcal{M}$. Using the relation

$$v_a n^a = -n^a h^{bc} D_a(\delta\hat{g}_{bc}) \tag{2.54}$$

which we can be derived by assuming that $\delta\hat{g}_{ab} = 0$ on Σ and the variation of the trace of the extrinsic curvature of the boundary,

$$\delta K = \delta({g^a}_b D_a n^b) = \frac{1}{2}n^c g^{ad} D_c(\delta\hat{g}_{ad}), \tag{2.55}$$

Eq. (2.51) gets modified to

$$\delta S_{\widehat{G}} = \int_{\mathcal{M}} \widehat{G}_{ab} \delta \hat{g}^{ab} - 2 \int_{\Sigma} \delta K. \tag{2.56}$$

Using

$$\delta K_{ab} = \frac{1}{2}(K_{ab} - K g_{ab}) \tag{2.57}$$

as can be easily verified, we finally obtain

$$\delta S_{\widehat{G}} = \int_{\mathcal{M}} \widehat{G}_{ab} \delta \hat{g}^{ab} - \int_{\Sigma} (K_{ab} - K g_{ab}) \delta g^{ab} \tag{2.58}$$

with g_{ab} the metric on Σ as usual. In fact, the above equation continues to hold if we relax the boundary variation to $\delta g_{ab} = 0$ instead of $\delta \hat{g}_{ab} = 0$. This can be understood by using the fact that if $\delta g_{ab} = 0$ on the boundary, we can find a gauge transformation $\delta g_{ab} = D_{(a} l_{b)}$ with $l_b = 0$ on the boundary which makes $\delta \hat{g}_{ab} = 0$. Since Eq. (2.58) holds for all variations with $\delta \hat{g}_{ab} = 0$ on Σ and since all terms in it are invariant under such gauge transformations, this equation must continue to hold for variations which merely satisfy $\delta g_{ab} = 0$ on the boundary. Thus, the variation of $S_{\widehat{G}}$ with respect to variations with $\delta \hat{g}_{ab} = 0$ or $\delta g_{ab} = 0$ on the boundary contains an additional unwanted term. However, this can be remedied by simply adding the Gibbons-Hawking term [23], $2 \int_{\Sigma} K$, to $S_{\widehat{G}}$:

$$S_{\widehat{G}} \to S'_{\widehat{G}} = S_{\widehat{G}} + 2 \int_{\Sigma} K. \tag{2.59}$$

As the variation of $S'_{\widehat{G}}$ gives the Einstein equation, it is the appropriate action to use for general relativity

So far, we have considered Σ to be the boundary of the manifold $\mathcal{M}$. However, the theory can easily be extended to the case where Σ is a hypersurface (brane) embedded in $\mathcal{M}$ (bulk). In this case, Σ may be regarded as a common boundary of two pieces, $\mathcal{M}_1$ and $\mathcal{M}_2$, of $\mathcal{M}$, and we should simply add the actions of the form of the first term in Eq. (2.59) for these two pieces. Keeping in mind that the extrinsic curvatures of Σ in $\mathcal{M}_1$ and $\mathcal{M}_2$ are allowed to be different, the total action is

$$S_{\widehat{G}} = \sum_{i=1}^{2} \left(\int_{\mathcal{M}_i} R + 2 \int_{\Sigma_i} K \right) \tag{2.60}$$

where Σ_1 and Σ_2 are both Σ but viewed from different $\mathcal{M}_i$. In varying $S_{\widehat{G}}$, an important thing that we must keep in mind is that δg_{ab} needs not vanish on Σ as Σ is no longer the boundary of the whole manifold $\mathcal{M}$. A straightforward calculation reveals that the variation gets modified to

$$\delta \left(\int_{\mathcal{M}_i} R + 2\delta \int_{\Sigma_i} K \right) = - \int_{\mathcal{M}_i} \widehat{G}^{ab} \delta \hat{g}_{ab} - \int_{\Sigma_i} (K^{ab} - K g^{ab}) \delta g_{ab}. \tag{2.61}$$

If we add matter fields confined in the brane with the action

$$S_m = \int_{\Sigma} \mathcal{L}_m \tag{2.62}$$

whose variation is simply

$$\delta S_m = \int_{\Sigma} T^{ab} \delta g_{ab} \tag{2.63}$$

with the energy-momentum tensor

$$T^{ab} \equiv \frac{1}{\sqrt{-g}} \frac{\delta S_m}{\delta g_{ab}}, \tag{2.64}$$

then the variation of the total action, $\delta S = \delta S_g + \delta S_m$, vanishes if

$$\sum_{i=1}^{2} \int_{\Sigma_i} (K^{ab} - K g^{ab}) \delta g_{ab} = \int_{\Sigma} T^{ab} \delta g_{ab}. \tag{2.65}$$

Noting that $\int_{\Sigma_1} = - \int_{\Sigma_2}$, we finally obtain the Israel's matching condition

$$[K^{ab} - K g^{ab}] = T^{ab}. \tag{2.66}$$

We end this chapter with the general 5-dimensional action with a 3-brane Σ embedded:

$$\begin{aligned} S \;=\; & m_5^3 \left[\int_{\mathcal{M}} (\widehat{R} - 2\Lambda_5) + 2 \int_{\Sigma} K \right] + \int_{\mathcal{M}} \mathcal{L}_5(\hat{g}_{ab}, \Phi) + m_4^2 \int_{\Sigma} (R - 2\Lambda_4) \\ & + \int_{\Sigma} \mathcal{L}_4(g_{ab}, \phi). \end{aligned} \tag{2.67}$$

This action leads to all equations of motion considered in the rest of this thesis.

Chapter 3

The Cosmological Solutions

In this chapter, we will obtain the cosmological solutions to the braneworld model with the scalar field as the matter field, putting an amphasis on the inflation solutions. We begin by discussing the standard 4-dimensional cosmology and then proceed to discuss the braneworld inflation. This chapter will end with the discussion of the Hamilton-Jacobi formulations, both in the standard 4-dimensional model and the braneworld model.

3.1 Basic Equations in the Friedmann-Lemaitre-Robertson-Walker model

3.1.1 The Einstein Equations

From the cosmological principle, the universe is homogeneous and isotropic. Therefore, in describing the spacetime geometry of the universe on the large scale, we may choose the diffeomorphism, Eq. (2.1), such that the time coordinate is separated and orthogonal to the spacelike hypersurface of constant curvature with the conformal factor $a(t)$. The result is known as the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric [24, 25, 26, 27]:

$$ds^2 = g_{ab}dx^a dx^b = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{3.1}$$

with $k = 0, \pm 1$. Using the above metric in the Einstein equation, we obtain the relations between the scale factor $a(t)$ and the energy-momentum tensor T_{ab}. Roughly speaking, the evolution of the universe can be determined by the behavior of the scale factor $a(t)$. We now go into some detail. Recall the Einstein equation,

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R, \tag{3.2}$$

where R_{ab} and R are the Ricci tensor and the scalar curvature defined by

$$R_{ab} = R^c{}_{acb} \qquad R = R^a{}_a = g^{ab}R_{ab}, \tag{3.3}$$

respectively. As is well known, the Riemann curvature tensor that describes the curvature can be obtained from the Christoffel symbols as

$$R^d{}_{abc} = \partial_b\Gamma^d_{ca} - \partial_c\Gamma^d_{ba} + \Gamma^d_{bf}\Gamma^f_{ca} - \Gamma^d_{cf}\Gamma^f_{ba}, \tag{3.4}$$

where the Christoffel symbol is defined by

$$\Gamma^c_{ab} = \frac{1}{2}g^{cf}(\partial_a g_{bf} + \partial_b g_{af} - \partial_f g_{ab}). \tag{3.5}$$

One can compute the Christoffel symbols from the metric, Eq. (3.1), as follows:

$$\begin{aligned}
&\Gamma^0_{11} = \frac{a\dot{a}}{1-kr^2}\,, \qquad \Gamma^0_{22} = a\dot{a}r^2\,, \qquad \Gamma^0_{33} = a\dot{a}r^2\sin^2\theta\,,\\
&\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20}\,, = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a}\,,\\
&\Gamma^1_{22} = -r(1-kr^2)\,, \qquad \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta\,,\\
&\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}\,,\\
&\Gamma^2_{33} = -\sin\theta\cos\theta\,, \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta.
\end{aligned} \tag{3.6}$$

Then the nonzero components of the Ricci tensor and the scalar curvature are given by

$$\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a},\\
R_{11} &= \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2},\\
R_{22} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k),\\
R_{33} &= r^2(a\ddot{a} + 2\dot{a}^2 + 2k)\sin^2\theta,\\
R &= \frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k).
\end{aligned} \tag{3.7}$$

Introducing the maximally symmetric 3-dimensional metric γ_{ij} , with either negative, vanishing or positive spatial curvature (respectively labeled by $k = -1, 0$

or 1). The metric in Eq. (3.1) can be rewritten as

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j, \tag{3.8}$$

and the Ricci tensor for 3-dimensional spatial indices (i, j, k) takes the form

$$R_{ij} = \gamma_{ij}a^2 \left[\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + \frac{2k}{a^2}\right]. \tag{3.9}$$

We now compute the Einstein tensor, Eq. (3.2). The component G_{00} is

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2}g_{00}R \\ &= -3\frac{\ddot{a}}{a} - \frac{1}{2}(-1)\frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k) \\ &= 3H^2 + \frac{3k}{a^2}, \end{aligned} \tag{3.10}$$

where the Hubble parameter H is defined as

$$H = \frac{\dot{a}}{a} \tag{3.11}$$

and the spatial components of the Einstein tensor G_{ij} are

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{1}{2}g_{ij}R \\ &= \gamma_{ij}(a\ddot{a} + 2\dot{a}^2 + 2k) - \frac{1}{2}(a^2\gamma_{ij}\frac{6}{a^2}(a\ddot{a} + \dot{a}^2 + k) \\ &= a^2\gamma_{ij}\left(-2\frac{\ddot{a}}{a} - \frac{\dot{a}}{a^2}\right) - k\gamma_{ij}. \end{aligned} \tag{3.12}$$

3.1.2 The Continuity Equation

In many cosmological models we can assume that matter and energy contained in the universe are perfect fluid, which is isotropic in the local rest frame,

$$\begin{aligned} T_{00} &= \rho, \\ T_{ij} &= pg_{ij}, \end{aligned} \tag{3.13}$$

where ρ and p are the energy density and pressure respectively. In the generic inertial frame one can write down the energy-momentum tensor for the perfect fluid by using the 4-velocity of the fluid, u^a, as [19, 24, 25, 26, 27, 28]:

$$T_{ab} = (\rho + p)u_a u_b + pg_{ab}. \tag{3.14}$$

It is useful to consider the energy-momentum tensor in the comoving coordinates in which $u^a = (1, 0, 0, 0)$, where the trace is given by

$$T = T^a{}_a = -\rho + 3p. \tag{3.15}$$

From the local conservation of the energy-momentum tensor, $\nabla_a T^a{}_b = 0$, the $b = 0$ component in the FLRW spacetime takes the form

$$\begin{aligned} 0 &= \partial_a T^a{}_0 + \Gamma^a_{a0} T^0{}_0 - \Gamma^f_{a0} T^a{}_f, \\ &= \dot{\rho} + 3H(\rho + p). \end{aligned} \tag{3.16}$$

We thus arrive at the continuity equation:

$$\dot{\rho} = -3H(\rho + p). \tag{3.17}$$

Notice that if $p = -\rho$ (such as in the case in which the cosmological constant term is treated as the whole energy-momentum tensor), then the energy density ρ does not change with time.

3.1.3 The Friedmann Equation

We now obtain a useful differential equation describing the dynamics of the scale factor $a(t)$, known as the Friedmann equation [19, 24, 25, 26, 27, 28]: , from the relation between G_{00} and T_{00} in the Einstein equation,

$$G_{ab} = 8\pi G_N T_{ab} = \frac{1}{m_4^2} T_{ab}, \tag{3.18}$$

where the 4-dimensional gravitational constants, G_N and κ_4, are related to the 4-dimensional Planck mass m_4 by

$$\begin{aligned} \kappa_4^2 &= 8\pi G_N, && (3.19) \\ m_4 &= \kappa_4^{-1}. && (3.20) \end{aligned}$$

Now we can write the 0-0 component explicitly,

$$\begin{aligned} G_{00} &= 3H^2 + 3\frac{k}{a^2}, \\ &= \frac{1}{m_4^2}T_{00} = \frac{1}{m_4^2}(\rho + m_4^2\Lambda_4), \end{aligned} \tag{3.21}$$

where we have included the cosmological constant Λ_4 in the energy-momentum tensor, i.e., by replacing $T_{ab} \to T_{ab} + m_4^2\Lambda_4 g_{ab}$ [19, 24, 25, 26, 27, 29, 30]. We thus obtain the 4-dimensional Friedmann equation:

$$H^2 = \frac{\Lambda_4}{3} + \frac{1}{3m_4^2}\rho - \frac{k}{a^2}. \tag{3.22}$$

3.1.4 The Raychaudhuri Equation

As we have seen in the Chapter 2, the Raychaudhuri equation describes the evolution of the hypersurface along the congruence of geodesics passing through it [10, 11, 21, 22]. Thus this equation can be derived directly by considering the geodesic deviation of the congruence through the hypersurface without the necessity of using the Einstein equation . Interestingly, the Raychaudhuri equation may be obtained from the Friedmann equation, Eq. (3.22), and the continuity equation, Eq. (3.17), as follows:

$$\begin{aligned} 2H\dot{H} &= \frac{1}{3m_4^2}\dot{\rho} + \frac{2k\dot{a}}{a^3} \\ \dot{H} &= \frac{1}{2H}\left[\frac{1}{3m_4^2}3H(-\rho - p) + \frac{k}{a^2}2H\right] \\ &= -\frac{1}{2m_4^2}(\rho + p) + \frac{1}{3m_4^2}\rho - H^2 + \frac{\Lambda_4}{3} \\ &= -H^2 + \frac{\Lambda_4}{3} - \frac{1}{6m_4^2}(\rho + 3p). \end{aligned} \tag{3.23}$$

Above the continuity equation, Eq. (3.17), and the Friedmann equation, Eq. (3.22), were used on the second and third lines respectively. We thus obtain the Raychaudhuri equation:

$$\dot{H} = -H^2 + \frac{\Lambda_4}{3} - \frac{1}{6m_4^2}(\rho + 3p). \tag{3.24}$$

Note that the cosmological constant term, Λ_4, may be absorbed into the energy density ρ so that Eq. (3.24) can be reduced to the usual form obtained in the last chapter.

3.1.5 The Acceleration Equation

With the Raychaudhuri equation, the acceleration equation is easily derived as

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = \frac{\Lambda_4}{3} - \frac{1}{6{m_4}^2}(\rho + 3p). \tag{3.25}$$

Note that the trace of the energy-momentum tensor, $\rho + 3p$, plays an important role in the acceleration equation.

3.2 Basic Equations in the Braneworld Models

For the 5-dimensional braneworld models, we can apply the calculational methods used in last section to the 5-dimensional metric. However, the notable difference in this chapter is that we have to impose some conditions on the energy-momentum tensor for separating the energy-momentum tensor into two parts: one concentrated on the 3-brane and the other in the bulk. In addition, for the Randall-Sundrum models, we need to impose a Z_2 symmetry before solving the Einstein equations [3, 4].

3.2.1 The Einstein Equations in the Braneworld Models

In the braneworld models, we consider our universe as an infinitesimally thin wall of constant spatial curvature [8, 31, 32, 33, 34, 35, 36] . This means that the 4-dimensional hypersurface describing our universe is still the FLRW model. The general 5-dimensional metric which is compatible with the FLRW in four dimensions can be written in appropriate coordinate systems in the form

$$d\hat{s}^2 = \hat{g}'_{ab}(z^c)dz^a dz^b + a^2(z^c)\gamma_{ij}dx^i dx^j, \tag{3.26}$$

where $\hat{g}'_{ab}$ is a two-dimensional metric depending only on the two coordinates, z^a, which span time and the extra spatial dimension. For convenience we can choose a Gaussian normal coordinate system such that the time coordinate is orthogonal to the 5th dimension as

$$\hat{g}'_{ab}dz^a dz^b = -n^2(\tau, y)d\tau^2 + b^2(\tau, y)dy^2. \tag{3.27}$$

Then we can write the metric, Eq. (3.26), in a Gaussian normal coordinate system where the brane is located at $y = 0$, in the form

$$d\hat{s}^2 = \hat{g}_{ab}dx^a dx^b = -n^2(\tau, y)d\tau^2 + a^2(\tau, y)\gamma_{ij}dx^i dx^j + b^2(\tau, y)dy^2. \tag{3.28}$$

At $y = 0$, the metric Eq. (3.28) reduces to the FLRW metric,

$$ds^2 = -dt^2 + a^2\gamma_{ij}dx^i dy^j, \tag{3.29}$$

where we have set

$$n^2(\tau, 0)d\tau^2 = dt^2. \tag{3.30}$$

In this coordinate system, the components of the Einstein tensor read

$$\begin{aligned}
\widehat{G}_{00} &= 3\frac{\dot{a}^2}{a^2} + 3\frac{kn^2}{a^2} + \left[\frac{\dot{a}\dot{b}}{ab} - \frac{n^2}{b^2}\left(\frac{a''}{a} + \frac{a'}{a}\left(\frac{a'}{a} - \frac{b'}{b}\right)\right)\right], \\
\widehat{G}_{ij} &= \frac{a^2}{n^2}\gamma_{ij}\left(-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) - k\gamma_{ij} \\
&+ \frac{a^2}{n^2}\gamma_{ij}\left[-2\frac{\dot{a}\dot{n}}{an} + \frac{\dot{b}}{b}\left(-2\frac{\dot{a}}{a} + \frac{\dot{n}}{n}\right) - \frac{\ddot{b}}{b}\right] \\
&+ \frac{a^2}{b^2}\gamma_{ij}\left[\frac{a'}{a}\left(\frac{a'}{a} + 2\frac{n'}{n}\right) - \frac{b'}{b}\left(\frac{n'}{n} + 2\frac{a'}{a}\right) + 2\frac{a''}{a} + \frac{n''}{n}\right], \\
\widehat{G}_{05} &= 3\left(\frac{n'}{n}\frac{\dot{a}}{a} + \frac{a'}{a}\frac{\dot{b}}{b} - \frac{\dot{a}'}{a}\right), \\
\widehat{G}_{55} &= 3\left[\frac{a'}{a}\left(\frac{a'}{a} + \frac{n'}{n}\right) - \frac{b^2}{n^2}\left(\frac{\dot{a}}{a}\left(\frac{\dot{a}}{a} - \frac{\dot{n}}{n}\right) + \frac{\ddot{a}}{a}\right) - k\frac{b^2}{a^2}\right],
\end{aligned} \tag{3.31}$$

where a prime denotes a derivative with respect to the 5th coordinate y, and a dot denotes a derivative with respect to time τ. As in the last section, the dynamics of the scale factor $a(\tau)$ comes from the 5-dimensional field equation:

$$\widehat{G}_{ab} = \kappa_5^2 \widehat{T}_{ab}. \tag{3.32}$$

It is convenient to decompose the energy-momentum tensor into two parts located in the bulk and on the 3-brane repectively:

$$\widehat{T}^a{}_b = \widehat{T}^a{}_b|_{bulk} + T^a{}_b|_{brane}. \tag{3.33}$$

We are interested in the universe which is dominated by a perfect fluid, then the energy-momentum tensors of the bulk and the brane matter are assumed to be of the form

$$\widehat{T}^a{}_b|_{bulk} = \mathrm{diag}(-\rho_B, p_B, p_B, p_B, p_T), \tag{3.34}$$

$$T^a{}_b|_{brane} = \frac{\delta(y)}{b}\,\mathrm{diag}(-\rho_b, p_b, p_b, p_b, 0). \tag{3.35}$$

Note that the delta function in Eq. (3.35) comes from the discontinuous of the first deivative with respect to y. With the energy-momentum tensor on the brane, one obtains the Israel's junction conditions as

$$\left.\frac{[a']}{ab}\right|_{y=0} = -\frac{\kappa_5}{3}\rho_b, \tag{3.36}$$

$$\left.\frac{[n']}{nb}\right|_{y=0} = \frac{\kappa_5}{3}(2\rho_b + 3p_b). \tag{3.37}$$

In some braneworld models, especially the Randall-Sundrum models, the orbifold nature of the extra dimension (specifically a Z_2 symmetry across the brane) is imposed [3, 4, 37, 38, 39]:

$$\begin{aligned} a(y) &= a(-y), & a'(y) &= -a'(-y), \\ b(y) &= b(-y), & b'(y) &= -b'(-y), \\ n(y) &= n(-y), & n'(y) &= -n'(-y). \end{aligned} \tag{3.38}$$

Then Eqs. (3.36) and (3.37) become

$$\left.\frac{a'}{ab}\right|_{y=0} = -\frac{\kappa_5}{6}\rho_b, \tag{3.39}$$

$$\left.\frac{n'}{nb}\right|_{y=0} = \frac{\kappa_5}{6}(2\rho_b + 3p_b). \tag{3.40}$$

3.2.2 The Continuity Equation for the Braneworld Models

One commonly starts the derivation of the continuity equation from the relation $\nabla_a T^a{}_0 = 0$. However the Eqs. (3.39) and (3.40) provide a more interesting way to obtain the equation.

We begin by differentiating both sides of Eq. (3.39),

$$-\frac{\kappa_5}{6}\dot{\rho}_b = \left[\frac{\dot{a}'}{ab} - \frac{\dot{a}a'}{a^2 b} - \frac{a'\dot{b}}{ab^2}\right], \tag{3.41}$$

and multiplying Eq. (3.40) with $\dot{a}/a$,

$$\left(\frac{\dot{a}}{a}\right)\Big|_{y=0}\frac{\kappa_5}{6}(2\rho_b + 3p_b) = \frac{\dot{a}n'}{abn}\Big|_{y=0}. \tag{3.42}$$

Combine these results together, we obtain

$$\begin{aligned}\frac{\kappa_5^2}{6}\left[\dot{\rho}_b + 3H(\rho_b + p_b)\right]\Big|_{y=0} &= -\left[\frac{\dot{a}'}{ab} - \frac{a'\dot{a}}{a^2 b} - \frac{a'\dot{b}}{ab^2} - \frac{\dot{a}n'}{abn} + \frac{a'\dot{a}}{a^2 b}\right]\Bigg|_{y=0}, \\ &= \frac{1}{3b}\widehat{G}_{05}. \end{aligned} \tag{3.43}$$

We see that the density energy ρ_b and the pressure p_b satisfy the 4-dimensional continuity equation, Eq. (3.17), under the condition $\widehat{G}_{05} = 0$ which can be satisfied if $\widehat{T}_{05} = 0$, i.e., when there is no energy flow in the extra dimension.

3.2.3 The Friedmann Equation

From the Einstein equations, Eq. (3.32), and the Einstein tensor, Eq. (3.31), we can rewrite G_{00} in the form

$$F' = \frac{2a'a^3}{3}\kappa_5^2 \widehat{T}^0{}_0, \tag{3.44}$$

where the function F is defined as

$$F(\tau, y) \equiv \frac{(a'a)^2}{b^2} - \frac{(\dot{a}a)^2}{n^2} - ka^2. \tag{3.45}$$

Since $\widehat{T}^0{}_0 = -\rho_B$ is here independent of y, we can integrate Eq. (3.44) to obtain

$$F + \frac{\kappa_5^2}{6} a^4 \rho_B + \varepsilon = 0, \tag{3.46}$$

where ε is a constant of integration, or more explicitly,

$$\left(\frac{\dot{a}}{na}\right)^2 = \frac{\kappa_5^2}{6}\rho_B + \left(\frac{a'}{ab}\right)^2 - \frac{k}{a^2} + \frac{\varepsilon}{a^4}. \tag{3.47}$$

At $y = 0$, we can use the explicit form of a'/ab in Eq. (3.39) and choose $n(\tau, 0) = 1$ to obtain

$$\left.\frac{\dot{a}^2}{a^2}\right|_{y=0} = H^2 = \left.\left[\frac{\kappa_5^2}{6}\rho_B + \frac{\kappa_5^4}{36}\rho_b^2 - \frac{k}{a^2} + \frac{\varepsilon}{a^4}\right]\right|_{y=0}. \tag{3.48}$$

As we will be mainly concerned with the equations on the 3-brane, then all the following equations are understood to be evaluated on the brane ($y = 0$) and the notation $|_{y=0}$ will be dropped for convenience.

In the Randall-Sundrum scenario [3, 4], the brane tension λ plays the role of the cosmological constant on brane. Treating the cosmological constant as a part of the energy-momentum tensor on the brane, we may write [19, 25, 26, 27, 29, 34]

$$\rho_b = \rho + \lambda, \tag{3.49}$$

$$p_b = p - \lambda, \tag{3.50}$$

where ρ and p are the energy density and pressure of the matter concentrated on the brane. In the Randall-Sundrum braneworld, the gravitational constant κ_5 and the 5-dimensional Planck mass are related by

$$m_5 = \kappa_5^{-2/3}. \tag{3.51}$$

Another important property of the Randall-Sundrum models is that the bulk contains only the 5-dimensional cosmological constant, Λ_5, without any matter, so that

$$\Lambda_5 = \kappa_5^2 \rho_B. \tag{3.52}$$

Subsituting Eqs. (3.51) and (3.52) into the Friedmann equation on the 3-brane, Eq. (3.48), we obtain

$$\begin{aligned} H^2 &= \frac{\Lambda_5}{6} + \frac{1}{36m_5^6}(\rho^2 + 2\lambda\rho + \lambda^2) - \frac{k}{a^2} + \frac{\varepsilon}{a^4} \\ &= \frac{\Lambda_4}{3} + \frac{1}{3m_4^2}\rho + \frac{1}{36m_5^6}\rho^2 - \frac{k}{a^2} + \frac{\varepsilon}{a^4}, \end{aligned} \tag{3.53}$$

where the 4-dimenstional Planck mass, m_4, and the 4-dimensional cosmological constant, Λ_4, are defined as

$$m_4^2 = \frac{6}{\lambda}m_5^6, \tag{3.54}$$

$$\Lambda_4 = \frac{\Lambda_5}{2} + \frac{\lambda^2}{12m_5^6} = \frac{\Lambda_5}{2} + \frac{\lambda}{2m_4^2}. \tag{3.55}$$

We thus rewrite the Friedmann equation, Eq. (3.53), in terms of λ and m_4 as

$$H^2 = \frac{1}{3m_4^2}\rho\left(1 + \frac{\rho}{2\lambda}\right) + \frac{\Lambda_4}{3} - \frac{k}{a^2} + \frac{\varepsilon}{a^4}. \tag{3.56}$$

The last term in the above equation appears as a form of “dark radiation” since it is decoupled from matter on the brane. In the case of no dark radiation, $\varepsilon = 0$, and in the low energy limit, $\lambda \to \infty$, Eq. (3.56) reduces to the standard 4-dimensional Friedmann equation in the FLRW model.

3.2.4 The Raychaudhuri Equation

The Raychaudhuri equation for the braneworld can be obtained from the Friedmann equation, Eq. (3.53), as

$$\begin{aligned} \dot{H} &= \frac{1}{2H}\left[\dot{\rho}\left(\frac{1}{3m_4^2} + \frac{\rho}{18m_5^6}\right) + \frac{k(2H)}{a^2} - \frac{2\varepsilon(2H)}{a^4}\right] \\ &= -H^2 + \frac{\Lambda_4}{3} - \frac{1}{6m_4^2}(\rho + 3p) - \frac{1}{36m_5^6}(2\rho^2 + 3\rho p) - \frac{\varepsilon}{4}, \end{aligned} \tag{3.57}$$

or in terms of λ and m_4,

$$\dot{H} = -H^2 + \frac{\Lambda_4}{3} - \frac{1}{6m_4^2}\left[(\rho + 3p) + \frac{1}{\lambda}(2\rho^2 + 3\rho p)\right] - \frac{\varepsilon}{4}. \tag{3.58}$$

Again in the case of $\lambda \to \infty$ and $\varepsilon = 0$, this equation reduces to the FLRW Raychaudhuri equation in 4 dimensions.

3.2.5 The Acceleration Equation

From the usual relation $\ddot{a}/a = \dot{H} + H^2$ and the Raychaudhuri equation, Eq. (3.57), we obtain the acceleration equation:

$$\frac{\ddot{a}}{a} = \frac{\Lambda_4}{3} - \frac{1}{6m_4^2}\left[(\rho + 3p) + \frac{1}{\lambda}(2\rho^2 + 3\rho p)\right] - \frac{\varepsilon}{4}. \tag{3.59}$$

Notice that the extra term in the bracket apart from the trace of T_{ab} $(\rho + 3p)$ is multiplied by $1/\lambda$. This means that the effect from the bulk vanishes when $\lambda \to \infty$, i.e., in the low energy limit [34, 35, 36, 40].

3.3 Basic Equations for the Scalar Fields

Nowadays most people agree that the inflationary cosmology is the most satisfactory theory of the physics of early universe [41, 42, 43, 44, 45, 46]. Most of the inflationary models use the scalar field ϕ, known as the inflaton field, for driving the inflation. As the inflaton is just a regular scalar field, then its action in curved spacetime takes the generic form [28, 47]:

$$S_\phi = \int d^4x\sqrt{-g}\mathcal{L} = \int d^4x\sqrt{-g}\left[\frac{1}{2}\partial_a\phi\partial^a\phi + V(\phi)\right]. \tag{3.60}$$

From the action principle, the equation of motion can be obtained by setting the variation $\delta S = 0$ with the result

$$\ddot{\phi} + 3H\phi - \frac{\nabla^2\phi}{a^2} + V'(\phi) = 0, \tag{3.61}$$

where $V'(\phi)$ denotes the derivative with respect to ϕ. During inflation we assume a spatially homogeneous universe, then $\partial_i\phi = 0$ and the equation of motion for the inflaton reduces to

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \tag{3.62}$$

which is simply a harmonic oscillator equation with a damping term $3H\dot{\phi}$. The corresponding energy-momentum tensor of the scalar field is well known:

$$T_{ab} = \partial_a\phi\partial_b\phi - g_{ab}\mathcal{L}. \tag{3.63}$$

Then the energy density and the pressure of the scalar field are

$$\rho = T_{00} = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad (3.64)$$

$$p = T_{ii} = \frac{1}{2}\dot{\phi}^2 - V(\phi). \quad (3.65)$$

Making use of Eqs. (3.64) and (3.65), we can write $\dot{\phi}^2$ and $V(\phi)$ in terms of the energy density and the pressure as

$$\dot{\phi}^2 = \rho + p, \quad (3.66)$$

$$V(\phi) = \frac{1}{2}(\rho - p). \quad (3.67)$$

Differentiating the energy density in Eq. (3.64), we obtain

$$\dot{\rho} = \dot{\phi}\ddot{\phi} + V'\dot{\phi}. \quad (3.68)$$

Using Eq. (3.62), we finally arrive at the continuity equation for the scalar field,

$$\dot{\rho} + 3H(\rho + p) = 0. \quad (3.69)$$

Note that our derivation of the continuity equation holds for the scalar field with the specified energy momentum tensor and does not depend on the spacetime dimension.

3.4 The Slow-roll Parameters

From the definition of the Hubble parameter [18, 19, 28, 42],

$$\begin{aligned}\frac{\ddot{a}}{a} &= \dot{H} + H^2 \\ &= (1-\epsilon)H^2, \quad (3.70)\end{aligned}$$

where we have defined the slow-roll parameter ϵ as

$$\epsilon \equiv -\frac{\dot{H}}{H^2}. \quad (3.71)$$

Now the condition for inflation to take place, $\ddot{a} > 0$, is equivalent to the condition that this slow-roll parameter is less than one ($\epsilon < 1$). Thus the condition for inflation can be written in the various forms as

$$\text{INFLATION} \Longleftrightarrow \ddot{a} > 0 \Longleftrightarrow \frac{d}{dt}\frac{H^{-1}}{a} < 0 \Longleftrightarrow \epsilon < 1. \tag{3.72}$$

When one says that the inflation is a rapid expansion of the universe somehow the definition of "rapid expansion" is not clear. What is the expansion rate that should be called "rapid"? Then the comoving Hubble length, H^{-1}/a, gives more physical interpretation. If this characteristic scale is decreasing with time, it means the observable universe becomes smaller when viewed in the coordinates fixed with the expansion. That is inflation occurs.

Apart from ϵ, there are more slow-roll parameters that are commonly used for analyzing the inflation models. We end this section by quoting two slow-roll parameters, ϵ and η [18, 19, 28, 42]:

$$\epsilon \equiv -\frac{\dot{H}}{H^2}, \tag{3.73}$$

$$\eta \equiv -\frac{\ddot{\phi}}{H\dot{\phi}}. \tag{3.74}$$

3.5 Conditions of the Inflation in the FLRW Model

From the acceleration equation, Eq. (3.25),

$$\frac{\ddot{a}}{a} = -\frac{1}{3m_4^2}(\rho + 3p), \tag{3.75}$$

the inflation happens when $\ddot{a} > 0$ or equivalently

$$p < -\frac{1}{3}\rho. \tag{3.76}$$

We can rewrite this condition in terms of the scalar field ϕ and the potential $V(\phi)$ by making use of Eqs. (3.64) and (3.65) as

$$p < -\frac{1}{3}\rho \Longleftrightarrow \dot{\phi} < V. \tag{3.77}$$

We now consider the slow-roll conditions for the FLRW model. Taking the extremely case of the condition (3.77),

$$\dot{\phi} \ll V(\phi), \tag{3.78}$$

then when the scalar field dominates the Friedmann equation, Eq. (3.22), becomes

$$\begin{aligned} H^2 &= \frac{1}{3m_4^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right] \\ &\simeq \frac{1}{3m_4^2}V(\phi) \end{aligned} \tag{3.79}$$

where we have used $\simeq$ to denote equality within the slow-roll approximation. Assuming the potential $V(\phi)$ is flat or more quantitatively $\ddot{\phi} \ll 3H\dot{\phi}$, then the equation of motion of the scalar field gives another approximation:

$$3H\dot{\phi} \simeq -V'(\phi) \Longrightarrow \dot{\phi} \simeq -\frac{V'(\phi)}{3H}. \tag{3.80}$$

Rewriting these slow-roll conditions in terms of the potential $V(\phi)$:

$$\dot{\phi}^2 \ll V(\phi) \implies \frac{V'^2}{V} \ll H^2, \tag{3.81}$$

$$\ddot{\phi} \ll 3H\dot{\phi} \implies V'' \ll H^2. \tag{3.82}$$

Using the above results, one straightforwardly obtains the slow-roll parameters in the slow-roll approximation as

$$\epsilon_{GR} \equiv \frac{m_4^2}{2}\left(\frac{V'}{V}\right)^2, \tag{3.83}$$

$$\eta_{GR} \equiv m_4^2\left(\frac{V''}{V}\right). \tag{3.84}$$

3.6 The Braneworld Inflation

The Friedmann equation in the braneworld in the absence of the dark energy ($\varepsilon = 0$) can be written as [34, 40, 48]

$$H^2 = \frac{1}{3m_4^2}\rho\left[1 + \frac{\rho}{2\lambda}\right]. \tag{3.85}$$

From the acceleration equation in the braneworld cosmology, Eq. (3.59),

$$\frac{\ddot{a}}{a} = -\frac{1}{6m_4^2}\left[(\rho + 3p) + \frac{1}{\lambda}(2\rho^2 + 3\rho p)\right], \tag{3.86}$$

the condition that makes $\ddot{a} > 0$ reads

$$3p\left(1 + \frac{\rho}{\lambda}\right) + \rho\left(1 + \frac{2\rho}{\lambda}\right) < 0. \tag{3.87}$$

The inflation thus occurs when

$$\ddot{a} > 0 \Rightarrow p < -\left[\frac{\lambda + 2\rho}{\lambda + \rho}\right]\frac{\rho}{3}. \tag{3.88}$$

As $\rho/\lambda \to \infty$, the above condition reduces to the violation of the strong energy condition

$$p < -\frac{1}{3}\rho \quad \underset{\lambda \to \infty}{\Longleftarrow} \quad p < -\left[\frac{\lambda + 2\rho}{\lambda + \rho}\right]\frac{\rho}{3} \quad \underset{\rho/\lambda \to \infty}{\Longrightarrow} \quad p < -\frac{2}{3}\rho. \tag{3.89}$$

When the only matter in the universe is a scalar field, we can rewrite the condition for inflation, Eq. (3.87), as follows:

$$\begin{aligned} -3p - \rho - \frac{\rho}{\lambda}(3p + 2\rho) &> 0 \\ 2\rho - 3(\rho + p) + \frac{2\rho}{4\lambda}\left[2\rho - 6(\rho + p)\right] &> 0. \end{aligned} \tag{3.90}$$

In terms of $\dot{\phi}^2$ and V,

$$\begin{aligned} \dot{\phi}^2 + 2V - 3\dot{\phi}^2 + \frac{\dot{\phi}^2 + 2V}{4\lambda}(\dot{\phi}^2 + 2V - 6\dot{\phi}^2) &> 0 \\ -\dot{\phi}^2 + V + \frac{\dot{\phi}^2 + 2V}{8\lambda}(-5\dot{\phi}^2 + 2V) &> 0 \\ \dot{\phi}^2 - V + \frac{\dot{\phi}^2 + 2V}{8\lambda}(5\dot{\phi}^2 - 2V) &< 0. \end{aligned} \tag{3.91}$$

This condition reduces to $\dot{\phi}^2 < V(\phi)$ in the standard cosmology when

$$\dot{\phi}^2 + 2V \ll \lambda. \tag{3.92}$$

We now consider the slow-roll parameters in the braneworld model. From the Friedmann equation, Eq. (3.85), in case that the scalar field dominates, the slow-roll approximations are [40, 48, 49, 50, 51, 52]

$$H^2 \simeq \frac{1}{3m_4^2} V \left[1 + \frac{V}{2\lambda}\right], \tag{3.93}$$

$$\dot{\phi} \simeq -\frac{V'}{3H}, \tag{3.94}$$

where we have used $\simeq$ to denote equality within the slow-roll approximation as usual. To find the explicit form of the slow-roll parameter

$$\epsilon \equiv -\frac{\dot{H}}{H^2}$$

as in Eq. (3.71), we need to evaluate $\dot{H}$ by using the slow-roll approximation for H^2. Differentiating H^2 in Eq. (3.93), we get

$$\begin{aligned} H^2 &\simeq \frac{1}{3m_4^2}\left[V + \frac{V^2}{2\lambda}\right], \\ 2H\dot{H} &\simeq \frac{1}{3m_4^2}\left[V'\dot{\phi} + \frac{2VV'\dot{\phi}}{2\lambda}\right], \\ \dot{H} &\simeq \frac{1}{3m_4^2}\left(\frac{\dot{\phi}}{2H}\right) V' \left[\frac{\lambda + V}{\lambda}\right]. \end{aligned} \tag{3.95}$$

In the above equation, one may use Eq. (3.94) to express the $\dot{\phi}$ term as

$$\frac{\dot{\phi}}{2H} \simeq -\frac{V'}{6H^2}. \tag{3.96}$$

From the Friedmann equation, Eq. (3.93),

$$\frac{1}{H^2} \simeq (3m_4^2)\left(\frac{1}{V}\right)\left[\frac{2\lambda}{2\lambda + V}\right]. \tag{3.97}$$

With the above results, one obtains the slow-roll parameter in the slow-roll approximation:

$$\begin{aligned} \epsilon \equiv -\frac{\dot{H}}{H^2} &\simeq -\left(\frac{1}{m_4^2}\right)\left(-\frac{V'^2}{6}\right)\left(\frac{1}{H^2}\right)^2\left[\frac{\lambda + V}{\lambda}\right] \\ &\simeq (3m_4^2)\left(\frac{1}{6}\right)\left(\frac{V'}{V}\right)^2\left[\left(\frac{2\lambda}{2\lambda + V}\right)^2\left(\frac{\lambda + V}{\lambda}\right)\right] \\ &\simeq \frac{m_4^2}{2}\left(\frac{V'}{V}\right)^2\left[\frac{2\lambda(2\lambda + 2V)}{(2\lambda + V)^2}\right]. \end{aligned} \tag{3.98}$$

Similarly, one may obtain another slow-roll parameter η in the slow-roll approximation. We now quote our final forms of the slow-roll parameters for the braneworld inflation [40, 49, 50, 53, 54] :

$$\epsilon \equiv \frac{m_4^2}{2}\left(\frac{V'}{V}\right)^2\left[\frac{2\lambda(2\lambda+2V)}{(2\lambda+V)^2}\right] = \frac{m_4^2}{2}\left(\frac{V'}{V}\right)^2\left[\frac{1+V/\lambda}{(2+V/\lambda)^2}\right], \tag{3.99}$$

$$\eta \equiv \frac{m_4^2}{8\pi}\left(\frac{V''}{V}\right)\left[\frac{2\lambda}{2\lambda+V}\right]. \tag{3.100}$$

At low energy $V \ll \lambda$, the above result reduces to the slow-roll parameters in standard cosmology as it should,

$$\epsilon_{GR} \equiv \frac{m_4^2}{2}\left(\frac{V'}{V}\right)^2, \tag{3.101}$$

$$\eta_{GR} \equiv \frac{m_4^2}{8\pi}\left(\frac{V''}{V}\right). \tag{3.102}$$

At high energies $\lambda \ll V$, the parameters however become

$$\epsilon_{high\ E} = \epsilon_{GR}\left[\frac{\lambda}{V}\right], \tag{3.103}$$

$$\eta_{high\ E} = \eta_{GR}\left[\frac{2\lambda}{V}\right]. \tag{3.104}$$

By definition, the number of e-foldings during inflation is given by

$$\begin{aligned} N &= \int_{t_i}^{t_f} H dt \\ &= H\int_{\phi_i}^{\phi_f} \frac{d\phi}{\dot{\phi}}. \end{aligned} \tag{3.105}$$

In the slow-roll limit, the number of e-foldings becomes

$$\begin{aligned} N &\simeq -3H^2\int_{\phi_i}^{\phi_f} \frac{d\phi}{V'} \\ &\simeq -\frac{8\phi}{m_4^2}\int_{\phi_i}^{\phi_f} \frac{V}{V'}\left[1+\frac{V}{2\lambda}\right] d\phi. \end{aligned} \tag{3.106}$$

3.7 The Hamilton-Jacobi Formulation of Inflation in the FLRW Model

So far, we have considered inflations only within the slow-roll approximation. It is, however, well known that the slow-roll approximation is not always valid for all models of inflation. There are some models such as the inverted potential inflation models and the complicated cases of the hybrid inflations in which the slow-roll approximation breaks down at some point before the inflation ends [55]. However, there is another powerful way to analyze these models, that is, rewriting the equation of motion during inflation in terms of the functions of the scalar field ϕ rather than the time t. This method is called the Hamilton-Jacobi formulation [19, 55, 56, 57] which we now consider.

In the slow-roll approximation, we think of the inflaton as rolling down the potential hill in some specific direction. In other words, the scalar field in this approximation is a strictly increasing function of time. Consequently, we can treat the scalar field itself as the time variable to rewrite the whole set of inflation equations in the solvable forms. We begin with the Friedmann equation when the scalar field dominates,

$$H^2 = \frac{1}{6m_4^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right). \qquad (3.107)$$

Differentiating Eq. (3.107) with respect to time,

$$2H(\phi)H'(\phi)\dot{\phi} = \frac{1}{3m_4^2}(\ddot{\phi} + V'(\phi))\dot{\phi}, \qquad (3.108)$$

where the prime denotes differentiation with respect to ϕ, and substituting the equation of motion $V' + \ddot{\phi} = -3H\dot{\phi}$, we obtain

$$\dot{\phi} = -2m_4^2 H'(\phi). \qquad (3.109)$$

Substituting the expression for $\dot{\phi}$ in Eq. (3.109) into the Friedmann equation, Eq. (3.107), we obtain the Friedmann equation in the Hamilton-Jacobi form,

$$\begin{aligned} \frac{1}{2}\dot{\phi}^2 - 3m_4^2 H^2(\phi) &= -V(\phi) \\ [H'(\phi)]^2 - \frac{3}{2m_4^2}H^2(\phi) &= -\frac{1}{2m_4^4}V(\phi). \end{aligned} \qquad (3.110)$$

Given the explicit form of $V(\phi)$, this equation can be solved for $H(\phi)$ as a function of ϕ. Once the explicit form of $H(\phi)$ is obtained, Eq. (3.109) then gives $\dot{\phi}$ and hence the energy density ρ as functions of ϕ.

In the Hamilton-Jacobi formalism, the continuity equation can be rewritten by subsituting $\rho + p = \dot{\phi}^2$ into Eq. (3.69). We obtain

$$\dot{\rho} = -3H\dot{\phi}^2 \Longrightarrow \rho' = -3H\dot{\phi}, \tag{3.111}$$

where we have used $\dot{\rho} = \rho'\dot{\phi}$. Also from Eq. (3.109), we find

$$H' = -\frac{1}{2m_4^2}\dot{\phi} \Longrightarrow H'a' = -\frac{1}{2m_4^2}Ha. \tag{3.112}$$

where the relation $\dot{\phi}a' = \dot{a} = Ha$ has been used. The slow-roll parameter ϵ_H in this formalism can be derived in the usual way,

$$\begin{aligned} \frac{\ddot{a}}{a} &= H'\dot{\phi} + H^2 \\ &= H'(-2m_4^2 H') + H^2 \\ &= H^2(\phi)(1 - \epsilon_H(\phi)), \end{aligned} \tag{3.113}$$

where we have defined the slow-roll parameter as

$$\epsilon_H(\phi) \equiv 2m_4^2 \left(\frac{H'(\phi)}{H(\phi)}\right)^2. \tag{3.114}$$

Similarly, one can define

$$\eta_H(\phi) \equiv 2m_4^2 \frac{H''}{H}. \tag{3.115}$$

The relations between these parameters and the slow-roll parameters (ϵ, η) in the slow-roll limit are not hard to derive. The result is

$$\epsilon_H \simeq \epsilon, \tag{3.116}$$

$$\eta_H \simeq \eta - \epsilon. \tag{3.117}$$

Similarly, one can find the number of e-folding in the Hamilton-Jacobi formalism,

$$N = \int_{\phi_0}^{\phi_{end}} d\phi \frac{H}{\dot{\phi}} = -\frac{1}{2m_4^2}\int_{\phi_0}^{\phi_{end}} d\phi \frac{H}{H'}. \tag{3.118}$$

Note that we have used the fact that the Hubble parameter $H(\phi)$ is not exactly a constant but varies as the field evolves along the potential, in contast with many models which assume $H(\phi)$ constant during inflation. Thus it makes sense to have $H(\phi)$ instead of $V(\phi)$ in the expressions for the slow-roll parameters and the number of e-folding.

3.8 The Braneworld Cosmology in the Hamilton-Jacobi Framework

We now turn to the Hamilton-Jacobi formalism of the braneworld cosmology. Since the Friedmann equation in the braneworld model contains the square of the energy density term, then it looks difficult to find an exact solution. However, by introducing a new variable y as a function of the density, the simplest form of the Friedmann equation, Eq. (3.85), can be solved exactly [58]. As we shall see, this new variable turns out to play the same role as the Hubble parameter $H(\phi)$ does in the Hamilton-Jacobi formulation of the 4-dimensional case, and the method used in the previous section can be applied to formulate the slow-roll parameters γ, β analogous to ϵ, η. The mimic Hamilton-Jacobi formulation in the braneworld is convenient for the calculation of the energy scale of inflation [59].

We now go into the detail. Let $x = \rho/2\lambda$, then the Friedmann equation on the braneworld, Eq. (3.85), reads

$$H^2 = \frac{2\lambda}{3m_4^2}x(1+x). \qquad (3.119)$$

Since ρ can be expressed in terms of the scalar field ϕ and its time derivative, and the equation of motion of ϕ is the same as in the FLRW case, then the method analogous to the one used in the previous section leads to the equation analogous to Eq. (3.110) for determining H as a function of ϕ, which in turn leads to the expression of ρ (and hence x) as a function of ϕ.

Introducing a new variable y defined as

$$x = \frac{y^2}{1-y^2} \Longrightarrow x(x+1) = \frac{y^4}{(1-y^2)^2}, \tag{3.120}$$

we obtain

$$H(y) = \left(\frac{2\lambda}{3m_4^2}\right)^{1/2} \frac{y}{1-y^2}\,. \tag{3.121}$$

Note that the density energy ρ can be expressed as a function of y:

$$\rho = 2\lambda x = \frac{2\lambda y^2}{1-y^2}\,. \tag{3.122}$$

From

$$\dot{\phi} = -\frac{1}{3H}\rho' \tag{3.123}$$

where a prime denotes differentiation with respect to ϕ as usual, then using the above results, we find

$$\dot{\phi} = -\left(\frac{8\lambda m_4^2}{3}\right)^{1/2} \frac{y'}{1-y^2}\,. \tag{3.124}$$

We now obtain an important relation analogous to the relation $H'a' = -Ha/2m_4^2$ in the FLRW case:

$$\begin{aligned} y'a' &= -(a'\dot{\phi})\left(\frac{3}{8\lambda m_4^2}\right)^{1/2}(1-y^2) \\ &= -\left(\frac{3}{8\lambda m_4^2}\right)^{1/2}\left(\frac{2\lambda}{3m_4^2}\right)^{1/2}(\dot{a})\frac{y}{H(y)} \\ &= -\frac{1}{2m_4^2}ya \end{aligned} \tag{3.125}$$

which can be integrated to give

$$a(\phi) = \exp\left[-\frac{1}{2m_4^2}\int_{\phi_0}^{\phi_{end}} d\phi \frac{y}{y'}\right]. \tag{3.126}$$

This means the number of e-foldings can be obtained once we know the explicit form of $y(\phi)$.

Comparing the role of $y(\phi)$ in the above results with $H(\phi)$ in the FLRW case, we see that they play very similar roles. Thus the slow-roll parameters in the

Hamilton-Jacobi formalism for the braneworld model can be defined as follows:

$$\epsilon_{H}(\phi) = 2m_4^2\frac{H'^2}{H^2} \implies \beta_{H} \equiv 2m_4^2\frac{y'^2}{y^2}, \tag{3.127}$$

$$\eta_{H}(\phi) = 2m_4^2\frac{H''}{H} \implies \gamma_{H} \equiv 2m_4^2\frac{y''}{y}. \tag{3.128}$$

Their relationships are

$$\gamma_{H} = \beta_{H} - \sqrt{\frac{m_4^2}{2}}\frac{\beta'_{H}}{\sqrt{\beta_{H}}}, \tag{3.129}$$

$$\beta_{H} = \left(\frac{1-y^2}{1+y^2}\right)\epsilon, \tag{3.130}$$

where

$$\epsilon = -H'/H^2 \tag{3.131}$$

is the slow-roll parameter in the FLRW Hamilton-Jacobi formalism. The number of e-foldings can similarly be obtained:

$$\begin{aligned} N &= \frac{1}{2m_4^2}\int_{\phi_0}^{\phi_{end}} d\phi \frac{y}{y'} \\ &= \sqrt{\frac{1}{2m_4^2}}\int_{\phi_0}^{\phi_{end}} \frac{d\phi}{\sqrt{\beta_{H}(\phi)}}. \end{aligned} \tag{3.132}$$

To end this section, we note that the parameters β_{H} and γ_{H} are useful in analyzing the behaviour of the inflationary solutions [59]. However, Ramirez and Liddle [60] found that the numerical values of funcions β_{H} and γ_{H} defined above do not satisfy the expected values of the slow-roll parameters, for example, β_{H} is not exactly equal to one when the inflation ends. So they modified these parameters by adding some correction terms in order to get the better numerical values close to the standard ones.

Chapter 4

The Cosmological Perturbations

Having obtained the cosmological solutions in the previous chapter, we now turn to the issue of cosmological perturbations which is a very important topic in cosmology [19, 20, 61, 62, 63, 64]. We will begin with the discussion of the perturbations for the flat FLRW model with the inflaton as the matter field. We then proceed to discuss the issue of gauge invariance. Finally, we will end this chapter with the discussion of the density perturbation in braneworld cosmology [65, 66, 67, 68, 69, 70].

4.1 The Metric Perturbations for the Flat FLRW Model

Consider the flat FLRW model (3.1) which describes the universe during inflation. In terms of the conformal time τ defined by $d\tau = dt/a$, the metric reads

$$ds^2 = g^{(0)}_{ab} dx^a dx^b = -a^2(\tau)\left[d\tau^2 - \delta_{ij} dx^i dx^j\right]. \tag{4.1}$$

From the definition of τ, we find [19, 28]

$$a' = \dot{a}a \Longrightarrow H = \frac{\dot{a}}{a} = \frac{a'}{a^2} = \frac{\mathcal{H}}{a} \tag{4.2}$$

where a prime denotes differentiation with respect to τ. Here $\mathcal{H}$ plays the same role as the Hubble constant H. The general form of the first order perturbed metric is [28, 61, 20, 71]

$$\begin{aligned} ds^2 &= a^2(\tau)\left[-(1+2A)d\tau^2 + 2(B_{|i} - S_i)d\tau dx^i + ((1-2\psi)\delta_{ij}\right. \\ &\quad \left. + E_{|ij} + F_{i|j} + E_{ij})dx^i dx^j\right] \end{aligned} \tag{4.3}$$

where we have defined $B_{|i} \equiv \nabla_i B$, $F_{i|j} \equiv \nabla_j F_i$ and a symmetric traceless tensor

$$E_{|ij} = (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)E. \tag{4.4}$$

Here, S_i and F_i are divergenceless ($\nabla \cdot S = \nabla \cdot F = 0$) and E_{ij} is transverse traceless ($\nabla_i E^{ij} = E^i{}_i = 0$). The functions A, B, ψ and E are known as the scalar perturbations, the vectors S_i and F_i are the vector perturbations, and E_{ij} is called the tensor perturbation. All of these functions are treated as small perturbations so that, from now on, only terms up to the linear order in these functions will be retained in all calculations. In this thesis, we will however pay our attention only to the scalar perturbations, leaving the topics of vector and tensor perturbations untouch.

4.2 The Scalar Perturbation of the Einstein Tensor

Paying attention only to the scalar perturbations, the metric tensor in Eq. (4.3) can be expressed in the matrix form as

$$g_{ab} = g^{(0)}_{ab} + \delta g_{ab} = a^2 \begin{pmatrix} -(1+2A) & B_{|i} \\ B_{|i} & (1-2\psi)\delta_{ij} + E_{|ij} \end{pmatrix}. \tag{4.5}$$

We then can find the inverse metric from the identity

$$g^{ac} g_{cb} = \delta^a{}_b \tag{4.6}$$

with the result

$$g^{ab} = g^{(0)ab} + \delta g^{ab} = a^2 \begin{pmatrix} -1+2A & B_|{}^i \\ B_|{}^i & (1+2\psi)\delta^{ij} - E_|{}^{ij} \end{pmatrix}. \tag{4.7}$$

To write the Einstein equation in the perturbed metric, we need to evaluate the perturbed Einstein tensor to the linear order in perturbations. Our first step is to evaluate the perturbed Christoffel symbol, $\Gamma^a_{bc} = {}^{(0)}\Gamma^a_{bc} + \delta\Gamma^a_{bc}$, where

$$\delta\Gamma^a_{bc} = \frac{1}{2}\delta g^{ae}\left(\frac{\partial g^{(0)}_{ec}}{\partial x^b} + \frac{\partial g^{(0)}_{be}}{\partial x^c} - \frac{\partial g^{(0)}_{bc}}{\partial x^e}\right) + \frac{1}{2} g^{(0)ae}\left(\frac{\partial \delta g_{ec}}{\partial x^b} + \frac{\partial \delta g_{be}}{\partial x^c} - \frac{\partial \delta g_{bc}}{\partial x^e}\right). \tag{4.8}$$

and the unperturbed connections take the forms

$$
\begin{aligned}
{}^{(0)}\Gamma^0_{00} = \frac{a'}{a} \qquad &; \qquad {}^{(0)}\Gamma^i_{0j} = \frac{a'}{a}\delta^i{}_j \\
{}^{(0)}\Gamma^0_{ij} = \frac{a'}{a}\delta_{ij} \qquad &; \quad {}^{(0)}\Gamma^i_{00} = {}^{(0)}\Gamma^0_{0i} = {}^{(0)}\Gamma^i_{jk} = 0.
\end{aligned} \tag{4.9}
$$

To calculate $\delta\Gamma^a_{bc}$ requires some quite tedious but straightforward calculations. For Γ^i_{jk}:

$$
\begin{aligned}
\delta\Gamma^i_{jk} &= \frac{1}{2}\delta g^{ie}\left(\frac{\partial g^{(0)}_{ek}}{\partial x^j} + \frac{\partial g^{(0)}_{je}}{\partial x^k} - \frac{\partial g^{(0)}_{jk}}{\partial x^e}\right) + \frac{1}{2}g^{(0)ie}\left(\frac{\partial \delta g_{ek}}{\partial x^j} + \frac{\partial \delta g_{je}}{\partial x^k} - \frac{\partial \delta g_{jk}}{\partial x^e}\right) \\
&= \frac{1}{2}\delta g^{i0}\left(\frac{\partial g^{(0)}_{0k}}{\partial x^j} + \frac{\partial g^{(0)}_{j0}}{\partial x^k} - \frac{\partial g^{(0)}_{jk}}{\partial x^0}\right) + \frac{1}{2}g^{(0)i0}\left(\frac{\partial \delta g_{0k}}{\partial x^j} + \frac{\partial \delta g_{j0}}{\partial x^k} - \frac{\partial \delta g_{jk}}{\partial x^0}\right) \\
&\quad + \frac{1}{2}\delta g^{il}\left(\frac{\partial g^{(0)}_{lk}}{\partial x^j} + \frac{\partial g^{(0)}_{jl}}{\partial x^k} - \frac{\partial g^{(0)}_{jk}}{\partial x^l}\right) + \frac{1}{2}g^{(0)il}\left(\frac{\partial \delta g_{lk}}{\partial x^j} + \frac{\partial \delta g_{jl}}{\partial x^k} - \frac{\partial \delta g_{jk}}{\partial x^l}\right) \\
&= \frac{1}{2}\left(\frac{B_|{}^i}{a^2}\right)\left\{2a^2\partial_j\partial_k B - [2aa'((1-2\psi)\delta_{jk} + E_{|jk}) - 2a^2\psi'\delta_{jk} + a^2 E'{}_{|jk}]\right\} \\
&\quad + \frac{1}{2}\left(\frac{B_|{}^i}{a^2}\right)\left\{2a^2\partial_j\partial_k B - [2aa'(-2\psi\delta_{jk} + E_{|jk}) - 2a^2\psi'\delta_{jk} + a^2 E'{}_{|jk}]\right\} \\
&\quad + \frac{1}{2a^2}(2\psi\delta^{il} - E_|{}^{il})a^2\left\{-2\psi_{|j}\delta_{lk} + \frac{\partial E_{|lk}}{\partial x^j} - 2\psi_{|k}\delta_{jl} + \frac{\partial E_{|jl}}{\partial x^k} + 2\psi_{|l}\delta_{jk}\right. \\
&\quad \left. - \frac{\partial E_{|jk}}{\partial x^l}\right\} + \frac{1}{2a^2}\left((1+2\psi)\delta^{il} - E_|{}^{il}\right)a^2\left\{ -2\psi_{|j}\delta_{lk} + \frac{\partial E_{|lk}}{\partial x^j} - 2\psi_{|k}\delta_{jl}\right. \\
&\quad \left. + \frac{\partial E_{|jl}}{\partial x^k} + 2\psi_{|l}\delta_{jk} - \frac{\partial E_{|jk}}{\partial x^l}\right\}.
\end{aligned}
$$

Keeping terms up to the first order in the perturbations, we obtain

$$
\begin{aligned}
\delta\Gamma^i_{jk} &= -\frac{a'}{a}B_|{}^i\delta_{jk} - \psi_{|j}\delta^i{}_k - \psi_{|k}\delta^i{}_j + \psi_|{}^i\delta_{jk} \\
&\quad + \frac{1}{2}\partial_j E_|{}^i{}_k + \frac{1}{2}\partial_k E_|{}^i{}_j - \frac{1}{2}\partial^i E_{|jk},
\end{aligned} \tag{4.10}
$$

which is just one component of $\delta\Gamma^a_{bc}$ that we have to compute. By the same calculation, we can obtain the remaining components as follows.

For Γ^0_{00}:

$$
\delta\Gamma^0_{00} = \frac{1}{2}\left(\frac{2A}{a^2}\right)\left[-2aa'(1+2A) - 2a^2A'\right] + \frac{1}{2a^2}(-1+2A)
$$

$$\left[-4aa'A - 2a^2A'\right] + \frac{1}{2}\left(\frac{B_|{}^i}{a^2}\right)\left[4aa'B_{|i} + 2a^2B'{}_{|i} + 2a^2A_{|i}\right]$$
$$+\frac{1}{2}\left(\frac{B_|{}^i}{a^2}\right)\left[4aa'B_{|i} + 2a^2B'{}_{|i} + 2a^2A_{|i}\right].$$

To the first order in perturbations,

$$\delta\Gamma^0_{00} = A'. \tag{4.11}$$

For Γ^0_{0i}:

$$\begin{aligned}
\delta\Gamma^0_{0i} &= \frac{1}{2}\left(\frac{2A}{a^2}\right)\left[-2a^2A_|i\right] + \frac{1}{2a^2}(-1+2A)\left[-2a^2A_|i\right] \\
&+\frac{1}{2}\left(\frac{B_|{}^j}{a^2}\right)\left[2aa'((1-2\psi)\delta_{ji} + E_{|ji}) - 2a^2\psi'\delta_{ji} + a^2E'{}_{|ji}\right] \\
&+\frac{1}{2}\left(\frac{B_|{}^j}{a^2}\right)\left[2aa'(-2\psi\delta_{ji} + E_{|ji}) - 2a^2\psi'\delta_{ji} + a^2E'{}_{|ji}\right].
\end{aligned}$$

To the first order in perturbations,

$$\delta\Gamma^0_{0i} = A_{|i} + \frac{a'}{a}B_{|i}. \tag{4.12}$$

For Γ^i_{00}:

$$\begin{aligned}
\delta\Gamma^i_{00} &= \frac{1}{2}\left(\frac{B_|{}^i}{a^2}\right)\left[-2aa'(1+2A) - 2a^2A'\right] \\
&+\frac{1}{2}\left(\frac{B_|{}^i}{a^2}\right)\left[-4aa'A - 2a^2A'\right] \\
&+\frac{1}{2a^2}(2\psi\delta^{ij} - E_|{}^{ij})\left[4aa'B_{|j} + 2a^2B'{}_{|j} + 2a^2A_{|j}\right] \\
&+\frac{1}{2a^2}\left(1+2\psi)\delta^{ij} - E_|{}^{ij}\right)\left[4aa'B_{|j} + 2a^2B'{}_{|j} + 2a^2A_{|j}\right].
\end{aligned}$$

To the first order in perturbations,

$$\delta\Gamma^i_{00} = \frac{a'}{a}B_|{}^i + B'_|{}^i + A_|{}^i. \tag{4.13}$$

Finally, for Γ^0_{ij}:

$$\delta\Gamma^0_{ij} = \frac{1}{2}\left(\frac{2A}{a^2}\right)\Big\{2a^2\partial_i\partial_jB - [2aa'((1-2\psi)\delta_{ij} + E_{|ij})$$

$$
\begin{aligned}
&-2a^2\psi'\delta_{ij} + a^2 E'_{|ij}]\Big\} \\
&+\frac{1}{2a^2}(-1+2A)\Big\{2a^2\partial_i\partial_j B - [2aa'(-2\psi\delta_{ij} + E_{|ij}) \\
&-2a^2\psi'\delta_{ij} + a^2 E'_{|ij}]\Big\} \\
&+\frac{1}{2}\left(\frac{B_|{}^k}{a^2}\right)a^2\Big\{-2\psi_{|i}\delta_{kj} + \frac{\partial E_{|kj}}{\partial x^i} \\
&-2\psi_{|j}\delta_{ik} + \frac{\partial E_{|ik}}{\partial x^j} + 2\psi_{|k}\delta_{ij} - \frac{\partial E_{|ij}}{\partial x^k}\Big\} \\
&+\frac{1}{2}\left(\frac{B_|{}^k}{a^2}\right)a^2\Big\{-2\psi_{|i}\delta_{kj} + \frac{\partial E_{|kj}}{\partial x^i} \\
&-2\psi_{|j}\delta_{ik} + \frac{\partial E_{|ik}}{\partial x^j} + 2\psi_{|k}\delta_{ij} - \frac{\partial E_{|ij}}{\partial x^k}\Big\}.
\end{aligned}
$$

To the first order in perturbations,

$$
\delta\Gamma^0_{ij} = -2\frac{a'}{a}A\delta_{ij} - \partial_i\partial_j B - 2\frac{a'}{a}\psi\delta_{ij} + \frac{a'}{a}E_{|ij} - \psi'\delta_{ij} + \frac{1}{2}E'_{|ij}. \tag{4.14}
$$

Now we are so close to obtain the perturbed Einstein tensor,

$$
G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R \Longrightarrow \delta G_{ab} = \delta R_{ab} - \frac{1}{2}\delta g_{ab}{}^{(0)}R - \frac{1}{2}g^{(0)}_{ab}\delta R \tag{4.15}
$$

where ${}^{(0)}R$ is the unperturbed scalar curvature. Our remaining tasks are thus the calculations of the unperturbed ${}^{(0)}R_{ab}$ and the variations δR_{ab} and δR. For the unperturbed ${}^{(0)}R_{ab}$, we can obtain it from the previous results in the last chapter by replacing $a' \to \dot{a}a$, or easily recalculate it again with the result:

$$
\begin{aligned}
{}^{(0)}R_{00} &= {}^{(0)}\Gamma^e_{00|e} - {}^{(0)}\Gamma^e_{0e|0} + {}^{(0)}\Gamma^e_{de}\,{}^{(0)}\Gamma^d_{00} - {}^{(0)}\Gamma^e_{d0}\,{}^{(0)}\Gamma^d_{0e} \\
&= -3\left[-\left(\frac{a'}{a}\right)^2 + \frac{a''}{a}\right] + \left(3\frac{a'}{a}\right)\left(\frac{a'}{a}\right) - 3\left(\frac{a'}{a}\right)^2 \\
&= -3\frac{a''}{a} + 3\left(\frac{a'}{a}\right)^2 \\
{}^{(0)}R_{0i} &= {}^{(0)}\Gamma^e_{0i|e} - {}^{(0)}\Gamma^e_{ie|0} + {}^{(0)}\Gamma^e_{de}\,{}^{(0)}\Gamma^d_{0i} - {}^{(0)}\Gamma^e_{di}\,{}^{(0)}\Gamma^d_{0e} \\
&= {}^{(0)}\Gamma^j_{0i|j} = 0 \\
{}^{(0)}R_{ij} &= \left[\frac{a''}{a} - \left(\frac{a'}{a}\right)^2\right]\delta_{ij}.
\end{aligned}
$$

With the above result, we obtain the unperturbed scalar curvature as

$$ {}^{(0)}R = \frac{6}{a^2}\frac{a''}{a}. \tag{4.16} $$

As for the perturbation of the Ricci tensor, we have

$$ \begin{aligned} \delta R_{ab} &= \delta\Gamma^e_{ab|e} - \delta\Gamma^e_{be|a} + \delta\Gamma^e_{de}\,{}^{(0)}\Gamma^d_{ab} + {}^{(0)}\Gamma^e_{de}\delta\Gamma^d_{ab} \\ &\quad -\delta\Gamma^e_{db}\,{}^{(0)}\Gamma^d_{ae} - {}^{(0)}\Gamma^e_{db}\delta\Gamma^d_{ae}. \end{aligned} \tag{4.17} $$

To the first order in perturbations, we find δR_{ab} and δR as

$$ \delta R_{00} = \frac{a'}{a}\partial_i\partial^i B + \partial_i\partial^i B' + \partial_i\partial^i A + 3\psi'' + 3\frac{a'}{a}\psi' + 3\frac{a'}{a}A' \tag{4.18} $$

$$ \delta R_{0i} = \frac{a''}{a}B_{|i} + \left(\frac{a'}{a}\right)^2 B_{|i} + 2\psi'_{|i} + 2\frac{a'}{a}A_{|i} + \frac{1}{2}\partial_k E'_{|}{}^{k}{}_{i} \tag{4.19} $$

$$ \begin{aligned} \delta R_{ij} &= \left(-\frac{a'}{a}A' - 5\frac{a'}{a}\psi' - 2\frac{a''}{a}A - 2\left(\frac{a'}{a}\right)^2 A - 2\frac{a''}{a}\psi\right)\delta_{ij} \\ &\quad - \left(2\left(\frac{a'}{a}\right)^2\psi - \psi'' + \partial_k\partial^k\psi - \frac{a'}{a}\partial_k\partial^k B\right)\delta_{ij} \\ &\quad -\partial_i\partial_j B' + \frac{a'}{a}E'_{|ij} + \frac{a''}{a}E_{|ij} + \left(\frac{a'}{a}\right)^2 E_{|ij} \\ &\quad +\frac{1}{2}E''_{|ij} + \partial_i\partial_j\psi - \partial_i\partial_j A - 2\frac{a'}{a}\partial_i\partial_j B \\ &\quad +\frac{1}{2}\partial_k\partial_i E_{|}{}^{k}{}_{j} + \frac{1}{2}\partial_k\partial_j E_{|}{}^{k}{}_{i} - \frac{1}{2}\partial_k\partial^k E_{|ij} \end{aligned} \tag{4.20} $$

$$ \begin{aligned} \delta R &= \frac{1}{a^2}\Big[-6\frac{a'}{a}\partial_i\partial^i B - 2\partial_i\partial^i B' - 2\partial_i\partial^i A - 6\psi'' - 6\frac{a'}{a}A' \\ &\quad -18\frac{a'}{a}\psi' - 12\frac{a''}{a}A + 4\partial_i\partial^i\psi + \partial_k\partial^i E_{|}{}^{k}{}_{i}\Big]. \end{aligned} \tag{4.21} $$

Putting all the results together, we obtain the perturbation of the Einstein tensor:

$$ \delta G_{00} = -2\frac{a'}{a}\partial_i\partial^i B - 6\frac{a'}{a}\psi' + 2\partial_i\partial^i\psi + \frac{1}{2}\partial_k\partial^i E_{|}{}^{k}{}_{i} \tag{4.22} $$

$$ \delta G_{0i} = -2\frac{a''}{a}\partial_i B + \left(\frac{a'}{a}\right)^2\partial_i B + 2\partial_i\psi' + \frac{1}{2}\partial_k E'_{|}{}^{k}{}_{i} + 2\frac{a'}{a}\partial_i A \tag{4.23} $$

$$ \begin{aligned} \delta G_{ij} &= \Big[2\frac{a'}{a}A' + 4\frac{a'}{a}\psi' + 4\frac{a''}{a}A - 2\left(\frac{a'}{a}\right)^2 A + 4\frac{a''}{a}\psi - 2\left(\frac{a'}{a}\right)^2\psi + 2\psi'' \\ &\quad -\partial_k\partial^k\psi + 2\frac{a'}{a}\partial_k\partial^k B + \partial_k\partial^k B' + \partial_k\partial^k A + \frac{1}{2}\partial_k\partial^m E_{|}{}^{k}{}_{m}\Big]\delta_{ij}. \end{aligned} \tag{4.24} $$

Using the explicit form of the unperturbed Einstein tensor,

$$
\begin{aligned}
{}^{(0)}G_{00} &= 3\left(\frac{a'}{a}\right)^2 && (4.25)\\
{}^{(0)}G_{0i} &= 0 && (4.26)\\
{}^{(0)}G_{ij} &= \left(-2\frac{a''}{a}+\left(\frac{a'}{a}\right)^2\right)\delta_{ij} && (4.27)
\end{aligned}
$$

we can obtain $\delta G^a{}_b$ from

$$\delta G^a{}_b = \delta\left(g^{ae}G_{eb}\right) = \delta g^{ae\,(0)}G_{eb} + g^{(0)ae}\delta G_{eb} \qquad (4.28)$$

with the result:

$$
\begin{aligned}
\delta G^0{}_0 &= 6\left(\frac{a'}{a}\right)^2 A + 6\frac{a'}{a}\psi' + 2\frac{a'}{a}\partial_i\partial^i B - 2\partial_i\partial^i\psi - \frac{1}{2}\partial_k\partial^i E_|{}^k{}_i && (4.29)\\
\delta G^0{}_i &= -2\frac{a'}{a}\partial_i A - 2\partial_i\psi' - \frac{1}{2}\partial_k E'_|{}^k{}_i && (4.30)\\
\delta G^i{}_j &= \Big[2\frac{a'}{a}A' + 4\frac{a''}{a}A - 2\left(\frac{a'}{a}\right)^2 A + \partial_i\partial^i A + 4\frac{a'}{a}\psi' + 2\psi'' \\
&\quad -\partial_i\partial^i\psi + 2\frac{a'}{a}\partial_i\partial^i B + \partial_i\partial^i B' + \frac{1}{2}\partial_k\partial^m E_|{}^k{}_m\Big]\delta^i{}_j \\
&\quad -\partial^i\partial_j A + \partial^i\partial_j\psi - 2\frac{a'}{a}\partial^i\partial_j B - \partial^i\partial_j B' + \frac{a'}{a}E'_|{}^i{}_j + \frac{1}{2}E''_|{}^i{}_j \\
&\quad +\frac{1}{2}\partial_k\partial^i E_|{}^k{}_j + \frac{1}{2}\partial_k\partial_j E_|{}^{ik} - \frac{1}{2}\partial_k\partial^k E_|{}^i{}_j\,. && (4.31)
\end{aligned}
$$

4.3 The Perturbed Energy-Momentum Tensor

Having calculated the perturbed Einstein tensor, we now turn to considering the source that causes the perturbations, the fluctuations of matter. As this thesis is mainly concerned with the perturbations in the inflationary epoch, we pay our attention to the matter that dominates in this epoch, i.e., the inflaton field. Then during inflation, the energy-momentum tensor of the inflaton is

$$T_{ab} = \phi_{|a}\phi_{|b} - g_{ab}\left(\frac{1}{2}g^{ed}\phi_{|e}\phi_{|d} - V(\phi)\right). \qquad (4.32)$$

The explicit forms of its components, with the conformal time as the time coordinate, are

$$T_{00} = \frac{1}{2}\phi'^2 + V(\phi)a^2 \qquad (4.33)$$

$$\begin{aligned} T_{0i} &= 0 & (4.34) \\ T_{ij} &= \left(\frac{1}{2}\phi'^2 - V(\phi)a^2\right)\delta_{ij}. & (4.35) \end{aligned}$$

Allowing the inflaton field to fluctuate, i.e., letting $\phi \to \phi + \delta\phi$, causes the fluctuations of the energy-momentum tensor,

$$\begin{aligned} \delta T_{ab} &= (\delta\phi)_{|a}\phi_{|b} + \phi_{|a}(\delta\phi)_{|b} - \delta g_{ab}\left(\frac{1}{2}g^{ed}\phi_{|e}\phi_{|d} - V(\phi)\right) \\ &\quad -g_{ab}\left(\frac{1}{2}\delta g^{ed}\phi_{|e}\phi_{|d} + \frac{1}{2}g^{ed}(\delta\phi)_{|e}\phi_{|d} + \frac{\partial V}{\partial\phi}\delta\phi\right). \qquad (4.36) \end{aligned}$$

More explicitly, the components are

$$\begin{aligned} \delta T_{00} &= \delta\phi'\phi' + 2AV(\phi)a^2 + a^2\frac{\partial V}{\partial\phi}\delta\phi & (4.37) \\ \delta T_{0i} &= (\delta\phi)_{|i}\phi' + \frac{1}{2}B_{|i}\phi'^2 - B_{|i}V(\phi)a^2 & (4.38) \\ \delta T_{ij} &= \left(\delta\phi'\phi' - A\phi'^2 - a^2\frac{\partial V}{\partial\phi}\delta\phi - \psi\phi'^2 + 2\psi V(\phi)a^2\right)\delta_{ij} \\ &\quad +\frac{1}{2}E_{|ij}\phi'^2 - E_{|ij}V(\phi)a^2. & (4.39) \end{aligned}$$

By raising first index,

$$\delta T^a{}_b = \delta(g^{ae}T_{eb}) = \delta g^{ae}T_{eb} + g^{ae}\delta T_{eb}, \qquad (4.40)$$

we arrive at

$$\begin{aligned} \delta T^0{}_0 &= A\phi'^2 - \delta\phi'\phi' - \delta\phi\frac{\partial V}{\partial\phi}a^2 & (4.41) \\ \delta T^i{}_0 &= B_|{}^i\phi'^2 + (\delta\phi)_|{}^i\phi' & (4.42) \\ \delta T^0{}_i &= -(\delta\phi)_|{}^i\phi' & (4.43) \\ \delta T^i{}_j &= \left(-A\phi'^2 + \delta\phi'\phi' - \delta\phi\frac{\partial V}{\partial\phi}a^2\right)\delta^i{}_j\,. & (4.44) \end{aligned}$$

4.4 Gauge Transformations in the FLRW Model

We now turn to the issue of gauge transformation in the cosmological perturbation theory [20, 61, 72, 73, 74]. The motivation is quite simple: How do the perturbations introduced in the pervious section change under a coordinate transformation? Suppose two or more coordinate systems give the same form of the perturbed metric, but with different perturbation functions, then these coordinate systems have to be treated on equal footing and this freedom in choosing the coordinate system would be an imbiguity that we have to deal with when formulating the cosmological perturbation theory.

Consider an infinitesimal coordinate (or gauge) transformation

$$x^\mu \to \widetilde{x^\mu} = x^\mu + \delta x^\mu. \tag{4.45}$$

Formally speaking, x^μ and $\widetilde{x^\mu}$ are just two diffeomorphisms on the same manifold. Thus the corresponding transformation of the perturbation of an arbitrary quantity Q is

$$\widetilde{\delta Q} = \delta Q + \pounds_{\delta x} Q_0 \tag{4.46}$$

where Q_0 is the value on the background spacetime and $\pounds_{\delta x}$ is the Lie derivative of Q along the vector δx^μ.

To be more specific, consider an infinitesimal transformation

$$\tilde{\tau} = \tau + \xi^0, \tag{4.47}$$

$$\tilde{x}^i = x^i + \xi_|{}^i + \bar{\xi}^i. \tag{4.48}$$

Since

$$d\xi^0 = \xi^{0\prime} d\tilde{\tau} + \xi^0{}_{|i} d\tilde{x}^i, \tag{4.49}$$

$$d\xi_|{}^i = \xi_|^{\prime i} d\tilde{\tau} + \xi_|{}^i{}_j d\tilde{x}^j, \tag{4.50}$$

$$d\bar{\xi}^i = \bar{\xi}^{i\prime} d\tilde{\tau} + \bar{\xi}^i_{|j} d\tilde{x}^j, \tag{4.51}$$

then we find

$$d\tau = d\tilde{\tau} - \xi^{0\prime} d\tilde{\tau} - \xi^0{}_{|i} d\tilde{x}^j, \tag{4.52}$$

$$dx^i = d\tilde{x}^i - (\xi'_|{}^i + \bar{\xi}^{i\prime}) d\tilde{\tau} - (\xi_|{}^i{}_j + \bar{\xi}^i_{|j}) d\tilde{x}^j. \tag{4.53}$$

Choosing the background metric to be of the FLRW type, the transformation rule analogous to Eq. (4.46) gives, for example,

$$a(\tau) = a(\tilde{\tau}) - \xi^0 a'(\tilde{\tau}). \tag{4.54}$$

We thus find that, in the new coordinate system, the perturbed metric becomes

$$\begin{aligned} ds^2 = a^2(\tilde{\tau})\Big\{ &- (1 + 2(A - \frac{a'}{a}\xi^0 - \xi^0]))d\tilde{\tau}^2 + 2(B + \xi^0 - \xi')_{|i} d\tilde{\tau} d\tilde{x}^i \\ &-2(S_i + \bar{\xi}'_i) d\tilde{\tau} d\tilde{x}^i + \Big[(1 - 2(\psi + \frac{a'}{a}\xi^0))\delta_{ij} + (E - \xi)_{|ij} \\ &+(F_{i|j} - \bar{\xi}_{i|j}) + E_{ij}\Big] d\tilde{x}^i d\tilde{x}^j \Big\} \end{aligned} \tag{4.55}$$

which is of the same form as the original perturbed metric if we write

$$ds^2 = a^2(\tilde{\tau})\Big\{ - (1 + 2\tilde{A})d\tilde{\tau}^2 + 2(\tilde{B}_{|i} - \tilde{S}_i) d\tilde{\tau} d\tilde{x}^i \tag{4.56}$$

$$\Big[(1 - 2\tilde{\psi})\delta_{ij} + \tilde{E}_{|ij} + \tilde{F}_{i|j} + \tilde{E}_{ij}\Big] d\tilde{x}^i d\tilde{x}^j \Big\}, \tag{4.57}$$

where

$$\tilde{A} = A - \frac{a'}{a}\xi^0 - \xi^{0\prime}, \tag{4.58}$$

$$\tilde{\psi} = \psi + \frac{a'}{a}\xi^0, \tag{4.59}$$

$$\tilde{B} = B + \xi^0 - \xi', \tag{4.60}$$

$$\tilde{E} = E - \xi. \tag{4.61}$$

Note that we have considered only the scalar perturbations. The corresponding transformation rule for other types of perturbation can be derived in the same way.

4.5 The Density Perturbations

The large-scale anisotropy in the cosmic microwave background (CMB) mainly comes from the density perturbations which can be obtained directly from the metric perturbations initiated by Lifshitz in 1946 [71]. However, there is another powerful approach for calculating the density perturbations developed by Hawking in 1966 [16] which involves the coordinate-free dynamics of fluid. Even though this approach is simpler than the metric perturbation formalism to obtain the density perturbation, it is however difficult to obtain many important quantities such as the tensor perturbation within this approach. The main idea of this fluid flow approach is to construct gauge-invariant quantities from the fluid flow equations in one hypersurface, usually the comoving hypersurface, and then generalize the results to another type of hypersurface [18, 19].

In this section, we use another method developed by Wands et al. [75] which is based on a gauge-invatiant quantity ζ. As will be shown, the value of this quantity remains constant during inflation. Therefore this quantity is more appropriate for representing the perturbation than other physical quantities which change during inflation. To begin with, recall from Eq. (4.59) that the perturbation ψ transforms according to

$$\psi \to \psi + \mathcal{H}\delta\tau = \psi + H\delta t \tag{4.62}$$

where $\mathcal{H} = a'/a$. For the energy density, the corresponding transformation can be obtained by using Eq. (4.46):

$$\delta\rho \to \delta\rho - \rho'\delta\tau = \delta\rho - \dot{\rho}\delta t. \tag{4.63}$$

From these transformations, we can construct a gauge invariant quantity [61, 75]

$$\zeta \equiv \psi - \frac{H}{\dot{\rho}}\delta\rho\,. \tag{4.64}$$

We note that constructing the gauge-invariant quantities is one way to cure the gauge ambiguity metioned earlier. From the conservation of energy-momentum tensor, $\nabla^a T_{ab} = 0$, the linear perturbation of the continuity equation is [61, 75]

$$\delta\dot{\rho} + 3H(\delta\rho + \delta p) + 3(\rho + p)\dot{\psi} \;\; = \;\; 0 \tag{4.65}$$

valid on large scales. On the uniform density hypersurfaces where $\delta\rho = 0 \longrightarrow \zeta = \psi$, then the continuity equation becomes

$$\dot{\zeta} \;\; = \;\; -H\frac{\delta p}{\rho + p}\,. \tag{4.66}$$

In any gauge, the pressure perturbation can be split into adiabatic and non-adiabatic parts as follows [75]:

$$\delta p \;\; = \;\; c_s^2\delta\rho + \delta p_{nad}, \tag{4.67}$$

where $c_s^2 \equiv \dot{p}/\dot{\rho}$. The non-adiabatic part is defined as [18, 19]

$$\delta p_{nad} \equiv \dot{p}\left(\frac{\delta p}{\dot{p}} - \frac{\delta\rho}{\dot{\rho}}\right) \tag{4.68}$$

which is, by construction, a gauge-invariant quantity. Since $\delta\rho = 0$, we get

$$\dot{\zeta} \;\; = \;\; -H\frac{\delta p_{nad}}{\rho + p}\,. \tag{4.69}$$

Since this relation was derived from the local conservation of energy-momentum, then it does not rely on any specific relativistic theory of gravity and we can thus use it in the braneworld model. Note that $\dot{\zeta} = 0$ and hence ζ is constant during inflation since inflation is an adiabatic process.

In a special circumstance in which $\delta\phi/\dot{\phi} = \delta\rho/\dot{\rho}$ and in the spatially flat gauge where $\psi_{flat} = 0$, Eq. (4.64) becomes

$$\zeta = -\frac{H\delta\phi}{\dot{\phi}}, \tag{4.70}$$

where $\delta\phi$ comes from the inflaton fluctuations at the Hubble crossing $k = aH$ with k being the wavenumber of $\delta\phi$. It can be shown that in the slow-roll limit [18, 19]

$$\langle\delta\phi^2\rangle \simeq \left(\frac{H}{2\pi}\right)^2. \tag{4.71}$$

Finally, we can calculate the amplitude of the scalar perturbation, an important quantity used to determine the characteristic length scale of the scalar field fluctuations, from the relation [42]

$$A_s^2 = \frac{4}{25}\langle\zeta^2\rangle. \tag{4.72}$$

In the slow-roll limit,

$$\begin{aligned} A_s^2 &= \frac{4}{25}\langle\zeta^2\rangle & (4.73)\\ &= \frac{4}{25}\frac{H^2}{\dot{\phi}^2}\langle\delta\phi^2\rangle|_{k=aH}. & (4.74)\end{aligned}$$

In the case of braneworld cosmology, we find

$$\begin{aligned} A_s^2 &\simeq \frac{4}{25}\frac{9H^4}{V'^2}\frac{H^2}{4\pi^2}|_{k=aH} \\ &\simeq \frac{9}{25\pi^2}\left(\frac{1}{3m_4^2}\right)^3\frac{V^3}{V'^2}\left[\frac{2\lambda+V}{2\lambda}\right]^3|_{k=aH} \\ &\simeq \left(\frac{64}{75m_4^2}\right)\frac{V^3}{V'^2}\left[\frac{2\lambda+V}{2\lambda}\right]^3\Big|_{k=aH}. \end{aligned} \tag{4.75}$$

This quantity plays a crucial role in comparing the large scale CMB anisotropy predicted from inflation with observations [42].

Chapter 5

Conclusions

In this thesis, we have studied some aspects of brane cosmology. It was found that the standard 4-dimensional cosmological equations were modified due to the presence of the brane tension λ, which is just the constant energy-momentum concentrated on the brane, and the effect of the localization of the brane in the extra dimension. As we have seen, the ρ^2 term in the modified Friedmann equation came from the discontinuity across the brane in the bulk. The modified square term of the energy gives a significant change in the expansion rate of the universe. This effect must occur only in the early universe, otherwise it will alter the prediction after the nucleosynthesis era. Generally, the differences between the standard 4-dimensional theory and the braneworld one came from the extra terms containing λ, such as ρ/λ. We found that the effects of the braneworld disappear when we take the limit $\lambda \rightarrow \infty$

A nice thing that the brane tension gives us is that it eases the slow-roll condition, so many potentials that cannot give inflation in the FLRW model can do so in the braneworld scenario. One can interpret this as the brane tension enhancement of the Hubble friction to make the slow-roll inflation possible for the more steep potentials. However the model with too steep potential typically ends with a kinetic-dominated term that can affect the nucleosynthesis.

When we took into account the cosmological density perturbations, it was found that the brane tension increases the scalar power spectrum relative to the one obtained from the standard model. However, the other types of perturbation were not considered in this thesis; it is our hope that we will do so in the future.

In this thesis, we did not discuss the more realistic cases such as the effects from the bulk or the other branes in the bulk. However, from the recent development, the braneworld models cannot provide new phenomena that distinguish the braneworld inflation from the inflation in four dimensions. The braneworld paradigm seemingly gives more parameter in the model that can be taken away in some way. Whether the braneworld cosmological models are real certainly needs experimental proof which we hope will be done in the near future.

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Vitae

Name: Mr. Itzadah Thongkool

Born: March 1, 1979

Education:

2000 Bachelor of Science (second class honour) in Physics
Chulalongkorn University