การจำลองเชิงตัวเลขสำหรับการหาสัดส่วนคงที่ของการลงทุนเหมาะที่สุดสำหรับพอร์ตโฟลิโอของสินค้าโภคภัณฑ์และสัญญาฟิวเจอร์สภายใต้แบบจำลองราคาของชวาร์ตซ

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NUMERICAL SIMULATION FOR AN OPTIMAL FIXED RATIO OF INVESTMENT IN HEDGED PORTFOLIO OF COMMODITIES AND THEIR FUTURES UNDER SCHWARTZ PRICING MODEL

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ในงานวิจัยนี้ได้นำเสนอการจัดรูปแบบของพอร์ตโฟลิโอที่มีการลงทุนในสินค้าโภคภัณฑ์ควบคู่กันไปกับสัญญาซื้อขายล่วงหน้าชนิดฟิวเจอร์สที่ตราบนสินค้าโภคภัณฑ์ชนิดนั้นๆ สินค้าโภคภัณฑ์เป็นสินค้าที่มีลักษณะพิเศษแตกต่างจากสินค้าทางการเงินรูปแบบอื่นๆ เนื่องจากมีต้นทุนในการเก็บรักษาและการให้ประโยชน์ในการถือครองแก่ผู้ถือครองสินค้าในช่วงเวลาหนึ่ง ดังนั้นในการศึกษานี้จึงได้มีการประยุกต์ใช้ตัวแบบเชิงสโทแคสติกชนิดสองตัวแปรในการอธิบายถึงพฤติกรรมราคาของสินค้าโภคภัณฑ์ซึ่งมีลักษณะของความไม่แน่นอนรวมทั้งผลกระทบที่เกิดจากอัตราผลตอบแทนความสะดวกอีกด้วย นอกจากนี้ในงานวิจัยยังได้นำเสนอระเบียบวิธีเชิงตัวเลขที่ใช้ในการประมาณค่าสัดส่วนคงที่ของการลงทุนเหมาะที่สุดสำหรับพอร์ตโฟลิโอชนิดนี้เมื่อพิจารณาถึงสภาวะของตลาดซื้อขายสินค้าโภคภัณฑ์ในภาวะที่มีอัตราผลตอบแทนความสะดวกที่แตกต่างกัน

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In this thesis, we are interested in a portfolio maximization problem and a numerical method solving for the optimal investment solution associated with our problem. Accordingly, we first present a specific kind of portfolio model which consisted of only two assets, one is a commodity asset, and another one is a futures contract written on this commodity. Through out this study, we assume that the behavior of commodity price in a perfect market is described by the Schwartz two-factor pricing model. In additions, by no-arbitrage assumption, the close-form formula of fair price can be used to derive an instantaneous futures price. In the first part of this study we formulate the portfolio model by considering the generating of portfolio incomes. We give an emphasize on a tradeoff between a proportion of commodities and commodity futures in the portfolio. In order to solve this problem, we employ Monte Carlo technique to simulate the process generating the return on our portfolio model and to approximate the fixed ratio solution of an investment that maximizes the portfolio final wealth. Some numerical examples are given to illustrate the procedure and additional applications are suggested.

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CHAPTER I

INTRODUCTION

In the past few decades, an evolution of financial markets, both in developed countries and in emerging countries, played a significant role in a rapid growth of global economy. Basically, the main responsibility of the market is to regulate and to facilitate the trade occurred in it. However, there are many types of financial market locates nearly every country in the world, for example, the most well-known stock exchange in United states-the New York Stock Exchange (NYSE). Within a financial sector, there exists another important market that provides an effective and efficient mechanism for a management of price risks-the derivatives market. Derivative instruments allow investors a great deal of flexibility and choice to determine their investment policies such as speculating and hedging. Nowadays, we can see the derivatives exchanges all around the world. For instance, the world largest exchanges, according to the number of contracts traded, are the Chicago Mercantile Exchange (CME) and Chicago Board of Trade (CBOT) [7].

In this research, we are mainly focusing on futures contracts, which are ones of the most popular financial instruments, traded actively in the derivatives markets. Futures contract is a standardized forward contract that is regulated according to the quality, quantity, delivery location, and when delivery will be made. These regulations are made under the exchange market to advance the customer reliance. Similar to forward contacts, futures contracts are all viewed as derivative contracts because their values are derived from value of another asset. The assets under the negotiation are known as underlying variable. In general, futures contracts are simply an agreement between two parties to buy or sell the underlying asset at a certain time in a future date for a certain price. One of the parties to a futures contracts that agreed to sell or deliver the underlying asset at the specific time in the future is assumed to hold a "short position", on the other hand, the party on the opposite side of the futures contracts assumes a "long position "to buy the assets at the same time with the same price. In contrast to forward contracts, the parties engaging the futures contracts do not need to deliver a physical asset but the futures contracts are marked to market every period. Marking to market is the feature that tracks and updates the balance of the investor's account depending on the change in futures contracts prices. The gains or losses are applied to it daily. This type of account is so called the margin account which normally provides investors the interest. However, in this thesis, we are considering the margin accounts that only marked to market without subjecting to other margin constraints.

1.1 Commodity Product and Convenience yield

When considering forward or futures contract, it is important to give a discussion on its underlying asset because the behavior of their prices is practically subjected to their underlying variable. Along with stock, bond, and other assets, commodity forms one of the major asset class and favorably taken to be the underlying variable of derivatives products such as option, forward, as well as futures contracts. By definition, commodities are physical goods used in commerce to satisfy basic wants or needs. Regardless of the producers, commodities are uniformly marketable items which have no differentiation across the market. Moreover, commodities are usually used as a primary resources in production process of other goods and services. The exchange of commodities represents one of the earliest forms of trading; begin with the direct trading of physical good in the past till the trade of future delivery today.

Commodity assets are fundamentally differed from equity assets and fixed income securities. First of all, the distinction of investment assets to consumption assets is interpreted by it possessing purposes. An investment asset is an asset that is held for investment purposes but consumption asset is an asset that is held primarily for consumption. Commodities, such as energy and agricultural product, are realized as consumption assets rather than investment assets. In fact, commodities are physical goods that take cost to store and maintenance, thereby, the disruption in prices and benefits when agents are hoarding or dis-hoarding the commodities becomes more significant. Although the commodities usually provide no incomes but can generate some advantageous factor to the holder of the physical commodity. This leads us to an essential concept of convenience yield. The term convenience yield, which was proposed by [16] is very abstract concept. In particular, the convenience yield is referred as a measure of the opportunity cost of holding inventory, on the other hand, sometimes convenience yield is considered as a benefit paying to the owner of physical assets. Furthermore, net convenience yield on a commodity can be thought of in the same way as dividend yield on a common stock and is normally quoted as a continuously compounded yield. Thus, for commodities-related problems, it is necessary to extend the simple model to include the impact of convenience yield to reflect benefit unique of physical commodity holders.

1.2 Hedged portfolio of commodity

In a recent year, the increasingly importance of commodity-related exchange and other commodity-based derivatives have had a tremendous economic impact on the nations and people. Many parts of economics use or consume commodity on a daily basis, for example, the soft commodities like grain, wheat, soy and corn are served as raw ingredients for a food production. In transportation sector, hard commodities, especially for crude oil and natural gas, are vital elements in a survival of the business. There are also commodities that are used for industrial and manufacturing purposes such as copper and other metals. Typically, the commodity markets have been suffered severely by many arguments such as climate changes, natural disaster, mismatch between demand and supply, financial crisis, new discovery of resource and etc. Weather condition and natural disaster might cause agricultural price increase dramatically due to the scarcity. The oversupply will lead to the lower price because of an excessive number of products in the market. More precisely, the commodity prices are subjected to the level of future supply and demand. According to unpredictable prices of commodity products, the market participants want to remove the risk associated with their investments or business. This because people are naturally risk-averse, so they manipulate their portfolio by applying a common risk management policy called a hedge. Hedging strategy is an investment position using to mitigate the exposure of changes in asset price. A hedge can be constructed by many types of financial instruments including forward contracts and futures contracts.

In the past, the first public futures market was established in order to archive an efficient hedging strategy for agricultural commodity products. By the time, farmers decided to insure the value of their crops which were fluctuated depending on the weather conditions in the harvest season. Therefore, futures contracts enter to this subject to fulfill the farmer requirement. On the other hand, the suppliers who hold the opposite position to the farmer also want to fix manufacturing costs. By locking in the price of which particular item investors intended to buy or sell, forward contracts could also help investors to eliminate the ambiguity when they are dealing with an unexpected expense in the future. Then hedging itself has expanded to others financial products and developed through the time.

1.3 Problem of Interests

In this study, we consider an agent who invests in physical commodities as a major product of his portfolio policy. The problem is that, at any time in an investment period $[t,T]$ where $0 \le t \le T < \infty$, investors attend to invest a portion $\alpha \in [0,1]$ of their portfolio value in the physical commodity assets and the rest $1-\alpha$ in commodity futures which mature at time T to do a long hedge[1] for the portfolio. The profitability of investors evolves stochastically over time depending on the changes in price of all assets and securities in portfolio. In order to maximize portfolio wealth at the final time T the investors are allowed to dynamically rebalance their assets holding in commodity spots and the futures contracts when they first enter to any adjustment period of length h, where h is a fixed small positive value, i.e. at the beginning of period $[t,t+h]$. This kind of a specific portfolio model is directly motivated by the way that [10] had examined his dynamical portfolio processes in 1970. Following his assumption, we consider in a situation of which the investors participate in the perfect market that is basically no investor taxes and transaction costs, and we assume further that any trading orders take place in the market are continuous and all securities are perfectly divisible. However, the borrowing capacity is restricted, and the short sells of all assets are not allowed in this study. This means that the proportion α is a real

[1] A long hedge is a situation where an investor takes a long position in futures contracts to hedge against price risks.

value belongs to the set of $[0,1]$. In case of describing an agent's preference, we suppose that the investors are risk-averse which means that investors naturally avoid from all kind of risks. The expected utility theory is taken into an account to exhibit investor satisfaction.

In order to explain random behavior of asset prices, we assume that the stochastic process of commodity prices, commodity futures and portfolio process are evolving in an uncertainty environment described by a complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$. The stochastic process is an important idea that was increasingly used to describe financial uncertainties. In general, a stochastic differential equation is used for the modeling of price fluctuation and another factor in financial problems. In this study, we employ the two-factor stochastic model first presented by [1] to determine the joint stochastic process of commodity spot price and instantaneous convenience yield. In addition, we can directly calculate the prices of futures (or forward) contracts, by supposing there is no-arbitrage opportunity in the market, through a closed-form formula of the price of contingent claims which was proposed in the work of [5, 15].

The objective of this thesis is to find an optimal fixed ratio investment, denoted by α, of the portfolio model such that maximizes an agent's preference by using an approximation method. First, we are in position to determine the continuous-time portfolio wealth process which is governed by the process of commodity price and the process of spot futures price based on the Schwartz's two-factor model. Then we are going to construct our interested problems in the form of stochastic control model where our attentive state variable, X_t, is described by a control diffusion process of portfolio value. Basically, it is often useful and reasonable to assume that the time horizon is finite. There are many ways to fix this class of control problem. The main method we use in this study is by considering a specific case of reduced control problem and approximating the problem numerically through the Monte Carlo method. We are applying Euler-Maruyama method to approximate the result of stochastic differential equation and then use this result to generate portfolio trajectories. We also illustrate path trajectories of spot commodity price, convenience yield and futures price. Moreover, by simulation of futures price, the results have shown that the price calculated by close-form formula of Schwartz and by approximation method are perfectly matched. In an experiment, all the parameters are specified according to the estimated parameters of Schwartz. Monte Carlo procedure is again employed to solve for an optimal fixed ratio of an investment. Another possible approach for finding optimal control is a PDE method which also known as Hamilton-Jacobi-Bellman equation. There are two excellent literature that we used as a central principle which are [12, 13]. We demonstrate the fundamental set up of the corresponding partial differential equation that will pave the way of solving the problem analytically in the future.

The aims of this thesis are three-fold

(I) The first contribution of this thesis is to construct the portfolio model of commodity asset and commodity futures.

(II) The second is to illustrate the firm objectives of maximizing its preference through the utility function of its final wealth.

(III) The third is to approximate the optimal fixed ration of this type of portfolio by Monte Carlo method.

The rest of this thesis is organized as follow. In the second chapter, we will give a discussion on some financial-related terms and productive theorems in stochastic process which we will be applying them to construct portfolio model. In addition, we also provide a commodity model describing the behavior of commodity price. Next, we devote the third chapter to clarify the structure of portfolio model and present the evolution of portfolio value. At the end of this chapter, we propose our interested portfolio into the form of stochastic control model and its corresponding partial differential equation. In the fourth chapter, we perform the simulations of portfolio value and the method solving for numerical solution are illustrated step by step.

For the rest of this study, we let $\mathbb{T}\in[0,T]$, $T<\infty$ be a time space. Then we fix a filtered probability $(\Omega,\mathcal{F},\mathbb{F}=(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$ satisfying the usual conditions and a $n-$ dimensional Brownian motion, W_t, with respects to $\mathbb{F}$. We define an Ito process as a process $X_t=(X_t^1,\ldots,X_t^n)$ value in $\mathbb{R}^n$ and satisfies

$$dX_t=\mu(t,X_t)dt+\sigma(t,X_t)dW_t \tag{1.1}$$

where $\mu(t,x,\omega)=(\mu_i(t,x,\omega))_{1\leq i\leq n}$, $\sigma(t,x,\omega)=(\sigma_{ij}(t,x,\omega))_{1\leq i\leq n,1\leq j\leq m}$ defined on $\mathbb{T}\times\mathbb{R}^n\times\Omega$, and have valued in $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$, respectively.

Chapter II

Preliminary

In this chapter, first of all, we will restate meaningful definitions of a perfectly competitive market which was assumed to be a grounded notion in many financial-related problems. Then we give an economics description of no-arbitrage assumption which was held in the derivative pricing context. The measurement of investors preferences are described in the form of utility functions. For the financial-related terms, see [3, 8]. Next, the commodity spot price and convenience yield are assumed to follow the Schwartz two-factor model. Then we introduce useful stochastic-related theorems which were applied to develop our portfolio model. The method for approximating numerical results of stochastic differential equation is also provided. Finally, we include background knowledge of stochastic control model such as Feynman-kac formula and Hamilton-Jacobi Bellman equation.

2.1 Perfectly competitive market

There are some properties of hypothetical market usually required for the study of financial model shown as follows:

2.1.1 The market participants are rational, and try to maximize their financial objectives.

2.1.2 There are no transaction costs incurred to market participants.

2.1.3 There are no taxes rates incurred to market participants.

2.1.4 There are no restrictions on lending, borrowing and short selling.

2.2 The absence of arbitrage opportunity

The abilities to make profits under a zero-cost portfolio without taking any risks and without initial costs are not allowed.

2.3 Investor's preferences and Utility function

According to a description in [3, 11], the rational investors are typically being able to compare between different situations and to decide which one is better. This kind of preferences of investors is the basic idea for a measurement tool called utility function. It represents a satisfaction experienced by the consumer of a good. A utility function, $U(\cdot)$, is a mapping from an uncertain choice of consumption into the real number. Due to an uncertainty of choices, it is normal to consider an average of utility, $E(U(\cdot))$, instead. In this study, we consider a basic utility in the class of the constant relative risk aversion (CRRA) denoted by the function $U(X)=\frac{X^{\gamma}}{\gamma}$, $0<\gamma<1$ and $X>0$. A larger γ indicates the investors who are risk-loving, on the other hand a smaller γ indicates the investors who are more risk-averse.

2.4 Schwartz Two-Factor Commodity Model

A general hypothesis of asset prices in perfect markets is described by the Geometric Brownian motion (GBM) .This widely used hypothesis indicates that the asset prices are log-normally distributed random variable and only assume positive values to its variable. Geometric Brownian motion process serves as a basic concept for modelling of financial asset prices including commodity. Furthermore, the development of commodity pricing model takes into an account of the second factor, convenience yield, to model a more realistic behavior of commodity.

This class of convenience yield models is the most popular choice for modeling the behavior of spot price of energy products such as crude oil and natural gas. In a recent year, there are many literatures focusing on the stochastic behavior of spot commodity price. Early models, proposed by Brennan and [1] in 1985, assumed a constant convenience yield with one-factor Brownian motion. The convenience yield was first considered as a dividend yield paid to holder. Subsequently, a commodity spot price and convenience yield are assumed to follow a joint stochastic process with a constant correlation. Gibson and Schwartz presented a benchmark model for commodity price which included a stochastic convenience yield, [15]. The instantaneous convenience yield is taken as a second factor following a mean reverting stochastic process of the Ornstein-Uhlenbeck process . The more general model was introduced by Schwartz again. Moreover, Schwartz referred to the work of [15] in order to evaluate closed-form solution of futures price.

Now we will introduce the Schwartz two-factor model which we use as a basic assumption in this study. The first factor is spot commodity price S_t and the second factor accounts for the convenience yield δ_t of the commodity . These factors are assumed to satisfy the joint stochastic process shown as follow:

$$
\begin{aligned}
dS &= (r-\delta)Sdt + \sigma_1 S dW^{(1)} \\
d\delta &= [\kappa(a-\delta)-\lambda]dt + \sigma_2 dW^{(2)}
\end{aligned}
\qquad (2.4.1)
$$

where the increments to standard Brownian motion, $dW^{(i)}$, are correlated with

$$dW^{(1)}dW^{(2)} = \rho dt \qquad (2.4.2)$$

together with further assumptions which we maintain throughout this study that the long-term mean of instantaneous convenience yield a, speed of mean reversion of instantaneous convenience yield κ, market price of convenience yield λ, interest rate r, ρ is a correlation coefficient between spot price and convenience yield, volatility of spot commodity price σ_1 and volatility of instantaneous convenience yield σ_2 are constant .Then, we denote $F(\tau,S,\delta)$ by the price per unit of futures contract at time t for a unit commodity delivery at time T where $\tau = T-t$ is referred to time to maturity.

Moreover, by an assumption of no-arbitrage arguments, [15] suggested that the close-form formula of futures price is obtained in the form of

$$F(\tau,S,\delta) = S\cdot e^{\left[-\delta\cdot\frac{1-e^{-\kappa\tau}}{\kappa}+A(\tau)\right]} \qquad (2.4.3)$$

where

$$
\begin{aligned}
A(\tau) = &\left(r - a - \frac{\lambda}{\kappa} + \frac{\sigma_2^2}{2\kappa^2} - \frac{\sigma_1\sigma_2\rho}{\kappa}\right)\tau + \frac{1}{4}\sigma_2^2\cdot\frac{(1-e^{-2\kappa\tau})}{\kappa^3} \\
&+\left((a-\frac{\lambda}{\kappa})\kappa + \sigma_1\sigma_2\rho - \frac{\sigma_2^2}{\kappa}\right)\cdot\frac{1-e^{-\kappa\tau}}{\kappa^2}.
\end{aligned}
\qquad (2.4.4)
$$

2.5 Multi-dimensional Ito Formula

The fundamental and useful tool for evaluating Ito integral is the Ito's lemma stated as follows

Lemma 2.5.1. [12] We recall $X(t)$ an n-dimensional Ito process satisfying equation (1.1) which can be written in a metric form as

$$d\begin{bmatrix} X_1(t,\varpi) \\ \vdots \\ X_n(t,\varpi) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix} dt + \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1m} \\ \vdots & & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nm} \end{bmatrix} \begin{bmatrix} dW_1(t,\varpi) \\ \vdots \\ dW_m(t,\varpi) \end{bmatrix}$$

Let $g(t,x) = (g^{(1)}(t,x),\dots,g^{(p)}(t,x))$ be a C^2 function from $[0,\infty]\times\mathbb{R}^n$ into $\mathbb{R}^p$. So, the process

$$Y(t,\varpi) = g(t, X(t,\varpi))$$

is again an Ito process, then the stochastic differential for k^{th} the component is given by

$$dY^{(k)} = \frac{\partial g^{(k)}}{\partial t}(t,X)dt + \sum_i \frac{\partial g^{(k)}}{\partial x_i}(t,X)dX_i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g^{(k)}}{\partial x_i \partial x_j}(t,X)dX_i dX_j \tag{2.5.1}$$

and

$$dtdW_i = dW_i dt = 0,\ dW_i dW_j = \delta_{ij} dt, \tag{2.5.2}$$

where $\delta_{ij} = \begin{cases} \rho, & i \neq j \\ 1, & i = j \end{cases}$.

2.6 Euler-Maruyama Method

The Euler-Maruyama scheme is the simplest time discrete approximation of the Ito process since it is a simple generalization of Euler method for ordinary differential equations. We employ this method to approximate numerical solution of the stochastic differential equation and to simulate time discrete trajectories.

Lemma 2.6.1. [9] Euler-Maruyama method

Let X_t be an Ito process satisfying the stochastic differential equation (1.1), i.e.

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

on the time interval $[t_0, T]$ with an initial value $X_{t_0} = x_0$. For a given time discretization

$$t_0 < t_1 < \ldots < t_n < \ldots < t_N = T \ .$$

Euler Maruyama scheme takes the form

$$X_{n+1} = X_n + u\Delta_n t + v\Delta_n W,$$

where

$$\Delta_n t = t_{n+1} - t_n$$

is the length of the time discretization of subinterval $[t_n, t_{n+1}]$, and

$$\Delta_n W = W_{t_{n+1}} - W_{t_n}$$

is the increment of Brownian motion which is normally distributed with mean zero and variance $t_{n+1} - t_n$; equivalently $W_{t_{n+1}} - W_{t_n} \sim \sqrt{t_{n+1} - t_n} \cdot N(0,1)$, where $N(0,1)$ is a normal distributed random variable with mean zero and unit variance.

2.7 Feyman-kac formula

The Feynman-Kac formula establishes a link between parabolic partial differential equations and stochastic processes. The main theoretical result of the Feynman-Kac formula suggests that the expectations of random processes can be computed by solving the corresponding parabolic partial differential equations with a given boundary condition.

Theorem 2.7.1 [6] Generalized Feynman-Kac formula.

Let $[0,T]$ where $T<\infty$ be a fixed time interval denote by $\mathbb{T}$ and D a domain in $\mathbb{R}^n$ i.e., an open connented subset of $\mathbb{R}^n$. We recall the stochastic differential equation (1.1) which is rewritten componentwise

$$dX_s^{t,x} = \mu(s, X_s^{t,x})ds + \sum_{j=1}^{m} \sigma_j(s, X_s^{t,x})dW_s^{(j)}, \qquad X_t^{t,x} = x \in D$$

where $\mu: \mathbb{T}\times D \to \mathbb{R}^n$ and $\sigma_j: \mathbb{T}\times D \to \mathbb{R}^n$ are continuous for $j=1,\dots,m$. For a given measurable function $g: D\to[0,\infty)$, $h:\mathbb{T}\times D\to(-\infty,0]$ and $c:\mathbb{T}\times D\to\mathbb{R}$. Then we define a function $u:\mathbb{T}\times D\to[0,\infty]$ by

$$u(t,x) := \mathrm{E}\left[g(X_T^{t,x}) e^{\int_t^T c(s,X_s^{t,x})ds} + \int_t^T h(s,X_s^{t,x}) e^{\int_t^s c(u,X_u^{t,x})du} ds \right].$$

and we also denote the operator $\mathcal{L}$ for a sufficiently smooth function $f:\mathbb{T}\times D\to\mathbb{R}$ by

$$(\mathcal{L} f)(t,x) = \sum_{i=1}^{n} \mu_i(t,x)\frac{\partial f}{\partial x_i}(t,x) + \frac{1}{2}\sum_{i,j=1}^{n} \sigma_{ij}(t,x)\frac{\partial^2 f}{\partial x_i \partial x_j}(t,x) \qquad (2.7.1)$$

where

$$\sigma_{ij}(t,x) = (\sigma(t,x)\sigma^T(t,x))_{ij} \qquad (2.7.2)$$

Then u satisfies the partial differential equation (PDE)

$$c(t,x)u(t,x) - \frac{\partial u(t,x)}{\partial t} - \mathcal{L}u(t,x) = h(t,x) \quad \text{on } \mathbb{T}\times D,$$

with terminal boundary condition

$$u(T,x) = g(x) \quad \text{for } x \in D.$$

2.8 Derivations of Stochastic Control

In this section, we discuss derivations of stochastic control model by following materials presented in the book of [13]. First, we consider a dynamical system where the state of the system at time $t \in \mathbb{T}$ is denoted by stochastic process $X_t(\varpi)$. This state of the system is governed by a stochastic differential equation (SDE) valued in $\mathbb{R}^n$ as equation (1.1),

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$$

The dynamic of the system can be influenced by control processes $\alpha = (\alpha_t)_t$, whose value is decided at any time $t \in \mathbb{T}$, which is a progressively measurable process with respect to $\mathbb{F}$ and valued in A, a subset of $\mathbb{R}^m$. In general, the controls that satisfy some constraints are called admissible controls which belong to the set of all admissible controls denoted by $\mathcal{A}(t,x)$ for $(t,x) \in \mathbb{T} \times \mathbb{R}^n$. After the perturbation of external factor, the evolution of our state of the system will be governed by a control diffusion process, which is a stochastic differential equation of the form

$$dX_t = \mu(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dW_t. \tag{2.8.1}$$

In this study, we are interested in a finite horizon problem whose objective is to maximize some performance criteria, described by a functional $J(t, X, \alpha)$, over all admissible control processes. For a finite horizontal problem, the gain function $J(t, X, \alpha)$ is defined by

$$J(t,x,\alpha) = E\left[\int_t^T f(s, X_s^{t,x}, \alpha_s)ds + g(X_T^{t,x})\right]$$

for all $(t,x) \in \mathbb{T} \times \mathbb{R}^n$ where the function f is a running profit function and g is a terminal reward function. The objective is to maximize the gain function over all admissible control processes, this will result in a function so called the value function defined by $v(t,x) = \sup_{\alpha \in \mathcal{A}} J(t,x,\alpha)$.

2.9 Hamilton-Jacobi-Bellman equation

We consider a diffusion

$$dX_t = \mu(X_t, \alpha_t)dt + \sigma(X_t, \alpha_t)dW_t,$$

and the associated operator $\mathcal{L}^{\alpha}$ with the constant control α, defined similar to (2.7.1) and (2.7.2) by

$$(\mathcal{L}^{\alpha} f)(t,x) = \sum_{i=1}^{n} \mu_i(t,x)\frac{\partial f}{\partial x_i}(t,x) + \frac{1}{2}\sum_{i,j=1}^{n} \sigma_{ij}(t,x)\frac{\partial^2 f}{\partial x_i \partial x_j}(t,x).$$

From [13], let $v(t,x)$ be a solution of the partial differential equation

$$-\frac{\partial v(t,x)}{\partial t} - \sup_{\alpha \in \mathcal{A}}\left[\mathcal{L}^{\alpha} v(t,x) + f(t,x,\alpha)\right] = 0, \quad \forall (t,x) \in [0,T]\times \mathbb{R}^n,$$

with terminal condition

$$v(T,x) = g(x), \quad \forall x \in \mathbb{R}^n.$$

Suppose that there exists a measurable function $\alpha^*(t,x)$ is an optimal control such that

$$\sup_{\alpha \in \mathcal{A}}\left[\mathcal{L}^{\alpha} v(t,x) + f(t,x,\alpha)\right] = \mathcal{L}^{\alpha^*(t,x)} v(t,x) + f(t,x,\alpha^*(t,x)),$$

i.e.,

$$\alpha^*(t,x) \in \arg\max_{\alpha \in A}\left[\mathcal{L}^{\alpha} v(t,x) + f(t,x,\alpha)\right].$$

Then we get

$$-\frac{\partial v(t,x)}{\partial t} - \mathcal{L}^{\alpha^*(t,x)} v(t,x) = f(t,x,\alpha^*(t,x)), \quad \forall (t,x) \in [0,T]\times \mathbb{R}^n,$$

$$v(T,x) = g(x), \quad \forall x \in \mathbb{R}^n,$$

and by Feynman-Kac formula

$$v(t,x) = E\left[\int_t^T f(X_s^*, \alpha_s^*(s, X_s^*))ds + g(X_T^*)\right],$$

where X_s^* is the solution to the stochastic differential equation

$$dX_s^* = \mu(X_s^*, \alpha_s^*(s, X_s^*))dt + \sigma(X_s^*, \alpha_s^*(s, X_s^*))dW_s,$$

$$t \le s \le T\,, \quad X_s^* = x.$$

Lemma 2.9.1. [13] Consider a singleton set, $\{\alpha_0\}$, of control space in a finite time.

Let w be a function in $C^{1,2}(\mathbb{T}\times\mathbb{R}^n)\cap C^0(\mathbb{T}\times\mathbb{R}^n)$ satisfying a quadratic growth condition, which is stated as follow

$$|w(t,x)| \le K(1+|x^2|)\,, \quad \forall (t,x)\in\mathbb{T}\times\mathbb{R}^n\,.$$

Suppose that the control space A is reduced to a singleton set $\{\alpha_0\}$ and w satisfies the reduced Hamilton-Jacobi-Bellman equation-the Linear-Cauchy problem:

$$-\frac{\partial w(t,x)}{\partial t} - \mathcal{L}^{\alpha_0} w(t,x) = f(t,x,\alpha_0), \quad \forall (t,x)\in\mathbb{T}\times\mathbb{R}^n,$$

$$w(T,x) = g(x), \quad \forall x\in\mathbb{R}^n\,,$$

Tthen w admits the representation

$$w(t,x) = E[\int_0^T f(X_s^{t,x},\alpha_0)ds + g(X_T^{t,x})].$$

Chapter III

MODEL PROBLEM

In this chapter, we explain in more details about the dynamic of our portfolio equation by using the notion provided in the second chapter. We begin with the calculation of generating of cash flows in a discrete formulation. Then we derive for a continuous-time portfolio wealth process. Equipped with Schwartz's commodity pricing model, we can setup our control model. Secondly, we will provide the PDEs related to our problem.

3.1 Portfolio Equation

Before constructing a portfolio process in a differential form, we investigate how the cash flows are generated in our portfolio model. First of all, we have to examine a discrete-time formulation of the model. In case of the infinitesimally short time interval $(t,t+h)$, where $t\in[0,T]$ and h is a small positive real value that is fixed throughout the study, we can derive the continuous-time formula. In order to explain the model, we first start by introduce some notations used in this study

$N(t)$	be a number of commodities holding at the beginning of period t, i.e., between t and $t+h$,
$\alpha(t)$	be a proportion of portfolio wealth invested in physical commodities at the beginning of period t, i.e., between t and $t+h$,
$1-\alpha(t)$	be a proportion of portfolio wealth invested in commodity futures at the beginning of period t, i.e., between t and $t+h$,
$X(t)$	be a portfolio value at the beginning of period t,
$S(t)$	be a spot price per unit of physical commodity at time t,
$F(t)$	be a spot price per unit of commodity futures at time t which matures at time T

Note that $F(t)$ is an abbreviation of $F(\tau, S, \delta)$, where $\tau = T - t$ is time to maturity of the contract. We continue our analysis by considering a period portfolio model with period of length $h > 0$. All investor's incomes are generated by the price changes of any assets in their portfolio. We assume that portfolio value $X(t)$, spot commodity price $S(t)$, instantaneous convenience yield $\delta(t)$, and futures price $F(t)$ are treated as the information receives at the beginning of each period t.

Then we consider two consecutive points in time, i.e., t and $t+h$. For the first time of the investment, the investors split their budgets into two part. They contribute the first amount of their budgets to buy physical commodities of quantity $N(t)$ with price per unit $S(t)$, so the value of this investment is $S(t)N(t) < X(t)$. Then they deposit the remainder amount $X(t) - S(t)N(t)$ into their margin account in order to go " long" for futures contracts whose spot futures price is $F(t)$. Thus, a number of futures contracts that can be bought at the same time is represented by

$$N_F(t) = \frac{X(t) - N(t)S(t)}{F(t)}.$$

Hence, the budget equation at this state of time is

$$\begin{aligned} X(t) &= N(t)S(t) + N_F(t)F(t) \\ &= N(t) \cdot S(t) + \left(\frac{X(t) - N(t)S(t)}{F(t)} \right) \cdot F(t). \end{aligned} \tag{3.1.1}$$

Then an investor comes into the next period $t+h$ with the number of each asset invested in previous stage. According to our assumption, an investor receives new information about commodity price $S(t+h)$ and futures contract price $F(t+h)$ while they are reaching to the new stage $t+h$. So, the portfolio wealth at the beginning of time period $t+h$ is easily obtained as

$$X(t+h) = N(t)S(t+h) + \left(\frac{X(t) - N(t)S(t)}{F(t)} \right) \cdot F(t+h). \tag{3.1.2}$$

To derive changes of portfolio value in the interval $(t, t+h)$, take (3.1.2) - (3.1.1), we have

$$X(t+h)-X(t)=N(t)[S(t+h)-S(t)]+\left(\frac{X(t)-N(t)S(t)}{F(t)}\right)\cdot[F(t+h)-F(t)]$$

$$=X(t)\left(\frac{N(t)S(t)}{X(t)}\right)\cdot\frac{[S(t+h)-S(t)]}{S(t)}$$

$$+X(t)\cdot\left(1-\frac{N(t)S(t)}{X(t)}\right)\cdot\frac{[F(t+h)-F(t)]}{F(t)}. \quad (3.1.3)$$

Let $\alpha(t)=\dfrac{N(t)S(t)}{X(t)}$ be a feedback control variable described by a proportion of portfolio invested in physical commodity at time t, then (3.1.3) becomes,

$$X(t+h)-X(t)=X(t)\alpha(t)\cdot\frac{[S(t+h)-S(t)]}{S(t)}+X(t)\left(1-\alpha(t)\right)\cdot\frac{[F(t+h)-F(t)]}{F(t)}$$

or equivalent to

$$\Delta X(t)=X(t)\alpha(t)\cdot\frac{\Delta S(t)}{S(t)}+X(t)\cdot\left(1-\alpha(t)\right)\cdot\frac{\Delta F(t)}{F(t)}.$$

In the case of limitation when h approaches to zero, we obtain the continuous-time portfolio process of the form

$$dX_t=\frac{\alpha(t)X_t}{S_t}dS_t+\frac{\left(1-\alpha(t)\right)X_t}{F_t}dF_t. \quad (3.1.4)$$

Next, we will simplify the equation above to the desired form of controlled diffusion process (2.8.1). First of all, we assume that a function of futures prices of commodity contingent claim (2.4.3) is a twice continuously differentiable function in $S,\ \delta$. Then, we can derive the instantaneous rate of changes of futures price, dF, by applying the Ito's lemma (2.5.1) which was mentioned in the previous chapter. Note that for the following calculation, only in this section, subscripts are referred to partial derivatives.

Using a simple application of generalized Ito's lemma (2.5.1) implies that the instantaneous rate of changes of futures price $F(\tau,S,\delta)$, where $\tau=T-t$, is given by

$$dF(\tau,S,\delta)=F_t dt+F_S dS+F_\delta d\delta+\frac{1}{2}F_{SS}(dS)^2+\frac{1}{2}F_{\delta\delta}(d\delta)^2+F_{S\delta}(dS\cdot d\delta)$$

then substitute the joint stochastic processes (2.4.1) into the above equation, we get

$$dF(\tau,S,\delta)=F_t dt+F_S\cdot((r-\delta)Sdt+\sigma_1 SdW^{(1)})+F_\delta\cdot\left([\kappa(a-\delta)-\lambda]dt+\sigma_2 dW^{(2)}\right)$$

$$+\frac{1}{2}F_{SS}\cdot\left(((r-\delta)Sdt)^2+2\sigma_1(r-\delta)S^2dtdW^{(1)}{}_1+\left(\sigma_1 SdW^{(1)}\right)^2\right)$$

$$+\frac{1}{2}F_{\delta\delta}\cdot\left(([\kappa(a-\delta)-\lambda]dt)^2+2[\kappa(a-\delta)-\lambda]\sigma_2 dtdW^{(2)}{}_2+\left(\sigma_2 dW^{(2)}\right)^2\right)$$

$$+F_{S\delta}\begin{pmatrix} S(r-\delta)[\kappa(a-\delta)-\lambda]\cdot(dt)^2+[\kappa(a-\delta)-\lambda]\sigma_1 SdtdW^{(1)} \\ +S(r-\delta)\sigma_2 dtdW^{(2)}+\sigma_1\sigma_2 SdW^{(1)}dW^{(2)} \end{pmatrix}$$

where $(dS)^2=dS\cdot dS,(d\delta)^2=d\delta\cdot d\delta$ and $dS\cdot d\delta=d\delta\cdot dS$ are computed according to the rule (2.5.2). Then, we have

$$dF(\tau,S,\delta)=F_t dt+F_S\cdot\left((r-\delta)S\cdot dt+\sigma_1 SdW^{(1)}\right)+F_\delta\cdot\left(\left[\kappa(a-\delta)-\lambda\right]dt+\sigma_2 dW^{(2)}\right)$$

$$+\frac{1}{2}F_{SS}\cdot\sigma_1^2 S^2 dt+\frac{1}{2}F_{\delta\delta}\cdot\sigma_2^2 dt+F_{S\delta}\cdot\sigma_1\sigma_2\rho Sdt.$$

By regrouping our result, the instantaneous rate of changes of futures price is

$$dF(\tau,S,\delta)=\begin{bmatrix} F_t+(r-\delta_t)S\cdot F_S+\left[\kappa(a-\delta)-\lambda\right]\cdot F_\delta \\ +\frac{1}{2}\sigma_1^2 S^2\cdot F_{SS}+\frac{1}{2}\sigma_2^2\cdot F_{\delta\delta}+\sigma_1\sigma_2\rho S\cdot F_{S\delta} \end{bmatrix}dt+\sigma_1 S\cdot F_S dW^{(1)}+\sigma_2\cdot F_\delta dW^{(2)}.$$

(3.1.5)

Subsequently, we invoke a close-form formula of futures prices (2.4.3) together with (2.4.4). We denote

$$F(\tau,S,\delta)=S\cdot e^{C(\tau,\delta)},$$

where

$$C(\tau,\delta) = -\delta \cdot \frac{1-e^{-\kappa\tau}}{\kappa} + A(\tau), \tag{3.1.6}$$

and note that $\tau = T - t$. Then we compute for partial derivatives of $C(\tau,\delta)$ in variables t and δ, respectively, we get

$$\frac{\partial C(\tau,\delta)}{\partial t} = \delta \cdot e^{-\kappa\tau} + \frac{dA(\tau)}{dt}, \tag{3.1.7}$$

where

$$\frac{dA(\tau)}{dt} = -\left[\left(r - \hat{\alpha} + \frac{\sigma_2^2}{2\kappa^2} - \frac{\sigma_1\sigma_2\rho}{\kappa}\right) + \frac{1}{2}\sigma_2^2 \cdot \frac{e^{-2\kappa\tau}}{\kappa^2} + \left(\hat{\alpha}\kappa + \sigma_1\sigma_2\rho - \frac{\sigma_2^2}{\kappa}\right) \cdot \frac{e^{-\kappa\tau}}{\kappa}\right],$$

and

$$\frac{\partial C(\tau,\delta)}{\partial \delta} = \left[\frac{e^{-\kappa\tau} - 1}{\kappa}\right]. \tag{3.1.8}$$

For abbreviation, we denote $\frac{\partial C(\tau,\delta)}{\partial t}$ and $\frac{\partial C(\tau,\delta)}{\partial \delta}$ by $C_t(\tau,\delta)$ and $C_\delta(\tau)$, respectively. Now, we continue to derive partial derivatives and second order partial derivatives in each component of $F(\tau,S,\delta)$, we get

$$\begin{aligned}
F_t &= S \cdot C_t(\tau,\delta) e^{C(\tau,\delta)} \\
F_S &= e^{C(\tau,\delta)} \\
F_\delta &= S \cdot C_\delta(\tau) e^{C(\tau,\delta)} \\
F_{SS} &= 0 \\
F_{\delta\delta} &= S \cdot C_\delta^2(\tau) e^{C(\tau,\delta)} \\
F_{S\delta} &= F_{\delta S} = C_\delta(\tau) e^{C(\tau,\delta)}.
\end{aligned} \tag{3.1.9}$$

Then substitute (3.1.6), (3.1.7), (3.1.8) and (3.1.9) into (3.1.5) the instantaneous rate of changes in futures price becomes

$$dF(\tau,S,\delta)=[S\cdot C_t e^C+(r-\delta_t)Se^C+\left(\kappa(a-\delta)-\lambda\right)C_\delta Se^C$$

$$+\frac{1}{2}\sigma_2^2 C_\delta^2 Se^C+(\sigma_1\sigma_2\rho)C_\delta Se^C]dt+\sigma_1 S\cdot e^C dW^{(1)}+\sigma_2 C_\delta Se^C dW^{(2)}.$$

Furthermore, we divide both sides of the equation above by $F(\cdot)=Se^C$, we then get a percentage rate of changes in futures price relative to itself as

$$\frac{dF}{F}=\left[C_t+(r-\delta_t)+\left(\kappa(a-\delta)-\lambda\right)C_\delta+\frac{1}{2}\sigma_2^2 C_\delta^2+\rho\sigma_1\sigma_2 C_\delta\right]dt+\sigma_1 dW^{(1)}+\sigma_2 C_\delta dW^{(2)}.$$

We substitute both instantaneous rate of change of commodity spot price and the result we obtained above into (3.1.4) to finally obtain a stochastic process of Ito type, called a controlled diffusion process

$$dX_t=\alpha(t)X_t\cdot\left[(r-\delta)dt+\sigma_1 dW^{(1)})\right]$$

$$+(1-\alpha(t))X_t\left\{\begin{array}{l}\left[C_t+(r-\delta)+\left(\kappa(a-\delta)-\lambda\right)C_\delta+\frac{1}{2}\sigma_2^2\cdot C_\delta^2+\sigma_1\sigma_2\rho C_\delta\right]dt\\+\sigma_1 dW^{(1)}+\sigma_2 C_\delta dW^{(2)}\end{array}\right\}$$

Hence,

$$dX_t=X_t\cdot\left[(r-\delta)+(1-\alpha(t))\cdot\left(C_t+\left(\kappa(a-\delta)-\lambda\right)C_\delta+\frac{1}{2}\sigma_2^2 C_\delta^2+\sigma_1\sigma_2\rho C_\delta\right)\right]dt$$

$$+X_t\cdot\sigma_1 dW^{(1)}+(1-\alpha(t))X_t\cdot\sigma_2 C_\delta dW^{(2)}$$

(3.1.10)

with initial condition $X_0=x$.

3.2 Maximization of Portfolio Model

The investor whose portfolio wealth evolves according to the stochastic differential equation (3.10) is assumed to have a power utility function defined on a portfolio value at the final date T. The firm has a utility function that exhibits constant relative risk aversion equal to $1-\gamma$. We denote the portfolio wealth process with an initial capital at time $t=0$ equal to $X_0 = x$ together with a starting convenience yield δ by $X^{x,\delta}$. Recall that the dynamic of the controlled system is governed by (2.4.1) and (3.1.10), the two dependent Brownian motions $W^{(1)}$ and $W^{(2)}$ can be constructed in terms of two independent Brownian motions $Z^{(1)}$ and $Z^{(2)}$ by defining $dW^{(1)} = Z^{(1)}$ and $dW^{(2)} = \rho dZ^{(1)} + \sqrt{1-\rho^2}dZ^{(2)}$. Then, the system can be written in form of two independent Brownian motions $Z^{(1)}$ and $Z^{(2)}$ as

$$
\begin{aligned}
dX_t &= X_t \cdot \left[(r-\delta_t) + (1-\alpha_t)\left((\delta_t - \hat{\alpha})e^{-\kappa\tau} - A_1 \right) \right] dt + X_t \cdot (\sigma_1 + (1-\alpha_t)\cdot \sigma_2 C_\delta \rho) dZ^{(1)} \\
&\quad + (1-\alpha_t) X_t \cdot \sigma_2 C_\delta \sqrt{1-\rho^2} dZ^{(2)} \\
d\delta_t &= [\kappa(a-\delta_t) - \lambda] dt + \sigma_2 \rho dZ^{(1)} + \sigma_2 \sqrt{1-\rho^2} dZ^{(2)}
\end{aligned}
\tag{3.1.11}
$$

We can write (3.1.11) in the matrix from as

$$
d\begin{bmatrix} X_t \\ \delta_t \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} dt + \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{bmatrix} dZ^{(1)} \\ dZ^{(2)} \end{bmatrix} \tag{3.1.12}
$$

Where

$$\mu_1(t,\alpha_t,\delta_t,X_t) = X_t \cdot \left[(r-\delta_t) + (1-\alpha_t)\left((\delta_t - \hat{\alpha})e^{-\kappa\tau} - A_1 \right) \right]$$

$$\mu_2(t,\alpha_t,\delta_t,X_t) = \kappa(a-\delta_t) - \lambda$$

$$\sigma_{11}(t,\alpha_t,\delta_t,X_t) = X_t \cdot (\sigma_1 + (1-\alpha_t)\cdot \sigma_2 C_\delta \rho)$$

$$\sigma_{12}(t,\alpha_t,\delta_t,X_t) = (1-\alpha_t) X_t \cdot \sigma_2 C_\delta \sqrt{1-\rho^2}$$

$$\sigma_{21}(t,\alpha_t,\delta_t,X_t) = \sigma_2 \rho$$

$$\sigma_{22}(t,\alpha_t,\delta_t,X_t) = \sigma_2 \sqrt{1-\rho^2}.$$

Note that $C_t(\tau,\delta_t) = \delta_t \cdot e^{-\kappa\tau} - \dfrac{dA(\tau)}{dt}$, $C_\delta(\tau) = \left[\dfrac{e^{-\kappa\tau} - 1}{\kappa} \right]$, and

$$\frac{dA(\tau)}{dt} = -\left[\left(r - \hat{\alpha} + \frac{\sigma_2^2}{2\kappa^2} - \frac{\sigma_1\sigma_2\rho}{\kappa}\right) + \frac{1}{2}\sigma_2^2 \cdot \frac{e^{-2\kappa\tau}}{\kappa^2} + \left(\hat{\alpha}\kappa + \sigma_1\sigma_2\rho - \frac{\sigma_2^2}{\kappa}\right)\cdot\frac{e^{-\kappa\tau}}{\kappa}\right].$$

Define $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$, we obtain

$$\Sigma \cdot \Sigma^T = \begin{bmatrix} X_t^2\left(\sigma_1^2 + 2(1-\alpha_t)\sigma_1\sigma_2 C_\delta \rho + (1-\alpha_t)^2\sigma_2^2 C_\delta^2\right) & X_t\left(\sigma_1 + (1-\alpha_t)\sigma_2^2 C_\delta\right) \\ X_t\left(\sigma_1 + (1-\alpha_t)\sigma_2^2 C_\delta\right) & \sigma_2^2 \end{bmatrix}.$$

Here the operator $\mathcal{L}^\alpha$ associated with (3.1.11) is

$$\begin{aligned} (\mathcal{L}^\alpha f)(\cdot) &= \sum_{i=1}^{n} \mu_i(\cdot)\frac{\partial f(\cdot)}{\partial x_i} + \frac{1}{2}\sum_{i,k=1}^{n}(\Sigma\cdot\Sigma^T(\cdot))_{ij}\frac{\partial^2 f(\cdot)}{\partial x_i \partial x_j} \\ &= x_t \cdot \left[(r-\delta_t) + (1-\alpha_t)\left((\delta_t - \hat{\alpha})e^{-\kappa\tau} - A_1\right)\right]\frac{\partial f(\cdot)}{\partial x_1} + (\kappa(a-\delta_t)-\lambda)\frac{\partial f(\cdot)}{\partial x_2} \\ &+ \frac{1}{2}x_t^2\left(\sigma_1^2 + 2(1-\alpha_t)\sigma_1\sigma_2 C_\delta\rho + (1-\alpha_t)^2\sigma_2^2 C_\delta^2\right)\frac{\partial^2 f(\cdot)}{\partial x_1^2} + \frac{1}{2}\sigma_2^2\frac{\partial^2 f(\cdot)}{\partial x_2^2} \\ &+ x_t\left(\sigma_1 + (1-\alpha_t)\sigma_2^2 C_\delta\right)\frac{\partial^2 f(\cdot)}{\partial x_1 x_2}. \end{aligned}$$

(3.1.13)

The investors' objective is to maximize their expected utility of the final period of portfolio wealth, X_T. Hence the gain function is defined by

$$J(t,x,\delta;\alpha) = E[U(X_T^{t,x,\delta})],$$

From dynamic programming principle, the investor solves the value function denoted by

$$v(t,x,\delta) = \max_{\alpha_t \in [0,1]} E[U(X_T^{t,x,\delta})], \qquad (3.1.14)$$

$$t \in [0,T],\ 0 < T < \infty$$

where $U(\cdot): D \to [0,\infty)$ is a convex measurable function and E is a conditional expectation with respect to information at time t.

Remark 3.2.1 [13] Then we consider a specific case when the control space is reduced to a singleton set $\{\alpha_0\}$, i.e. there is no control on the state space.

Suppose that the control space A is reduced to a singleton set-$\{\alpha_0\}$, then the value function in (3.1.14) is a solution to the reduced Hamilton-Jacobi-Bellman equation associated with the stochastic control problem (3.11), shown as follow;

$$
\begin{aligned}
&-v_t - x\cdot\Big[(r-\delta)+(1-\alpha_0)\Big((\delta-\hat{\alpha})e^{-\kappa\tau}-A_1\Big)\Big]v_x + (\kappa(a-\delta)-\lambda)v_\delta \\
&+\frac{1}{2}x_t^2\left(\sigma_1^2+2(1-\alpha_0)\sigma_1\sigma_2 C_\delta\rho+(1-\alpha_0)^2\sigma_2^2 C_\delta^2\right)v_{xx}+\frac{1}{2}\sigma_2^2 v_{\delta\delta} \\
&+x_t\left(\sigma_1+(1-\alpha_0)\sigma_2^2 C_\delta\right)v_{x\delta}=0
\end{aligned}
$$

$$
\forall (t,x)\in\mathbb{T}\times\mathbb{R}_+ \text{ and } v(T,x)=\frac{x^\gamma}{\gamma},\ \ \forall x\in\mathbb{R}_+,\ 0<\gamma<1.
$$

However, the method to solve for a solution of this PDEs is sometimes too much complicated. Therefore, we will introduce a basic idea of numerical simulation for solving this kind of maximization problem in the next chapter.

CHAPTER IV

NUMERICAL SIMULATION

In this section, we give a discussion on Monte Carlo method which we use to simulate the stochastic process obtained from the previous chapter. Monte Carlo method is a broad class of computational algorithms used to solve various problems, especially in physics and mathematics. The concepts rely on generating suitable random numbers and observing that fraction of the numbers that obeys some properties. This method is useful for obtaining numerical solutions to problems that are difficult to solve analytically. In this study, we employ the Euler-Maruyama scheme for pathwises simulation.

4.1 The Euler-Maruyama Scheme

In part of a computational method, we first discretize time interval $[0,T]$ into N subintervals with an equivalent step size defined by $\Delta t = \frac{T}{N}$, for some positive integer N. Hence, a time discretization is

$$t = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T$$

then, we denote $X_{t_n}, S_{t_n}, \delta_{t_n}, F_{t_n}$ and α_{t_n} by X_n, S_n, δ_n, F_n and α_n, respectively.

The recursive formula for spot commodity price process, convenience yield, and spot futures price process can be derived by using Euler-Maruyama scheme shown as follows:

$$S_{n+1}^{(i)} = S_n^{(i)} + (r - \delta_n^{(i)}) S_n^{(i)} \Delta_n^{(i)} t + \sigma_1 \Delta_n^{(i)} W^{(1)}$$

$$\delta_{n+1}^{(i)} = \delta_n^{(i)} + [\kappa(a - \delta_n^{(i)}) - \lambda] \Delta_n^{(i)} t + \sigma_2 \Delta_n^{(i)} W^{(2)}$$

$$F_{n+1}^{(i)} = F_n^{(i)} + F_n^{(i)} \cdot \begin{bmatrix} C_t(t_n, \delta_n^{(i)}) + (r - \delta_n^{(i)}) + \left(\kappa(a - \delta_n^{(i)}) - \lambda\right) C_\delta(t_n) \\ + \frac{1}{2}\sigma_2^2 C_\delta^2(t_n) + (\sigma_1 \sigma_2 \rho) C_\delta(t_n) \end{bmatrix} \Delta_n^{(i)} t \qquad (4.1.1)$$

$$+ \sigma_1 S_n^{(i)} \cdot e^{C(t_n, \delta_n^{(i)})} \Delta_n^{(i)} W^{(1)} + \sigma_2 C_\delta(t_n) \cdot S_n^{(i)} \cdot e^{C(t_n, \delta_n^{(i)})} \Delta_n^{(i)} W^{(2)}$$

where $i = 1, \ldots, M$ for a large integer M.

Note that $C_t(\tau,\delta_t)=\delta_t\cdot e^{-\kappa\tau}-\dfrac{dA(\tau)}{dt}$, $C_\delta(\tau)=\left[\dfrac{e^{-\kappa\tau}-1}{\kappa}\right]$, $\tau=T-t$, and

$$\frac{dA(\tau)}{dt}=-\left[\left(r-\hat{\alpha}+\frac{\sigma_2^2}{2\kappa^2}-\frac{\sigma_1\sigma_2\rho}{\kappa}\right)+\frac{1}{2}\sigma_2^2\cdot\frac{e^{-2\kappa\tau}}{\kappa^2}+\left(\hat{\alpha}\kappa+\sigma_1\sigma_2\rho-\frac{\sigma_2^2}{\kappa}\right)\cdot\frac{e^{-\kappa\tau}}{\kappa}\right].$$

Note that, the increments of Brownian motion are correlated with correlation coefficient ρ.

To calculate the approximate result of each process, there are 3 steps summarized as follows:

(M1) Simulate a sample path of spot commodity by using recursive formula (4.1) to obtain the first sample path $i=1$, $\tilde{S}^{(1)}(T)$.

(M2) Repeat the procedure (M1) to obtain $\tilde{S}^{(i)}(T)$, $i=2,\ldots,M$.

(M3) Let $\tilde{S}_M(T)$ be the estimator of $E^{\alpha_0}[S_T \mid S_0=s,\delta_0=\delta]$. Defined by

$$\tilde{S}_M(T)=\frac{1}{M}\sum_{i=1}^{M}\tilde{S}^{(i)}(T).$$

Note that the approximate value of convenience yield and futures price can be calculated in the same way.

In case of experimentation, we first discretized time space into 1,000 equivalent subintervals, and simulated numerical results over 2,000 sample paths. The value of the closest futures contract to maturity is chosen to be an initial price of spot commodity. An initial of convenience yield is chosen to represent the situation of high and low cost of carrying as 0.1 and 0.4 respectively. We use the value of fair price at time $t=0$ obtained from the close-form formulation of Schwartz model as a starting point for futures prices approximation. For the unknown parameters, we used the estimated parameters approximated by [15]. The necessary parameters of our model are represented in the table below:

Parameter	Copper	Oil	Gold
S_0	110.04	19.99	379.27
r	0.06	0.06	0.06
μ	0.326	0.315	0.039
κ	1.156	1.876	0.011
a	0.248	0.106	-0.002
λ	0.256	0.198	0.0067
σ_1	0.274	0.393	0.135
σ_2	0.280	0.527	0.016
ρ	0.818	0.766	0.056

Table 4.1 The model parameters for three different commodities, copper, oil and gold.

Next, we will show results of a numerical simulation of three different types of commodity which are copper, oil, and precious metal-gold in order to show the tendency of spot commodity price, convenience yield and futures price. The path simulation of price, convenience yield and futures price of copper, with $\delta = 0.1$, are shown in figures 4.1 , 4.2, and 4.3 respectively. Figures 4.4 and 4.5 compare the behavior of exact futures price and approximate futures price obtain from Euler-Maruyama method in the cases of gold and oil. Then in figures 4.6 and 4.7 show a convergence of futures price to spot commodity at maturity.

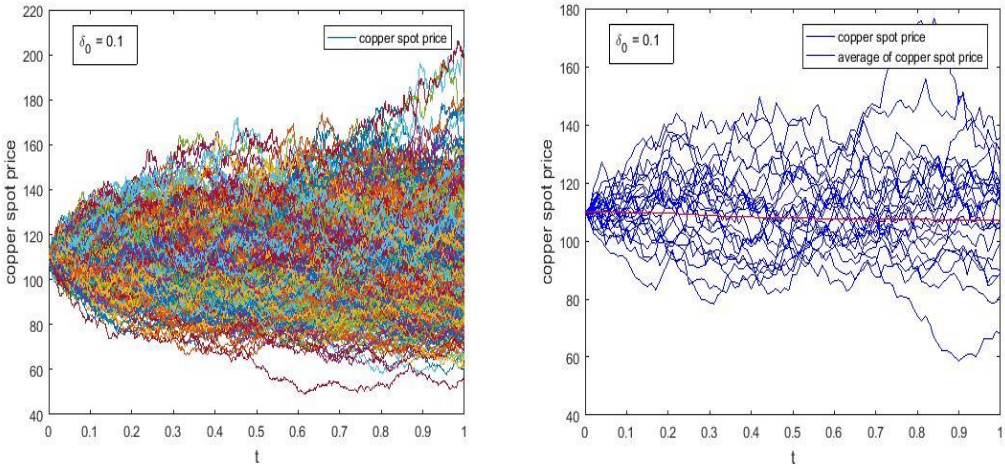


Figure 4.1 Graph of copper price sample paths with $\delta = 0.1$ *(left).*
Graph of copper price individual paths and the averages
with $\delta = 0.1$ *(right).*

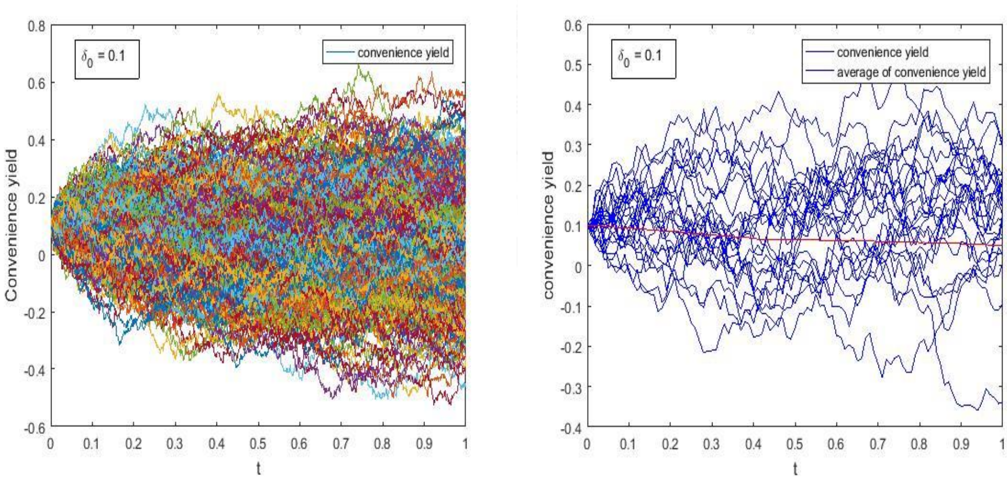


Figure 4.2 Graph of convenience yield sample paths with $\delta_0 = 0.1$ *(left).*
Graph of convenience yield individual paths and the averages
with $\delta_0 = 0.1$ *(right).*

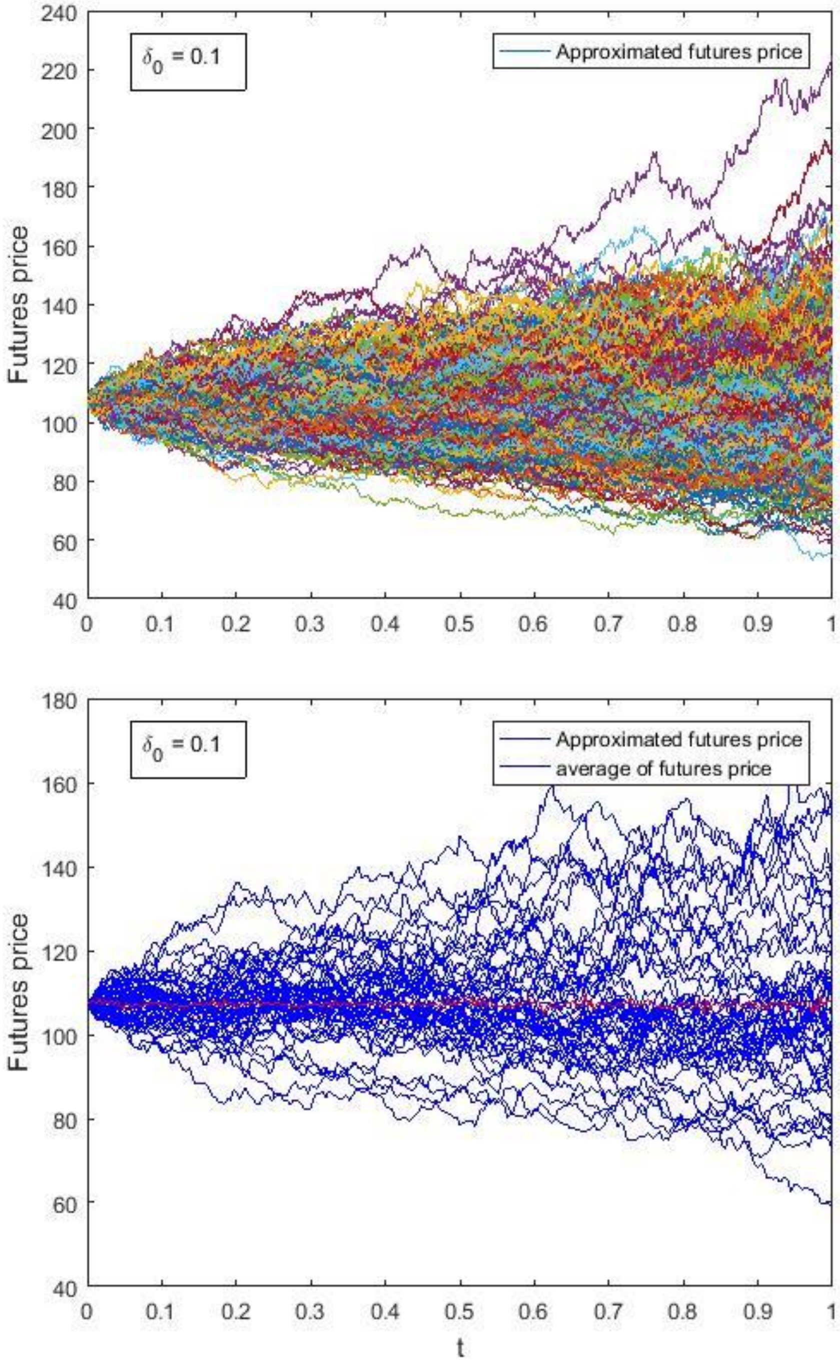


Figure 4.3 Graph of sample paths of copper futures price

and its average with $\delta_0 = 0.1$.

Graph of individual paths of copper futures price

and its average with $\delta_0 = 0.1$.

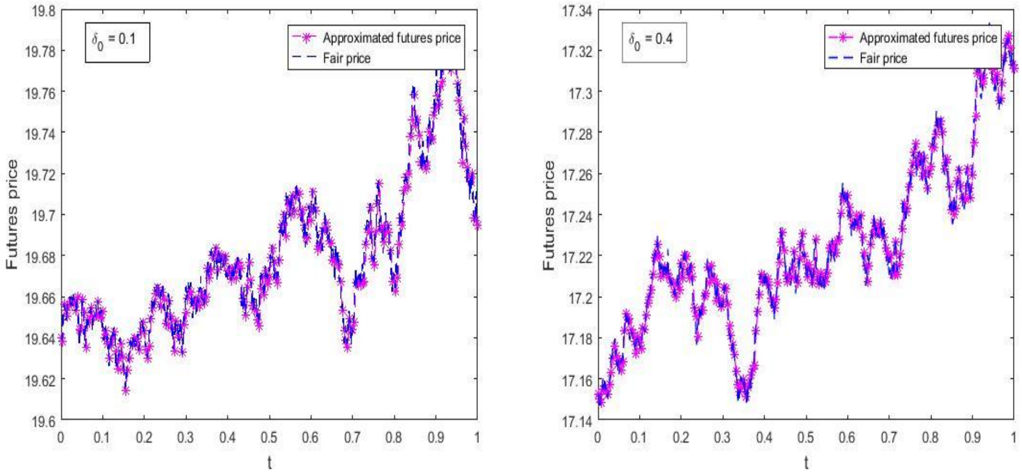


Figure 4.4 Graph of the approximate futures price and fair price of oil with $\delta_0 = 0.1$ *(left) and* $\delta_0 = 0.4$ *(right).*

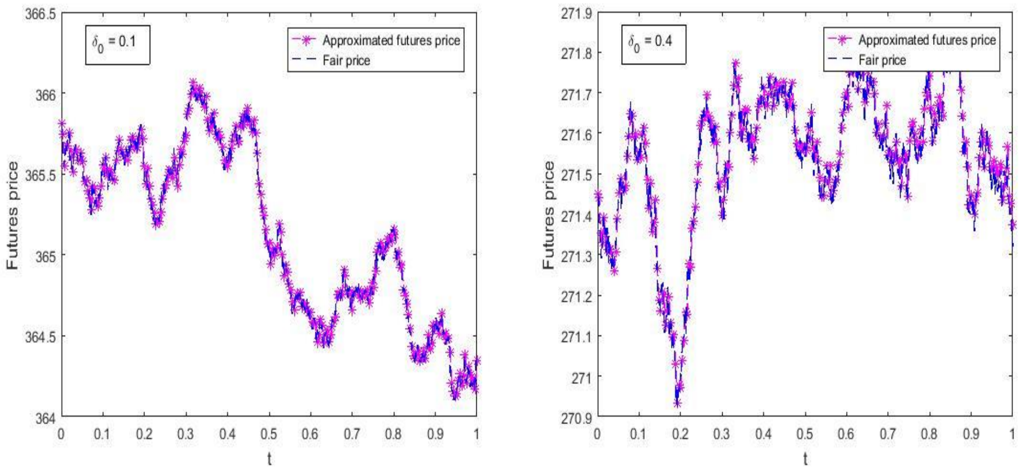


Figure 4.5 Graph of the approximate futures price and fair price of gold with $\delta_0 = 0.1$ *(left) and* $\delta_0 = 0.4$ *(right).*

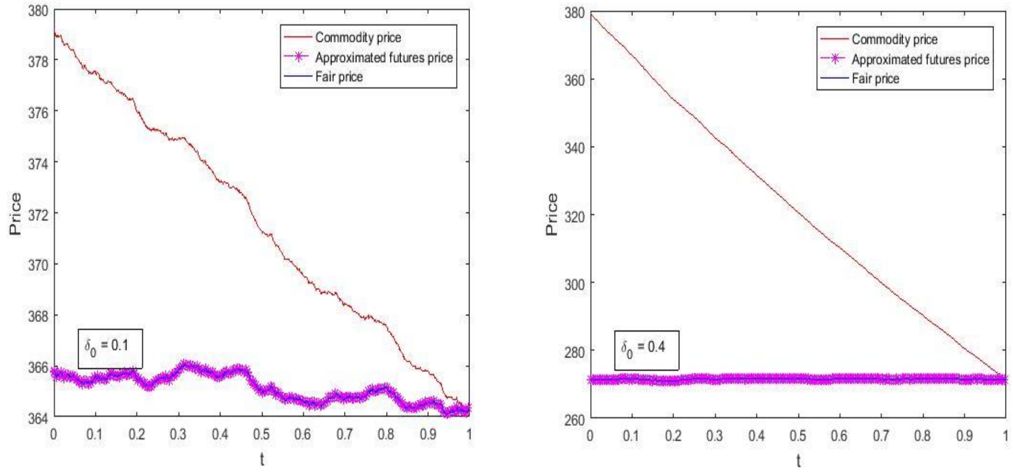


Figure 4.6 Graph of gold spot price and gold futures price with $\delta_0 = 0.1$ (left) and $\delta_0 = 0.4$ (right).

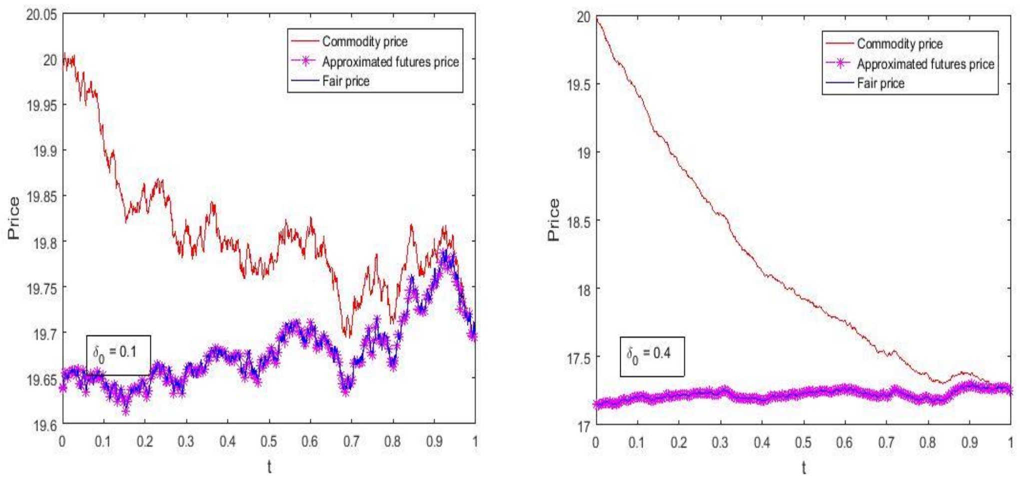


Figure 4.7 Graph of gold spot price and oil futures price with $\delta_0 = 0.1$ (left) and $\delta_0 = 0.4$ (right).

In general, commodity prices usually greater than commodity futures prices because the holder of physical assets want to earn more from the assets they kept in order to offset their cost of storage. This situation was shown in figures 4.6 and 4.7 above.

4.2 Portfolio Simulation

In this section, we first illustrate the behavior of our portfolio wealth when the proportion of an investment in physical commodity asset, α, is varying equal to 0.25, 0.5, and 0.75. The recursive formula for portfolio wealth process that we use to approximate numerical results is

$$X_{n+1} = X_n + X_n \cdot b(t_n, S_n, \delta_n, \alpha_n)\Delta_n t + X_n \eta_1(t_n, S_n, \delta_n, \alpha_n)\Delta_n W^{(1)}$$
$$+ X_n \eta_2(t_n, S_n, \delta_n, \alpha_n)\Delta_n W^{(2)}, \tag{4.2}$$

where

$$b(t_n, S_n, \delta_n, \alpha_n) = (r-\delta) + (1-\alpha(t_n)) \cdot \begin{pmatrix} C_t(t_n, \delta_n) + \left(\kappa(a-\delta_n) - \lambda\right) C_\delta(t_n) \\ + \frac{1}{2}\sigma_2^2 C_\delta^2(t_n) + \sigma_1\sigma_2\rho C_\delta(t_n) \end{pmatrix}$$

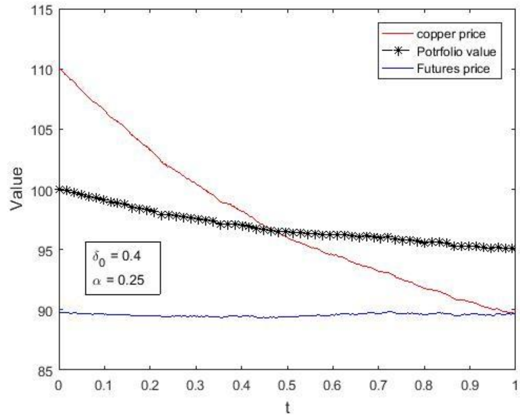
$$\eta_1(t_n, S_n, \delta_n, \alpha_n) = \sigma_1$$

$$\eta_2(t_n, S_n, \delta_n, \alpha_n) = (1-\alpha_n)\sigma_2 C_\delta(t_n).$$

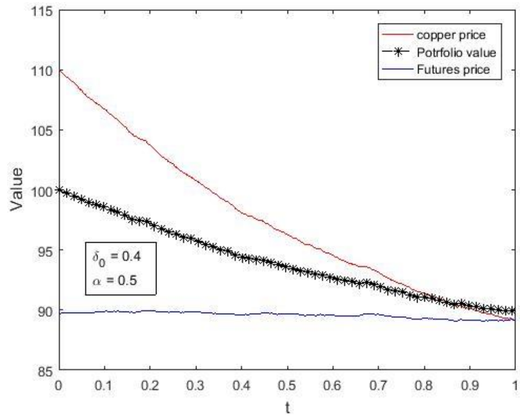
Note that $C_t(\tau, \delta_t) = \delta_t \cdot e^{-\kappa\tau} - \frac{dA(\tau)}{dt}$, $C_\delta(\tau) = \left[\frac{e^{-\kappa\tau} - 1}{\kappa}\right]$, and

$$\frac{dA(\tau)}{dt} = -\left[\left(r - \hat{\alpha} + \frac{\sigma_2^2}{2\kappa^2} - \frac{\sigma_1\sigma_2\rho}{\kappa}\right) + \frac{1}{2}\sigma_2^2 \cdot \frac{e^{-2\kappa\tau}}{\kappa^2} + \left(\hat{\alpha}\kappa + \sigma_1\sigma_2\rho - \frac{\sigma_2^2}{\kappa}\right) \cdot \frac{e^{-\kappa\tau}}{\kappa}\right].$$

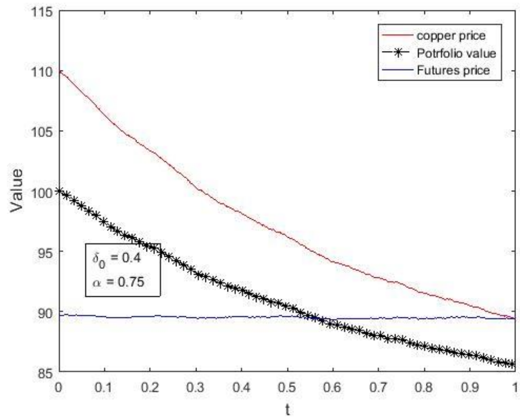
The starting value, X_0, for the portfolio of copper, gold and oil are chosen to be 100, 380 and 20, respectively. Figure 4.8 shows the result of portfolio of copper at time t when α is equal to 0.25, 0.5 and 0.75 under high convenience yield market. In Figure 4.9, we show the result of portfolio of gold at time t when α is equal to 0.25, 0.5 and 0.75 under high convenience yield market. Also, in figure 4.10, the result of portfolio of oil at time t when α is equal to 0.25, 0.5 and 0.75 under low convenience yield market is depicted.



(a)

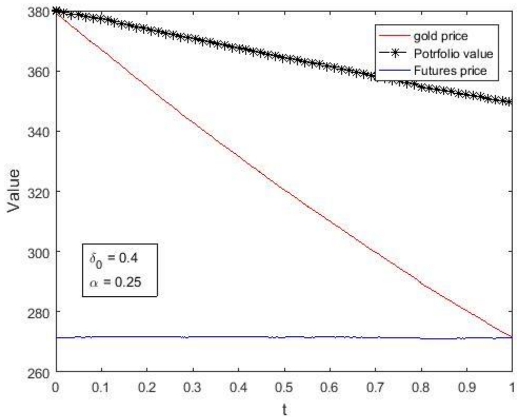


(b)

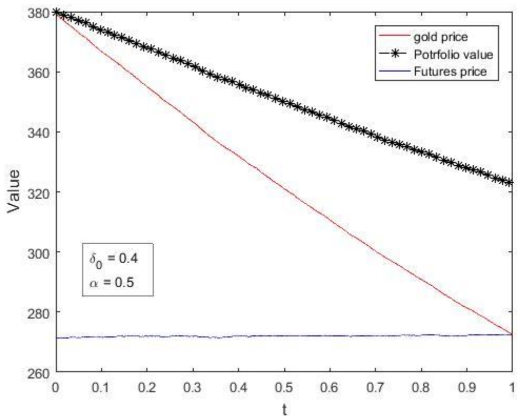


(c)

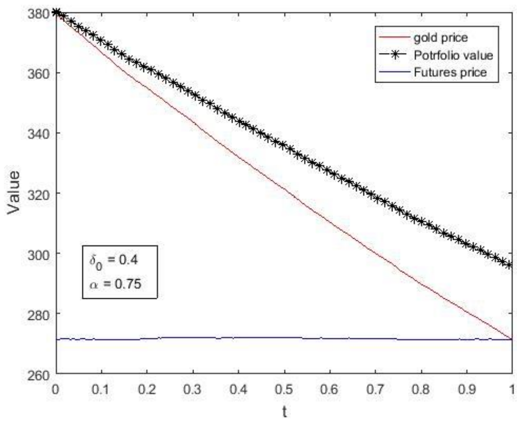
Figure 4.8 Graph of spot copper price, futures price and portfolio value when the convenience is high. (a) $\alpha = 0.25$. *(b)* $\alpha = 0.5$. *(c)* $\alpha = 0.75$.



(d)



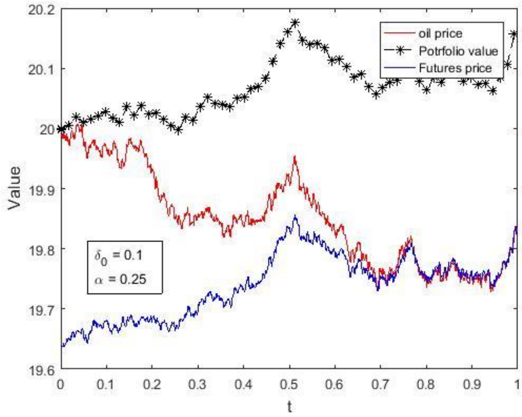
(e)



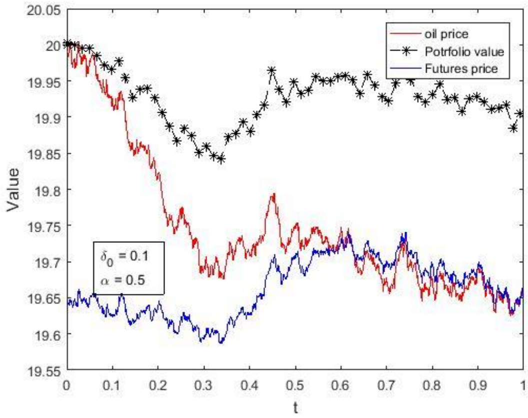
(f)

Figure 4.9 Graph of spot gold price, futures price and portfolio value when the convenience is high.

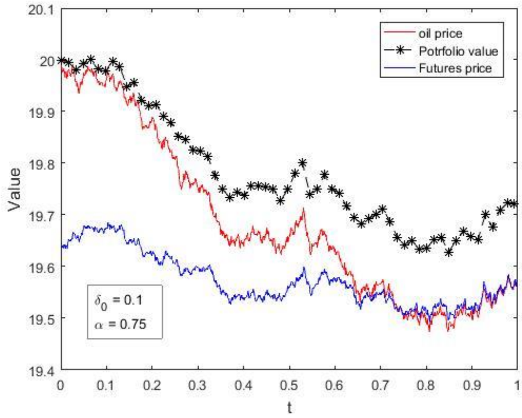
(d) $\alpha = 0.25$. *(e)* $\alpha = 0.5$. *(f)* $\alpha = 0.75$.



(g)



(h)



(i)

Figure 4.10 Graph of spot oil price, futures price and portfolio value when the convenience is low.

(g) $\alpha = 0.25$. *(h)* $\alpha = 0.5$. *(i)* $\alpha = 0.75$.

We see from figures 4.8 to 4.10 that when the proportion of an investment in physical commodity, α, approaches to one, the portfolio price behaves more like a price of commodity. On the other hand, the behavior of portfolio price becomes more similar to the futures price as the proportion α decreases.

4.3 Monte Carlo method solving for an optimal fixed ratio

In case of applying the Monte Carlo approach to solve for an optimal fixed ratio α. For a fixed positive integer J, let $\{\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(J)}\}$ be a randomly selected number in $[0,1]$. For a natural number M, let $\tilde{X}^{(j,i)}(T)$ be an approximate portfolio wealth at the final time T, associated with the ratio $\alpha^{(j)}$ for the i^{th} experimentation, where $i = 1, 2, \ldots, M,$ and satisfies SDEs (4.1)

The algorithm solving for a constant ratio is summarized as follows

(M1) For $j = 1$, picking $\alpha^{(j,i)}$ in the set $\{\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(J)}\}$

(M2) Simulate a sample path of spot commodity by using recursive formula (4.1) to obtain the first sample path $i = 1,\ \tilde{X}^{(1,1)}(T)$

(M3) Repeat the procedure (M2) to obtain $\tilde{X}^{(1,i)}(T)$ for $i = 2, \ldots, M$, and suppose that $\tilde{X}^{(1,i)}(T)$ are independent and identically distributed random sample with mean $E^{\alpha_1}[X_T \mid X_0 = x, \delta_0 = \delta]$.

(M4) Let $\tilde{X}_M^{(1)}(T)$ be the estimator of $E^{\alpha_1}[X_T \mid X_0 = x, \delta_0 = \delta]$. Defined by

$$\tilde{X}_M^{(1)}(T) = \frac{1}{M}\sum_{i=1}^{M} \tilde{X}^{(1,i)}(T).$$

(M5) Repeat the procedures (M1)-(M4) to obtain $\tilde{X}_M^{(j)}(T)$, $j = 2, \ldots, J$.

(M6) Then we solve for the maximum $\hat{X}_M = \max_{j \in [1,\ldots J]} \tilde{X}_M^{(j)}(T)$ and the fixed ratio, named $\hat{\alpha}_{opt}$ which depends only on J, and $\hat{\alpha}_{opt}$ corresponds to $\hat{X}_M$

Finally, we can find an approximate fixed ratio $\hat{\alpha}_{opt}$ that depends on an initial set up of our experiment.

Example	copper		gold		oil	
	$\hat{\alpha}_{opt}$	$\hat{X}_M$	$\hat{\alpha}_{opt}$	$\hat{X}_M$	$\hat{\alpha}_{opt}$	$\hat{X}_M$
$\delta_0 = 0.01$	0.8600	104.0218	0.9970	403.3404	0.5760	20.9316
$\delta_0 = 0.05$	0.8200	102.0466	0.9540	387.6894	0.7890	20.5550
$\delta_0 = 0.1$	0.0370	101.0809	0.052	383.1189	0.1980	20.4558
$\delta_0 = 0.2$	0.0350	101.0302	0.0130	381.4043	0.0640	20.2435
$\delta_0 = 0.3$	0.0210	100.6175	0.0040	380.9637	0.0230	20.2806
$\delta_0 = 0.4$	0.0070	100.6643	0.0020	380.5595	0.0070	20.3684
$\delta_0 = 0.5$	0.0060	100.6383	0.0070	381.0543	0.0080	20.2314

Table 4.2 The table shows the value of optimal investment ratio $\hat{\alpha}_{opt}$ *and the maximum value portfolio* $\hat{X}_M$ *when varying* δ_0.

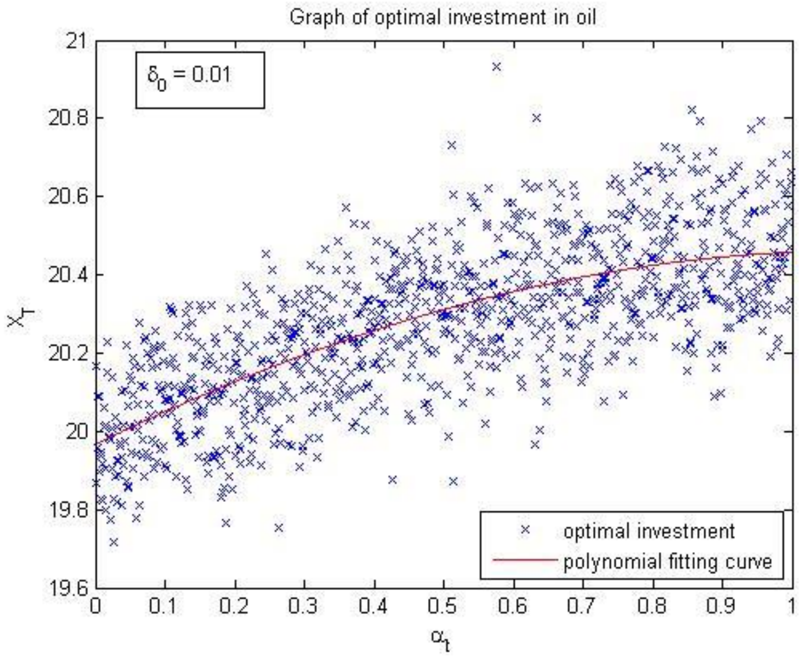


Figure 4.11 Graph portfolio value $\hat{X}_M$ *against optimal investment ratio* $\hat{\alpha}_{opt}$*, when* $\delta_0 = 0.01$

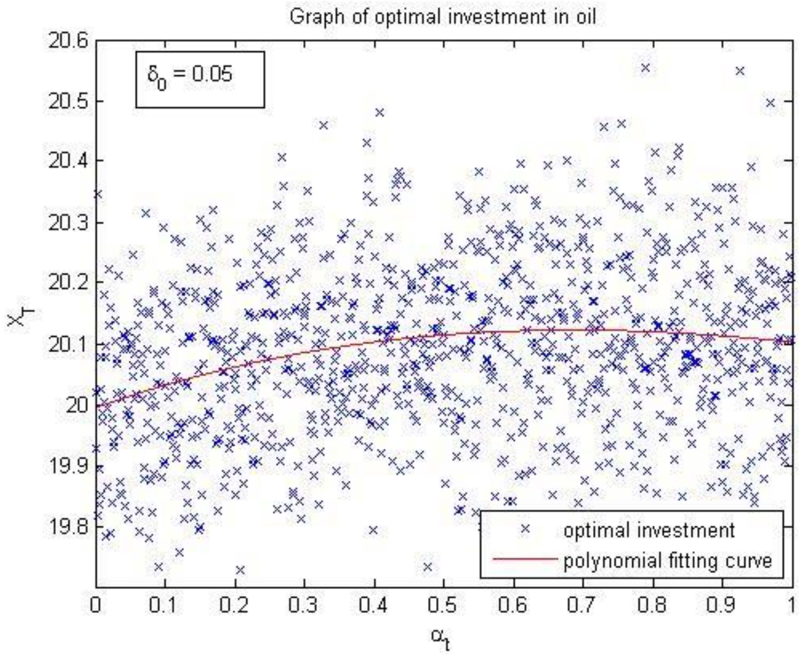


Figure 4.12 Graph portfolio value $\hat{X}_M$ *against optimal investment ratio* $\hat{\alpha}_{opt}$*, when* $\delta_0 = 0.05$

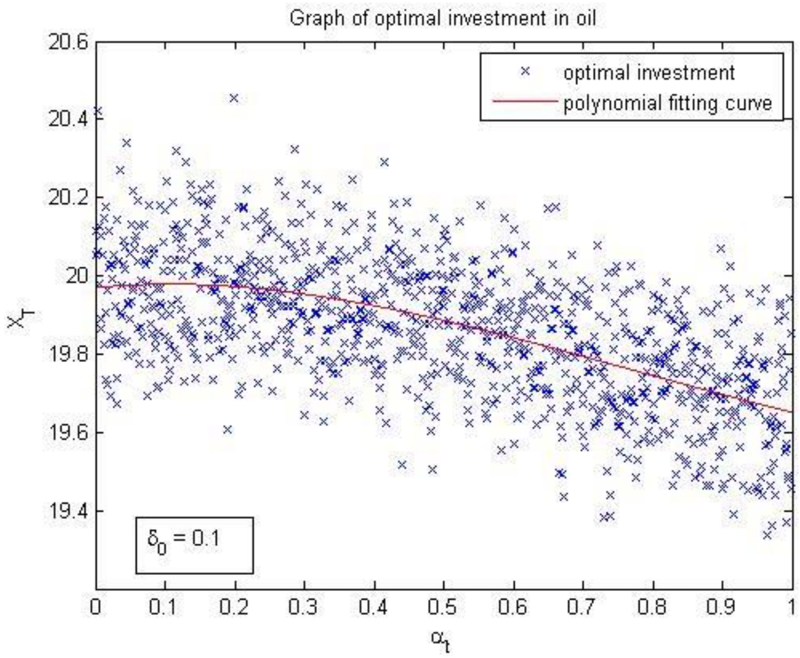


Figure 4.13 Graph portfolio value $\hat{X}_M$ *against optimal investment ratio* $\hat{\alpha}_{opt}$*, when* $\delta_0 = 0.1$

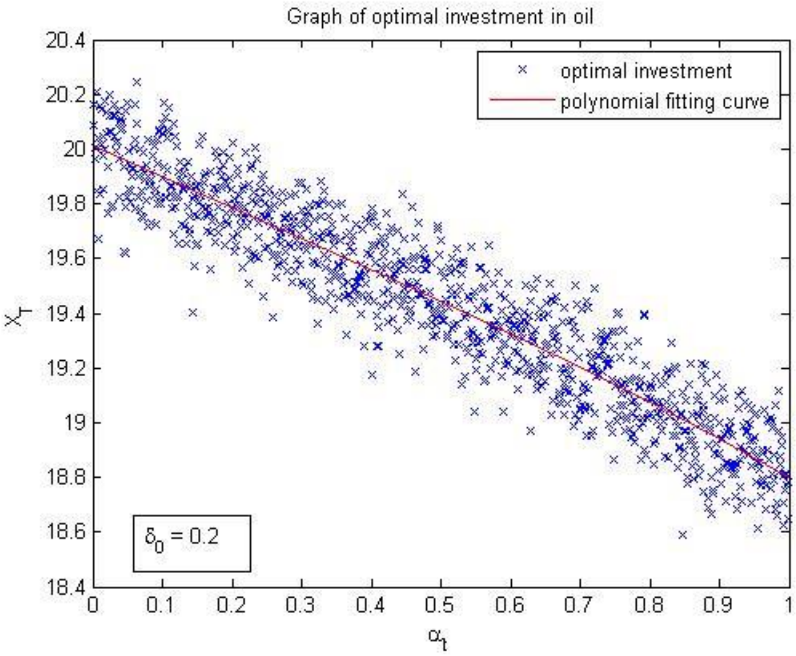


Figure 4.14 Graph portfolio value $\hat{X}_M$ *against optimal investment ratio* $\hat{\alpha}_{opt}$*, when* $\delta_0 = 0.2$

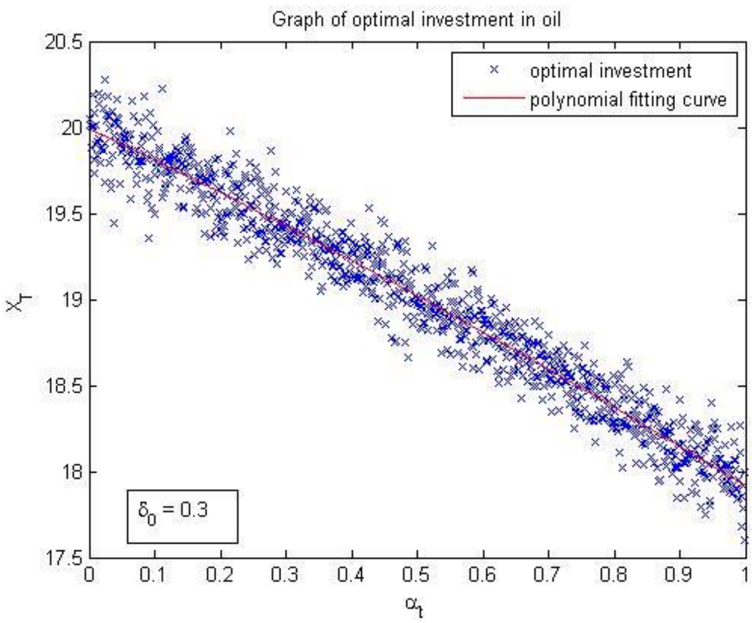


Figure 4.15 Graph portfolio value $\hat{X}_M$ *against optimal investment ratio* $\hat{\alpha}_{opt}$*, when* $\delta_0 = 0.3$

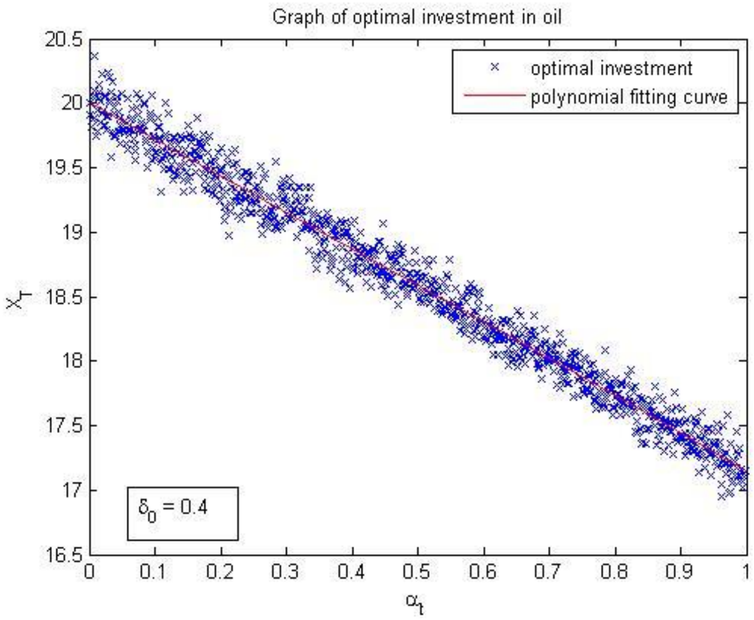


Figure 4.16 Graph portfolio value $\hat{X}_M$ *against optimal investment ratio* $\hat{\alpha}_{opt}$*, when* $\delta_0 = 0.4$

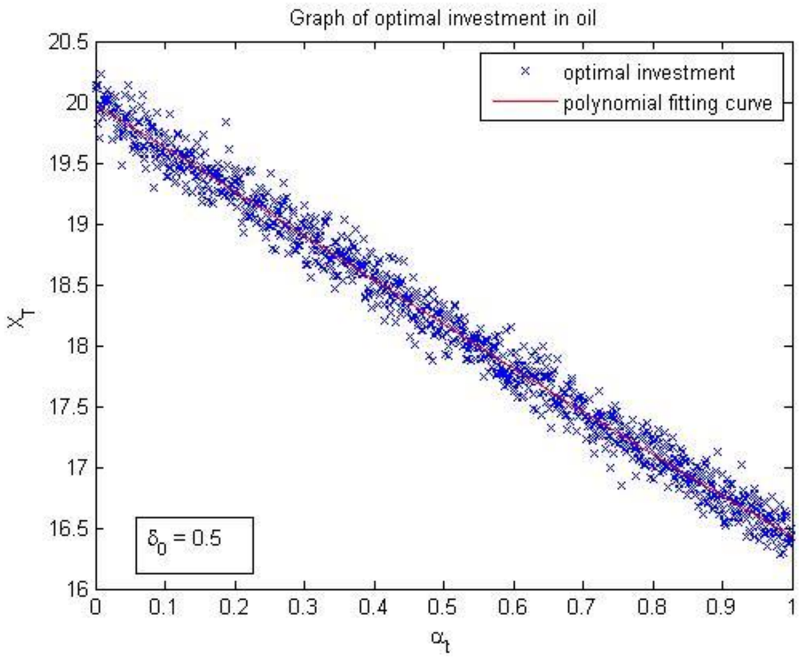


Figure 4.17 Graph portfolio value $\hat{X}_M$ *against optimal investment ratio* $\hat{\alpha}_{opt}$*, when* $\delta_0 = 0.5$

From figures 4.11 to 4.17, we illustrate the examples of the final wealth of portfolio when the proportion of an investment α is a constant belonged to the interval $[0,1]$. Since the result depicted in the graphs above is irregular, we then represent the trend of solution by using the interpolation function of degree two. We observe that the investment ratio of commodity tends to decrease when the convenience yield increases and tends to increase when the convenience yield decreases. This means that the investors avoid to hold the physical assets under a high convenience yield market and they are willing to store physical commodities when a convenience yield is low. Table 4.2 shows the result of an optimal investment $\hat{\alpha}_{opt}$, associated with a portfolio final wealth $\hat{X}_M$ when an initial of convenience yield δ_0 is various. This results in the same way as we observed from the graph above.

In order to maximize an investor utility of portfolio final wealth, we derive $Y_t = X_t^{\gamma}$ by applying Ito lemma to our process X_t, then we have

$$dY_t = \gamma Y_t \begin{pmatrix} (r-\delta_t)+(1-\alpha_t)\left(C_t + \left(\kappa(a-\delta)-\lambda\right)C_\delta + \frac{1}{2}\sigma_2^2 C_\delta^2 + \sigma_1\sigma_2\rho C_\delta \right) \\ +(\gamma-1)\left(\sigma_1^2 + (1-\alpha_t)^2\sigma_2^2 C_\delta^2 + \sigma_1\sigma_2\rho C_\delta (1-\alpha_t)\right) \end{pmatrix} dt$$

$$= \gamma Y_t \sigma_1 dW^{(1)} + \gamma Y_t \sigma_2 C_\delta (1-\alpha_t) dW^{(2)}.$$

We approximate this process through Euler-Maruyama method by following step (M1)-(M6) provided at the beginning of section 4.3 again.

Next, we illustrate the examples of the utility of final wealth of the investors' portfolio when the proportion of an investment α is a constant belonged to the interval $[0,1]$. First, the solution of an optimal investment α associated with the maximum of utility is shown in table 4.3. Also, from figures 4.18 to 4.24, we plot portfolio utilities of its wealth at time T on y-axis and the ratio of an investment α on x-axis. This results in the same way as the case of the maximization of portfolio final wealth shown previously.

Example	copper		gold		oil	
	$\hat{\alpha}_{opt}$	$\hat{U}_M$	$\hat{\alpha}_{opt}$	$\hat{U}_M$	$\hat{\alpha}_{opt}$	$\hat{U}_M$
$\delta_0 = 0.01$	0.9600	10.0418	0.9720	19.9956	0.9630	4.4282
$\delta_0 = 0.05$	0.9850	9.9342	0.9640	19.5948	0.7500	4.3902
$\delta_0 = 0.1$	0.0950	9.9091	0.0260	19.4780	0.3860	4.3750
$\delta_0 = 0.2$	0.0650	9.8818	0.0020	19.4804	0.0320	4.3545
$\delta_0 = 0.3$	0.0120	9.8695	0.0090	19.4373	0.0010	4.3518
$\delta_0 = 0.4$	0.0180	9.8620	0.0080	19.4489	0.0060	4.3528
$\delta_0 = 0.5$	0.0190	9.8771	0.0010	19.4399	0.0020	4.3481

Table 4.3 The table shows the value of optimal investment ratio $\hat{\alpha}_{opt}$ *and the maximum value portfolio* $\hat{U}_M$ *when varying* δ_0.

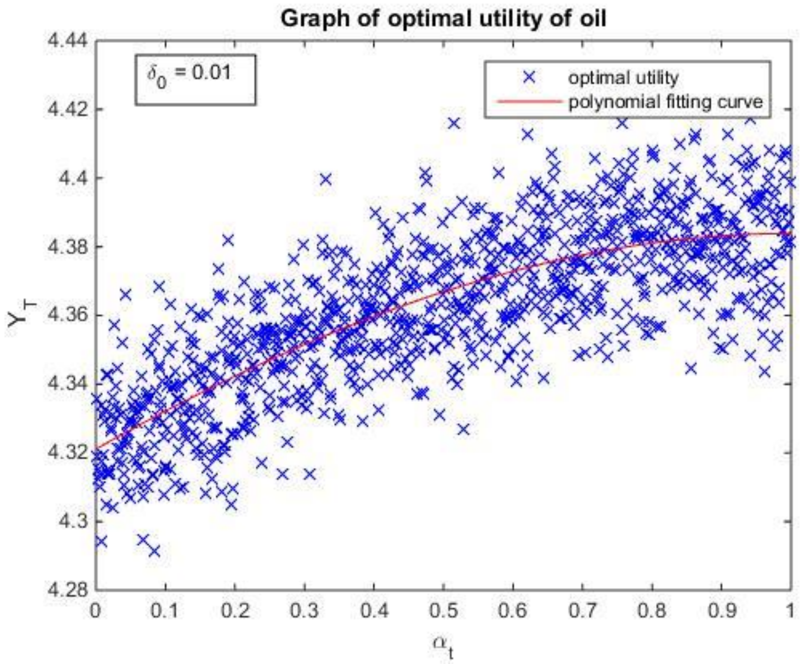


Figure 4.18 Graph utility $\hat{U}_M$ *against optimal investment ratio* $\hat{\alpha}_{opt}$, *when* $\delta_0 = 0.01$

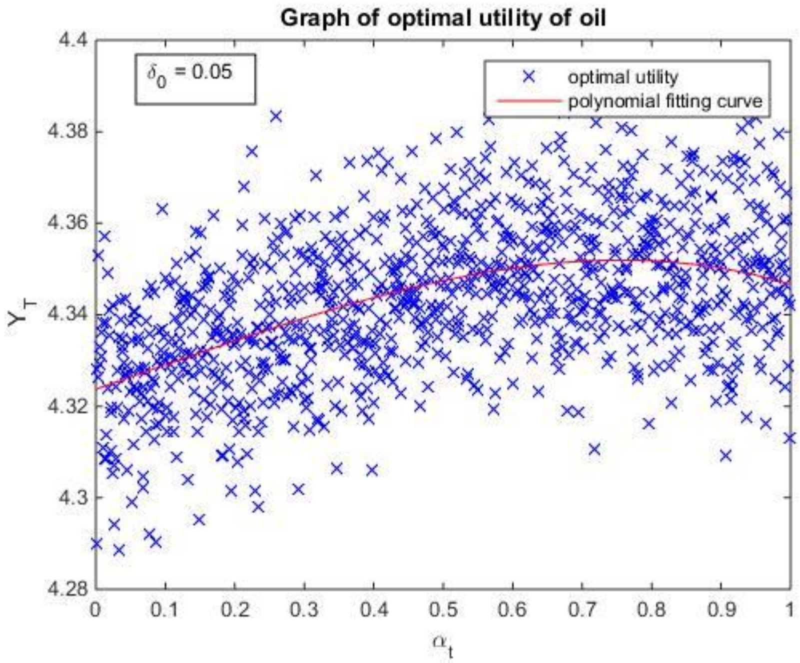


Figure 4.19 Graph utility $\hat{U}_M$ *against optimal investment ratio* $\hat{\alpha}_{opt}$, *when* $\delta_0 = 0.05$

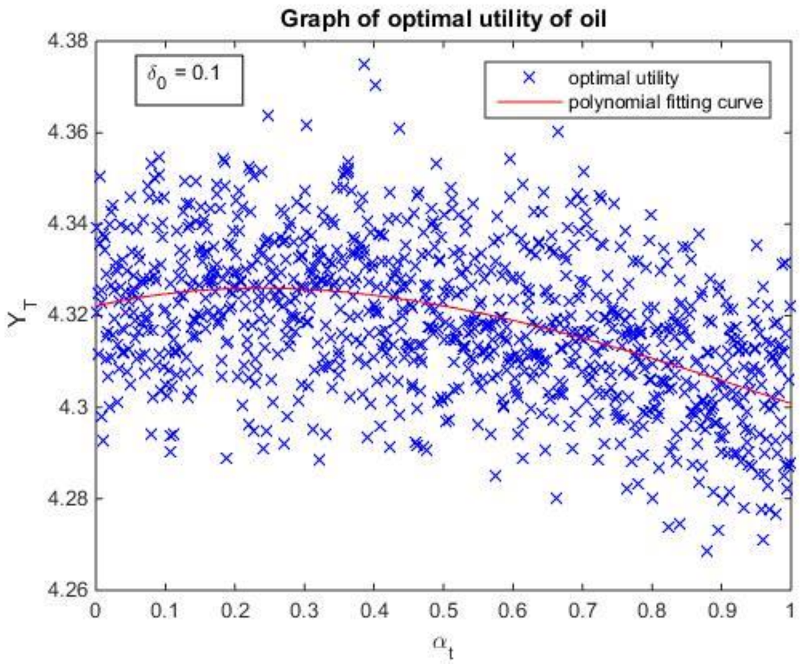


Figure 4.20 Graph utility $\hat{U}_M$ *against optimal investment ratio* $\hat{\alpha}_{opt}$, *when* $\delta_0 = 0.1$

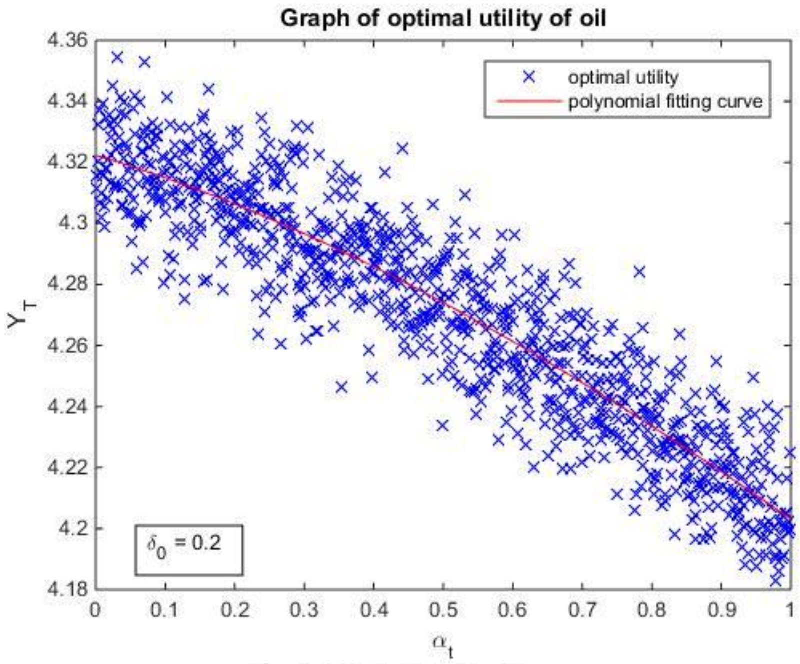


Figure 4.21 Graph utility $\hat{U}_M$ *against optimal investment ratio* $\hat{\alpha}_{opt}$, *when* $\delta_0 = 0.2$

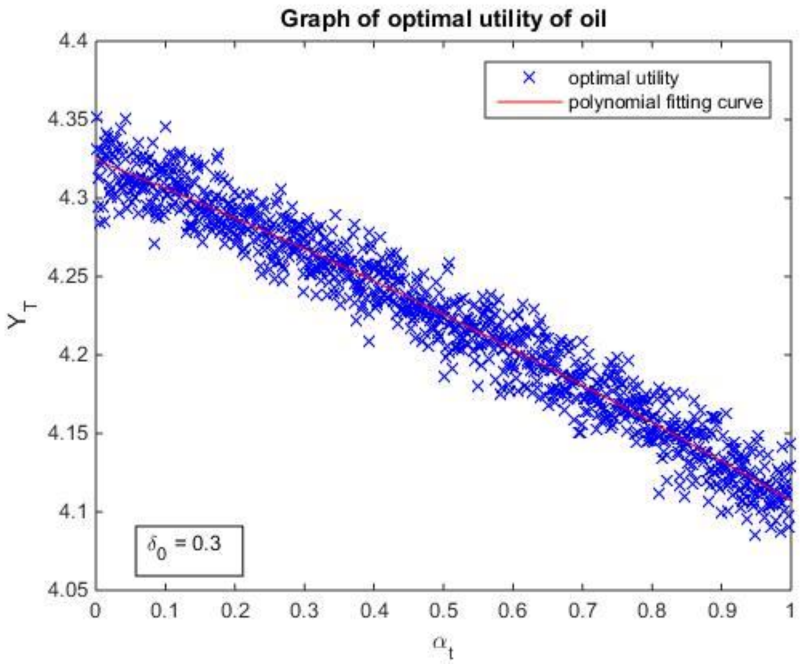


Figure 4.22 Graph utility $\hat{U}_M$ *against optimal investment ratio* $\hat{\alpha}_{opt}$, *when* $\delta_0 = 0.3$

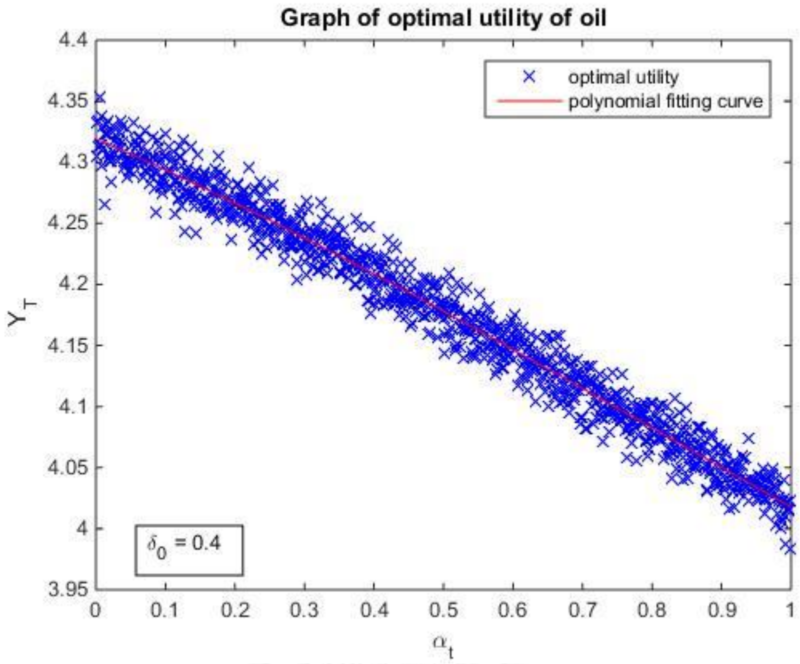


Figure 4.23 Graph utility $\hat{U}_M$ *against optimal investment ratio* $\hat{\alpha}_{opt}$*, when* $\delta_0 = 0.4$

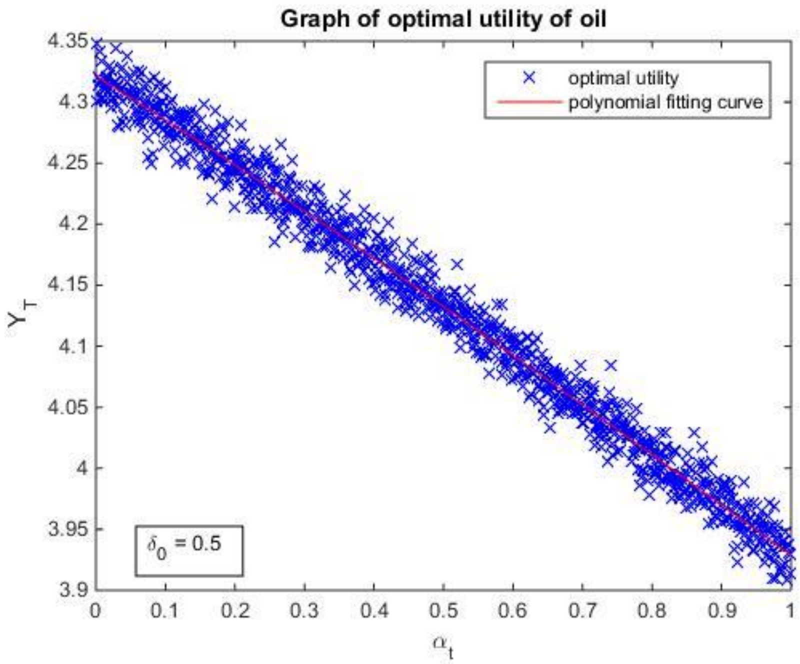


Figure 4.24 Graph utility $\hat{U}_M$ *against optimal investment ratio* $\hat{\alpha}_{opt}$*, when* $\delta_0 = 0.5$

Chapter V

Conclusion

In this thesis, we constructed a basic portfolio model of commodity asset and commodity futures. We have used the stochastic model proposed by Schwartz to describe the behavior of commodity products and the second factor convenience yield. In addition, we derived the instantaneous of futures price by applying Ito's lemma to the close-form formula of fair price of Schwartz two-factor model. The dynamics of our portfolio is a combination of spot commodity price process and futures price process. The basic problem is the investors want to maximize their final wealth by adjusting the weight of investment between the two assets in their portfolio. This problem can be explained in mathematical perspective as a derivation of control model. We introduced a straight forward method to approximate numerical results of this problem. The Euler-Murayama scheme is a fundamental method used to approximate stochastic processes in our study, and by observing patiently for optimal investment value finally we obtained the results.

In the experiment, we analyze three types of market, which are copper, oil, and gold, by using parameters estimated by Schwartz in 1997. We first illustrate the behavior of spot commodity price and futures price in high convenience yield and low convenience yield market. Furthermore, by simulation of futures price, the results have shown that the price calculated by close-form formula of Schwartz and by approximation method are perfectly matched. Then the convergence between spot commodity prices and futures prices are shown. Also, we illustrated the trait of portfolio processes when the optimal investment is fixed. The satisfaction of the investors is demonstrated in two ways. The first one is directly explained by portfolio final wealth and another one is through a utility function. In both case, Monte-Carlo is easy to implement due to it certain algorithm. However, it has reflected a lot of controversy on its accuracy. The change in value of the investor satisfaction when the amount of investment changed is depicted in figures 4.21-4.27. From the graph, it shows an irregularity of our portfolio when a proportion of physical commodity investment changed. We use a polynomial curve of degree two as a proxy to represent a tendency of a portfolio final wealth when α is varied. The numerical experiment resulted in two trivial cases, either commodity or futures contracts was invested. For this reason, we then observe for more information by tracking of the behavior of commodity spot price and futures spot price in every type of market we had provided. From the observation, when the convenience yield is high, the price of physical commodity is also higher than the futures price and then dramatically fail to meet the futures price at the maturity. This leads to the situation that the investors tend to invest almost of their portfolio value to the futures contracts in order to balance against the price risks in spot commodity that sharply failed. On the other hand, when the convenience yield is low, the price of physical commodity is also lower than the futures price and then increase to meet the futures price at the maturity. The rising in price of commodity attract the investors to invest more in physical assets rather than investing in futures contracts. Nonetheless, the observation only in convenience yield parameter is not a strong evidence to prove the sensitivity of portfolio model.

Future Work

For a future work, we want to solve our control problem (3.13) analytically through the PDEs,

$$-v_t - x\cdot\Big[(r-\delta)+(1-\alpha_0)\Big((\delta-\hat{\alpha})e^{-\kappa\tau}-A_1\Big)\Big]v_x + (\kappa(a-\delta)-\lambda)v_\delta$$

$$+\frac{1}{2}x_t^2\left(\sigma_1^2+2(1-\alpha_0)\sigma_1\sigma_2C_\delta\rho+(1-\alpha_0)^2\sigma_2^2C_\delta^2\right)v_{xx}+\frac{1}{2}\sigma_2^2v_{\delta\delta}$$

$$+x_t\left(\sigma_1+(1-\alpha_0)\sigma_2^2C_\delta\right)v_{x\delta}=0$$

$$\forall(t,x)\in\mathbb{T}\times\mathbb{R}_+ \text{ and } v(T,x)=\frac{x^\gamma}{\gamma},\ \ \forall x\in\mathbb{R}_+,\ 0<\gamma<1.$$

by using the same method as [14]. Furthermore, we want to improve our portfolio model to be a more realistic one by consider an optimal consumption of the investors instead.

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APPENDIX

VITA

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